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A strong form of β-$\mathcal{I}$-continuous functions

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Abstract. In this paper, β-$\mathcal{I}$-open sets are used to define and investigate a new class of functions called stongly β-$\mathcal{I}$-continuous functions in ideal topological spaces.

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1. Introduction

The subject of ideals in topological spaces has been introduced and studied by Kuratowski [5] and Vaidyanathasamy [6]. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) $A \in \mathcal{I}$ and $B \subset A$ implies $B \in \mathcal{I}$ and (ii) $A \in \mathcal{I}$ and $B \in I$ implies $A \cup B \in \mathcal{I}$. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\mathcal{P}(X)$ is the set of all subsets of X, a set operator $(.)^*$: $\mathcal{P}(X) \to \mathcal{P}(X)$, called the local function [6] of A with respect to τ and $\mathcal{I}$, is defined as follows: for $A \subset X$, $A^*(\tau, \mathcal{I}) = \{x \in X | U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau | x \in U\}$. A Kuratowski closure operator $\mathrm{Cl}^*(.)$ for a topology $\tau^*(\tau, \mathcal{I})$ called the $*$-topology, which is finer than τ is defined by $\mathrm{Cl}^*(A) = A \cup A^*(\tau, \mathcal{I})$, When there is no chance of confusion, $A^*(\mathcal{I})$ is denoted by A^*. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal topological space. By a space, we always mean a topological space (X, τ) with no separation properties assumed. If $A \subset X$, $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ will denote the closure and interior of A in (X, τ), repectively.

A point $x \in X$ is called a θ-cluster point of A if $\mathrm{Cl}(V) \cap A \neq \varnothing$ for every open set V of X containing x. The set of all θ-cluster points of A is said to be the θ-closure of A [7] and is denoted by $\mathrm{Cl}_\theta(A)$. If $A = \mathrm{Cl}_\theta(A)$, then the set A is said to be θ-closed [7]. The complement of a θ-closed set is said to be θ-open [7]. The union of all θ-open sets contained in a subset A is called the θ-interior of A and is denoted by $\mathrm{Int}_\theta(A)$. It follows from [7] that the collection of θ-open sets in a topological space (X, τ) forms a topology τ_θ on X. In this paper, the concept of stongly β-$\mathcal{I}$-continuous functions is introduced and studied. Some of their characteristic properties are investigated.

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2. Preliminaries

A subset S of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be β-$\mathcal{I}$-open [4] (resp. α-$\mathcal{I}$-open [4]) if $S \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}^*(S)))$ (resp. $S \subset \mathrm{Int}(\mathrm{Cl}^*(\mathrm{Int}(S)))$). The complement of a β-$\mathcal{I}$-open set is called β-$\mathcal{I}$-closed [4]. The intersection of all β-$\mathcal{I}$-closed sets containing S is called the β-$\mathcal{I}$-closure of S and is denoted by ${}_{\beta\mathcal{I}}\,\mathrm{Cl}(S)$. The β-$\mathcal{I}$-interior of S is defined by the union of all β-$\mathcal{I}$-open sets contained in S and is denoted by ${}_{\beta\mathcal{I}}\,\mathrm{Int}(S)$. A subset S of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be β-$\mathcal{I}$-regular [8] if it is both β-$\mathcal{I}$-open and β-$\mathcal{I}$-closed. The family of all β-$\mathcal{I}$-regular (resp. β-$\mathcal{I}$-open, β-$\mathcal{I}$-closed, α-$\mathcal{I}$-open) sets of $(X, \tau, \mathcal{I})$ is denoted by $\beta\mathcal{I}R(X)$ (resp. $\beta\mathcal{I}O(X)$, $\beta\mathcal{I}C(X)$, $\alpha\mathcal{I}O(X)$). The family of all β-$\mathcal{I}$-regular (resp. β-$\mathcal{I}$-open, β-$\mathcal{I}$-closed) sets of $(X, \tau, \mathcal{I})$ containing a point $x \in X$ is denoted by $\beta\mathcal{I}R(X, x)$ (resp. $\beta\mathcal{I}O(X, x)$, $\beta\mathcal{I}C(X, x)$). A point $x \in X$ is called the β-$\mathcal{I}$-θ-cluster point of S if ${}_{\beta\mathcal{I}}\,\mathrm{Cl}(U) \cap S \neq \varnothing$ for every β-$\mathcal{I}$-open set U of $(X, \tau, \mathcal{I})$ containing x. The set of all β-$\mathcal{I}$-θ-cluster points of S is called the β-$\mathcal{I}$-closure of S and is denoted by ${}_{\beta\mathcal{I}}\,\mathrm{Cl}_\theta(S)$. A subset S is said to be β-$\mathcal{I}$-θ-open if its complement is β-$\mathcal{I}$-θ-closed. A point $x \in X$ is called the β-$\mathcal{I}$-θ-interior point of S if there exists a β-$\mathcal{I}$-regular set U of X containing x such that $x \in U \subset S$. The set of all β-$\mathcal{I}$-θ-interior points of S and is denoted by ${}_{\beta\mathcal{I}}\,\mathrm{Int}_\theta(\mathrm{S})$.

