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On weakly BR-closed functions between topological spaces

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Abstract. In this paper, we offer a new class of functions called weakly BR-closed functions. Moreover, we investigate not only some of their basic properties but also their relationships with other types of already well-known functions.

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1. Introduction and preliminaries

Recently, Park [9] has introduced a new class of sets called b-θ-open sets. He showed that b-θ-cluster points can be characterized by b-regular sets and that the class of b-θ-open sets includes the class of b-regular sets. He also introduced the notion of strongly θ-b-continuous functions. In 2008, Ekici [4] continued the work of Park and also introduced a new class of functions called weakly BR-continuity. In this paper we define the notion of weakly BR-closedness as a natural dual to the weakly BR-continuity by using the notion of b-θ-open and b-θ-closed sets. We obtain some characterizations and properties of these functions. Moreover, we also study these functions comparing with other types of already known functions. It turns out that b-θ-closedness implies weak BR-closedness but not conversely. We show that under a certain condition the converse is also true.

Throughout the present paper, (X, τ) and (Y, σ) (or X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of a space X, the closure of A and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. A subset A is said to be regular open (resp. regular closed) if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). A subset A is said to be preopen [7] (resp. b-open [1], α-open [8]) if $A \subset Int(Cl(A))$ (resp. $A \subset Int(Cl(A)) \cup Cl(Int(A))$,

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$A \subset Int(Cl(Int(A))))$. A point $x \in X$ is called a θ-cluster point of A [13] if $A \cap Cl(U) \neq \emptyset$ for each open set U containing x. The set of all θ-cluster points of A is called the θ-closure of A and is denoted by $Cl_\theta(A)$. A subset A is called θ-closed [13] if $Cl_\theta(A) = A$. The complement of a θ-closed set is called a θ-open set. The complement of a b-open set is said to be b-closed. The intersection of all b-closed sets of X containing A is called the b-closure of A and is denoted by $Cl_b(A)$. The union of all b-open sets of X contained in a subset A is called b-interior [1] of A and is denoted by $Int_b(A)$. The family of all b-open (resp. b-regular i.e b-open and b-closed) sets is denoted by $BO(X)$ (resp. $BR(X)$). A point $x \in X$ is called a b-θ-cluster point of A [9] if $A \cap Cl_b(U) \neq \emptyset$ for each b-open set U containing x. The set of all b-θ-cluster points of A is called the b-θ-closure of A and is denoted by b-θ-$Cl(A)$. A subset A is said to be b-θ-closed if b-θ-$Cl(A) = A$. The complement of a b-θ-closed set is called a b-θ-open set. The family of all b-θ-open (resp. b-θ-closed) sets is denoted by $B\theta O(X)$ (resp. $B\theta C(X)$). The set $\{x \in X : x \in U \subset A$ for some b-regular set U of $X\}$ is called the b-θ-Interior of A and is denoted by b-θ-$Int(A)$.

Recall that for a subset U of a space X the following implications hold:

b-regular $\Rightarrow$ b-θ-open $\Rightarrow$ b-open, b-regular $\Rightarrow$ b-θ-closed $\Rightarrow$ b-closed and $B\theta O(X) \cap B\theta C(X) = BR(X)$ and the converses are not true in general (see [9]). Observe that $B\theta O(X) \cap B\theta C(X) = BR(X)$ is in fact Theorem 3.8 (b) of [9].

A function $f : (X, \tau) \to (Y, \sigma)$ is called:

(i) weakly closed [11, 12] if $Cl(f(Int(F)))) \subset f(F)$ for each closed $F \subset X$.

(ii) contra b-θ-open (resp. contra b-θ-closed) if $f(U)$ is b-θ-closed (resp. b-θ-open) in Y for each open (resp. closed) set U of X.

(iii) BR-open (resp. BR-closed) if $f(U)$ is b-regular in Y for each open (resp. closed) set U of X.

(iv) strongly continuous [5, 2] if for every subset A of X, $f(Cl(A)) \subset f(A)$.

2. Weakly BR-closed functions

Definition 1. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be weakly BR-closed if b-θ-$Cl(f(Int(F))) \subset f(F)$ for each closed set F of X.*

Definition 2. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be b-θ-closed if $f(F)$ is b-θ-closed in Y for each closed set F of X.*

Clearly, every b-θ-closed function is weakly BR-closed, but the converse is not true in general.

Example 1.