Definition 1. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be β-$\mathcal{I}$-continuous (see [4]) if $f^{-1}(V) \in \beta\mathcal{I}O(X)$ for every $V \in \sigma$, or equivalently, $f^{-1}(V) \in \beta\mathcal{I}C(X)$ for every closed set V of Y.*

Theorem 1 (see [4]). *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is β-$\mathcal{I}$-continuous if and only if for each $x \in X$ and each open set V of Y containing $f(x)$ there exists $U \in \beta\mathcal{I}O(X, x)$ such that $f(U) \subset V$.*

3. Strongly β-$\mathcal{I}$-continuous functions

We have introduced the following definition

Definition 2. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be strongly β-$\mathcal{I}$-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \beta\mathcal{I}O(X, x)$ such that $f({}_{\beta\mathcal{I}}\mathrm{Cl}(U)) \subset \mathrm{Cl}(V)$.*

Theorem 2. *Every β-$\mathcal{I}$-continuous function is strongly β-$\mathcal{I}$-continuous.*

Proof. Suppose that $x \in X$ and V is any open set of Y containing $f(x)$. Since f is β-$\mathcal{I}$-continuous and $\mathrm{Cl}(V)$ is closed in Y, $f^{-1}(V)$ is β-$\mathcal{I}$-open and $f^{-1}(\mathrm{Cl}(V))$ is β-$\mathcal{I}$-closed in X. Now, put $U = f^{-1}(V)$. Then we have $U \in \beta\mathcal{I}O(X, x)$ and ${}_{\beta\mathcal{I}}\,\mathrm{Cl}(U) \subset f^{-1}(\mathrm{Cl}(V))$. Therefore, we obtain $f({}_{\beta\mathcal{I}}\mathrm{Cl}(U)) \subset \mathrm{Cl}(V)$. This shows that f is strongly β-$\mathcal{I}$-continuous. □

The converse of Theorem 2 is not true as it can been seen from the following example.

Example 1. *Let $X = \{a, b, c\}$, $\tau = \{\varnothing, \{a\}, X\}$ and $\sigma = \{\varnothing,\ \{b\},\ X\}$ and $\mathcal{I} = \{\varnothing, \{a\}\}$. Then the identity function $f : (X, \tau, \mathcal{I}) \to (X, \sigma)$ is strongly β-$\mathcal{I}$-continuous but not β-$\mathcal{I}$-continuous.*

Theorem 3. *For a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ the following properties are equivalent:*

(i) f is strongly β-$\mathcal{I}$-continuous;

(ii) ${}_{\beta\mathcal{I}}\,\mathrm{Cl}_\theta(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}_\theta(B))$ for every subset B of Y;

(iii) $f({}_{\beta\mathcal{I}}\mathrm{Cl}_\theta(A)) \subset \mathrm{Cl}_\theta(f(A))$ for every subset A of X.

Proof. (i)⇒(iii): Let B be any subset of Y. Suppose that $x \notin f^{-1}(\mathrm{Cl}_\theta(B))$. Then $f(x) \notin \mathrm{Cl}_\theta(B)$ and there exists an open set V of Y containing $f(x)$ such that $\mathrm{Cl}(V) \cap B = \varnothing$. Since f is strongly $\beta - \mathcal{I}$-continuous, there exists $U \in \beta\mathcal{I}O(X, x)$ such that $f({}_{\beta\mathcal{I}}\mathrm{Cl}(U)) \subset \mathrm{Cl}(V)$. Therefore, we have $f({}_{\beta\mathcal{I}}\mathrm{Cl}(U)) \cap B = \varnothing$ and ${}_{\beta\mathcal{I}}\,\mathrm{Cl}(U) \cap f^{-1}(B) = \varnothing$. This shows that $x \notin_{\beta\mathcal{I}} \mathrm{Cl}_\theta(f^{-1}(B))$. Hence, we obtain ${}_{\beta\mathcal{I}}\,\mathrm{Cl}_\theta(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}_\theta(B))$.
(ii)⇒(iii): Let A be any subset of X. Then we have

$$ {}_{\beta\mathcal{I}}\,\mathrm{Cl}_\theta(A) \subset_{\beta\mathcal{I}} \mathrm{Cl}_\theta(f^{-1}(f(A))) \subset f^{-1}(\mathrm{Cl}_\theta(f(A))) $$

and hence $f({}_{\beta\mathcal{I}}\mathrm{Cl}_\theta(A)) \subset \mathrm{Cl}_\theta(f(A))$.
(iii)⇒(ii): Let B be a subset of Y. We have $f({}_{\beta\mathcal{I}}\mathrm{Cl}_\theta(f^{-1}(B))) \subset \mathrm{Cl}_\theta(f(f^{-1}(B))) \subset \mathrm{Cl}_\theta(B)$ and hence ${}_{\beta\mathcal{I}}\,\mathrm{Cl}_\theta(f^{-1}(B)) \subset f^{-1}(\mathrm{Cl}_\theta(B))$.
(ii)⇒(i): Let $x \in X$ and V be an open set of Y containing $f(x)$. Then we have $\mathrm{Cl}(V) \cap (Y - \mathrm{Cl}(V)) = \varnothing$ and $f(x) \notin \mathrm{Cl}_\theta(Y - \mathrm{Cl}(V))$. Hence, $x \notin f^{-1}(\mathrm{Cl}_\theta(Y - \mathrm{Cl}(V)))$ and $x \notin_{\beta mathcal I} \mathrm{Cl}_\theta(f^{-1}(Y - \mathrm{Cl}(V)))$. There exists $U \in \beta\mathcal{I}O(X, x)$ such that ${}_{\beta\mathcal{I}}\,\mathrm{Cl}(U) \cap f^{-1}(Y - \mathrm{Cl}(V)) = \varnothing$ and hence $f({}_{\beta\mathcal{I}}\mathrm{Cl}(U)) \subset \mathrm{Cl}(V)$. This shows that f is strongly $\beta - \mathcal{I}$-continuous. □

Theorem 4. *For a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ the following properties are equivalent:*

(i) f is strongly β-$\mathcal{I}$-continuous;

(ii) $f^{-1}(V) \subset {}_{\beta\mathcal{I}}\,\mathrm{Int}_\theta(f^{-1}(\mathrm{Cl}(V)))$ for every open set V of Y;

(iii) ${}_{\beta\mathcal{I}}\,\mathrm{Cl}_\theta(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$ for every open set V of Y.

Proof. (i)⇒(ii): Suppose that V is any open set of Y and $x \in f^{-1}(V)$. Then $f(x) \in V$ and there exists $U \in \beta\mathcal{I}O(X, x)$ such that $f({}_{\beta\mathcal{I}}\mathrm{Cl}(U)) \subset \mathrm{Cl}(U)$. Therefore, $x \in U \subset {}_{\beta\mathcal{I}}\,\mathrm{Cl}(U) \subset f^{-1}(\mathrm{Cl}(V))$. This shows that $x \in {}_{\beta\mathcal{I}}\,\mathrm{Int}_\theta(f^{-1}(\mathrm{Cl}(V)))$ for any open set V of Y. In consequence, $f^{-1}(V) \subset {}_{\beta\mathcal{I}}\,\mathrm{Int}_\theta(f^{-1}(\mathrm{Cl}(V)))$.
(ii)⇒(iii): Suppose that V is any open set of Y and $x \notin f^{-1}(\mathrm{Cl}(V))$. Then $f(x) \notin \mathrm{Cl}(V)$. It follows that there exists an open set U of Y such that $U \cap V = \varnothing$ and hence $\mathrm{Cl}(U) \cap V = \varnothing$. Therefore, we have $f^{-1}(\mathrm{Cl}(U)) \cap f^{-1}(V) = \varnothing$. Since $x \in f^{-1}(U)$, by (ii), $x \in {}_{\beta\mathcal{I}}\,\mathrm{Int}_\theta(f^{-1}(\mathrm{Cl}(U)))$. In consequence, there exists $W \in \beta\mathcal{I}O(X, x)$ such that ${}_{\beta\mathcal{I}}\,\mathrm{Cl}(W) \subset f^{-1}(\mathrm{Cl}(U))$. Thus, we have ${}_{\beta\mathcal{I}}\,\mathrm{Cl}(W) \cap f^{-1}(V) = \varnothing$ and hence $x \notin_{\beta\mathcal{I}} \mathrm{Cl}_\theta(f^{-1}(V))$. This shows that ${}_{\beta\mathcal{I}}\,\mathrm{Cl}_\theta(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$.
(iii)⇒(i): Suppose that $x \in X$ and V is any open set of Y containing $f(x)$. Then, $V \cap (Y - \mathrm{Cl}(V)) = \varnothing$ and $f(x) \notin \mathrm{Cl}(Y - \mathrm{Cl}(V))$. Therefore, $x \notin f^{-1}(\mathrm{Cl}(Y - \mathrm{Cl}(V)))$ and by (iii), $x \notin_{\beta\mathcal{I}} \mathrm{Cl}_\theta(f^{-1}(Y - \mathrm{Cl}(V)))$. In consequence, there exists $U \in \beta\mathcal{I}O(X, x)$ such that ${}_{\beta\mathcal{I}}\,\mathrm{Cl}(U) \cap f^{-1}(Y - \mathrm{Cl}(V)) = \varnothing$. Therefore, we obtain $f({}_{\beta\mathcal{I}}\mathrm{Cl}(U)) \subset \mathrm{Cl}(V)$. This shows that f is strongly β-$\mathcal{I}$-continuous. □

Definition 3. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be strongly θ-β-$\mathcal{I}$-continuous (see [9]) if for each point $x \in X$ and any open set V of Y containing $f(x)$, there exists $U \in \beta\mathcal{I}O(X, x)$ such that $f({}_{\beta\mathcal{I}}\mathrm{Cl}(U)) \subset V$.*

Theorem 5. *Let Y be a regular space. Then for a function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ the following properties are equivalent:*

(i) f is strongly θ-β-$\mathcal{I}$-continuous;

(ii) f is β-$\mathcal{I}$-continuous;

(iii) f is strongly β-$\mathcal{I}$-continuous.

Proof. (i)⇒(ii): This is obvious.
(ii)⇒(iii): Follows from Theorem 2.
(iii)⇒(i): Suppose that $x \in X$ and V is any open set of Y containing $f(x)$. Since Y is regular, there exists an open set W of Y such that $f(x) \in W \subset \mathrm{Cl}(W) \subset V$. Since f is strongly β-$\mathcal{I}$-continuous, there exists $U \in \beta\mathcal{I}O(X, x)$ such that $f({}_{\beta\mathcal{I}}\mathrm{Cl}(U)) \subset \mathrm{Cl}(W) \subset V$. This shows that f is strongly θ-β-$\mathcal{I}$-continuous. □

Definition 4 (see [3])**.** *Let A and X_0 be subsets of an ideal topological space $(X, \tau, \mathcal{I})$ such that $A \subset X_0 \subset X$. Then $(X_0, \tau_{|X_0}, \mathcal{I}_{|X_0})$ is an ideal topological space with an ideal $\mathcal{I}_{|X_0} = \{\mathcal{I} \in \mathcal{I} | \mathcal{I} \subset X_0\} = \{\mathcal{I} \bigcap X_0 | \mathcal{I} \in \mathcal{I}\}$.*

Lemma 1 (see [9])**.** *Let A and X_0 be subsets of an ideal topological space $(X, \tau, \mathcal{I})$. Then the following properties hold:*

(i) If $A \in \beta\mathcal{I}O(X)$ and $X_0 \in \alpha\mathcal{I}O(X)$, then $A \cap X_0 \in \beta\mathcal{I}O(X_0)$;

(ii) If $A \in \beta\mathcal{I}O(X_0)$ and $X_0 \in \alpha\mathcal{I}O(X)$, then $A \in \beta\mathcal{I}O(X)$.

Lemma 2 (see [9])**.** *Let A and X_0 be subsets of an ideal topological space $(X, \tau, \mathcal{I})$ such that $A \subset X_0 \subset X$. Let ${}_{\beta\mathcal{I}}\mathrm{Cl}_{X_0}(A)$ denote the β-$\mathcal{I}$-closure of A with respect to the subspace X_0. Then*

(i) If X_0 is α-$\mathcal{I}$-open in X, then ${}_{\beta\mathcal{I}}\mathrm{Cl}_{X_0}(A) \subset {}_{\beta\mathcal{I}}\mathrm{Cl}(A)$;

(ii) If $A \in \beta\mathcal{I}O(X_0)$ and $X_0 \in \alpha\mathcal{I}O(X)$, then ${}_{\beta\mathcal{I}}\mathrm{Cl}(A) \subset {}_{\beta\mathcal{I}}\mathrm{Cl}_{X_0}(A)$.

Theorem 6. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is strongly β-$\mathcal{I}$-continuous and X_0 is an α-$\mathcal{I}$-open subset of X, then the restriction $f_{|X_0}$: $(X_0, \tau_{|X_0}, \mathcal{I}_{|X_0}) \to (Y, \sigma)$ is strongly β-$\mathcal{I}_{|X_0}$-continuous.*

Proof. For any $x \in X_0$ and any open set V of Y containing $f(x)$, there exists $U \in \beta\mathcal{I}O(X, x)$ such that $f({}_{\beta\mathcal{I}}\mathrm{Cl}(U)) \subset \mathrm{Cl}(V)$ since f is strongly β-$\mathcal{I}$-continuous. Let $U_0 = U \cap X_0$, then by Lemmas 1 and 2, $U_0 \in \beta\mathcal{I}O(X_0, x)$ and ${}_{\beta\mathcal{I}}\mathrm{Cl}_{X_0}(U_0) \subset {}_{\beta\mathcal{I}}\mathrm{Cl}(U_0)$. Therefore, we obtain

$$(f_{|X_0})({}_{\beta\mathcal{I}}\mathrm{Cl}_{X_0}(U_0)) = f({}_{\beta\mathcal{I}}\mathrm{Cl}_{X_0}(U_0)) \subset f({}_{\beta\mathcal{I}}\mathrm{Cl}(U_0)) \subset f({}_{\beta\mathcal{I}}\mathrm{Cl}(U)) \subset \mathrm{Cl}(V).$$

This shows that $f_{|X_0}$ is strongly β-$\mathcal{I}_{|X_0}$-continuous. □

Theorem 7. *A function $f:(X,\tau,\mathcal{I})\to(Y,\sigma)$ is strongly β-$\mathcal{I}$-continuous if for each $x\in X$ there exists $X_0\in\alpha\mathcal{I}O(X,x)$ such that the restriction $f_{|X_0}\colon (X_0,\tau_{|X_0},\mathcal{I}_{|X_0})\to(Y,\sigma)$ is strongly β-$\mathcal{I}_{|X_0}$-continuous.*

Proof. Let $x\in X$ and V be an open set of Y containing $f(x)$. There exists $X_0\in\alpha\mathcal{I}O(X,x)$ such that $f_{|X_0}\colon (X_0,\tau_{|X_0},\mathcal{I}_{|X_0})\to(Y,\sigma)$ is strongly β-$\mathcal{I}$-continuous. Thus, there exists $U\in\beta\mathcal{I}O(X_0,x)$ such that $(f_{|X_0})\ ({}_{\beta\mathcal{I}}\mathrm{Cl}_{X_0}(U))\subset\mathrm{Cl}(V)$. By Lemmas 1 and 2, $U\in\beta\mathcal{I}O(X,x)$ and ${}_{\beta\mathcal{I}}\,\mathrm{Cl}(U)\subset{}_{\beta\mathcal{I}}\,\mathrm{Cl}_{X_0}(U)$. Hence, we have $f({}_{\beta\mathcal{I}}\mathrm{Cl}(U))=(f_{|X_0})\ ({}_{\beta\mathcal{I}}\mathrm{Cl}(U))\subset(f_{|X_0})\ ({}_{\beta\mathcal{I}}\mathrm{Cl}_{X_0}(U))\subset\mathrm{Cl}(V)$. This shows that f is strongly β-$\mathcal{I}_{|X_0}$-continuous. □

Corollary 1. *Let $\{U_\lambda:\lambda\in\Omega\}$ be an α-$\mathcal{I}$-open cover of an ideal topological space $(X,\tau,\mathcal{I})$. A function $f:(X,\tau,\mathcal{I})\to(Y,\sigma)$ is strongly β-$\mathcal{I}$-continuous if and only if the restriction $f_{|U_\lambda}\colon (U_\lambda,\tau_{|U_\lambda},\mathcal{I}_{|U_\lambda})\to(Y,\sigma)$ is strongly β-$\mathcal{I}_{|X_0}$-continuous for each $\lambda\in\Omega$.*

Proof. The proof follows from Theorems 6 and 7. □

Definition 5. *An ideal topological space $(X,\tau,\mathcal{I})$ is said to be:*

(i) β-$\mathcal{I}$-closed (resp. β-$\mathcal{I}$-Lindelof) if every cover of X by β-$\mathcal{I}$-open sets has a finite (resp. countable) subcover whose β-$\mathcal{I}$-closures cover X;

(ii) countably β-$\mathcal{I}$-closed if every countable cover of X by β-$\mathcal{I}$-open sets has a finite subcover whose β-$\mathcal{I}$-closures cover X.

Definition 6. *A topological space (X,τ) is said to be:*

(i) quasi H-closed (see [7]) (resp. almost Lindelof [2]) if every cover of X by open sets has a finite (resp. countable) subfamily whose closures cover X,

(ii) lightly compact (see [1]) if every countable cover of X by open sets has a finite subfamily whose closures cover X.

Definition 7. *A subset K of an ideal topological space $(X,\tau,\mathcal{I})$ is said to be β-$\mathcal{I}$-closed relative to X if for every cover $\{V_\lambda\colon \lambda\in\Omega\}$ of K by β-$\mathcal{I}$-open subsets of X, there exists a finite subset Ω_0 of Ω such that $K\subset\bigcup\ \{{}_{\beta\mathcal{I}}\mathrm{Cl}(V_\lambda)\colon \lambda\in\Omega_0\}$ (resp. $K\subset U\ \{\mathrm{Cl}(V_\lambda):\alpha\in\Omega_0\}$).*

Definition 8. *A subset K of a topological space (X,τ) is said to be quasi H-closed relative to X (see [7]) if every cover $\{V_\lambda:\lambda\in\Omega\}$ of K by open subsets of X, there exists a finite subset Ω_0 of Ω such that $K\subset\bigcup\{\mathrm{Cl}(U_\lambda):\lambda\in\Omega_0\}$.*

Theorem 8. *If $f:(X,\tau,\mathcal{I})\to(Y,\sigma)$ is a strongly β-$\mathcal{I}$-continuous function and K is β-$\mathcal{I}$-closed relative to X, then $f(K)$ is quasi-H-closed relative to Y.*

Proof. Suppose that $f:(X,\tau,\mathcal{I})\to(Y,\sigma)$ is strongly β-$\mathcal{I}$-continuous and K is β-$\mathcal{I}$-closed relative to X. Let $\{V_\lambda\colon \lambda\in\Omega\}$ be a cover of $f(K)$ by open sets of Y. For each point $x\in K$, there exists $\lambda(x)\in\Omega$ such that $f(x)\in V_{\lambda(x)}$. Since f is strongly β-$\mathcal{I}$-continuous, there exists $U_x\in\beta\mathcal{I}O(X,x)$ such that $f({}_{\beta\mathcal{I}}\mathrm{Cl}(U_x))\subset\mathrm{Cl}(V_{\lambda(x)})$. The family $\{U_x\colon x\in K\}$ is a cover of K by β-$\mathcal{I}$-open sets of X and hence there exists a finite subset K_1 of K such that $K \subset \bigcup_{x \in K_1} {}_{\beta\mathcal{I}}\operatorname{Cl}(U_x)$. Therefore, we obtain $f(K) \subset \bigcup_{x \in K_1} \operatorname{Cl}(V_{\lambda(x)})$. This shows that $f(K)$ is quasi-H-closed relative to Y. □

Remark 1. *If we change in the above Theorem the condition of a strongly β-$\mathcal{I}$-continuous function by a strongly strongly θ-β-$\mathcal{I}$-continuous function, we obtain that $f(K)$ is a compact subset of Y.*

Corollary 2. *If $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is a strongly β-$\mathcal{I}$-continuous surjection, then the following properties hold:*

(i) If X is β-$\mathcal{I}$-closed, then Y is quasi-H-closed;

(ii) If X is countably β-$\mathcal{I}$-closed, then Y is lightly compact;

(iii) If X is β-$\mathcal{I}$-Lindelof, then Y is almost Lindelof.

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