(i) A weakly BR-closed function need not be b-θ-closed.
Let $X = \{a, b\}$, $\tau = \{\emptyset, \{b\}, X\}$, $Y = \{x, y\}$ and $\sigma = \{\emptyset, \{x\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be given by $f(a) = x$ and $f(b) = y$. Then f is clearly weakly BR-closed, but it is not b-θ-closed since $f(a)$ is not a b-θ-closed set in Y.

(ii) A weakly-closed function need not be weakly BR-closed (and also not b-θ-closed).
Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}, \sigma = \{\emptyset, X, \{b\}, \{a, b\}, \{b, c\}\}$ and

$f : (X, \tau) \to (X, \sigma)$ be the identity function. Then f is weakly-closed but it is not weakly BR-closed since b-θ-$Cl(f(Int(\{b, c\}))) \not\subset f(\{b, c\})$.

Lemma 1 (see [9])**.** *Let A be a subset of a space X. Then:*

(1) b-θ-$Cl(A) = \cap\{V : A \subset V$ and $V \in BR(X)\}$.

(2) $x \in b$-θ-$Cl(A)$ if and only if $A \cap U \neq \emptyset$ for each b-regular set U of X containing x.

(3) b-θ-$Cl(A)$ is b-θ-closed.

(4) Any intersection of b-θ-closed sets is b-θ-closed and any union of b-θ-open sets is b-θ-open.

(5) A is b-θ-open in X if and only if for each $x \in A$, there exists a b-regular set U of X containing x such that $x \in U \subset A$.

Theorem 1. *For a function $f : (X, \tau) \to (Y, \sigma)$, the following conditions are equivalent:*

(1) f is weakly BR-closed.

(2) b-θ-$Cl(f(U)) \subset f(Cl(U))$ for every open set U of X.

(3) b-θ-$Cl(f(U)) \subset f(Cl(U))$ for each regular open subset U of X,

(4) For each subset F in Y and each open set U of X with $f^{-1}(F) \subset U$, there exists a b-θ-open set A in Y with $F \subset A$ and $f^{-1}(A) \subset Cl(U)$,

(5) For each point y in Y and each open set U in X with $f^{-1}(y) \subset U$, there exists a b-regular set A in Y containing y and $f^{-1}(A) \subset Cl(U)$,

(6) b-θ-$Cl(f(Int(Cl(U)))) \subset f(Cl(U))$ for each set U of X,

(7) b-θ-$Cl(f(U)) \subset f(Cl(U))$ for each preopen set U of X.

(8) b-θ-$Cl(f(U)) \subset f(Cl(U))$ for each α-open set U of X.

(9) b-θ-$Cl(f(Int(F))) \subset f(F)$ for each preclosed set F of X,

(10) b-θ-$Cl(f(Int(F))) \subset f(F)$ for each α-closed set F of X.

Proof. (1) $\Rightarrow$ (2): Let U be any open subset of X. Then b-θ-$Cl(f(U)) = b$-θ-$Cl(f(Int(U))) \subset b$-θ-$Cl(f(Int(Cl(U))) \subset f(Cl(U))$.
(2) $\Rightarrow$ (1): Let F be any closed subset of X. Then b-θ-$Cl(f(Int(F))) \subset f(Cl(Int(F)))$ $\subset f(Cl(F)) = f(F)$.
It is clear that: (1) $\Rightarrow$ (3), (4) $\Rightarrow$ (5), (1) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (8) $\Rightarrow$ (2) $\Rightarrow$ (9) $\Rightarrow$ (10) $\Rightarrow$ (1).
(3) $\Rightarrow$ (4): Let F be a subset of Y and U open in X with $f^{-1}(F) \subset U$. Then $f^{-1}(F) \cap Cl(X - Cl(U)) = \phi$ and consequently, $F \cap f(Cl(X - Cl(U))) = \phi$. Since

$X - Cl(U)$ is regular open, $F \cap b\text{-}\theta\text{-}Cl(f(X - Cl(U))) = \phi$ by (3). Let $A = Y - b\text{-}\theta\text{-}Cl(f(X - Cl(U)))$. Then A is b-θ-open with $F \subset A$ and $f^{-1}(A) \subset X - f^{-1}(b\text{-}\theta\text{-}Cl(f(X - Cl(U)))) \subset X - f^{-1}(f(X - Cl(U))) \subset Cl(U)$.
(5) $\Rightarrow$ (1): Let F be closed of X and $y \in Y - f(F)$. Since $f^{-1}(y) \subset X - F$, there exists a b-regular A in Y with $y \in A$ and $f^{-1}(A) \subset Cl(X - F) = X - Int(F)$ by (5). Therefore $A \cap f(Int(F)) = \phi$, such that $y \in Y - b\text{-}\theta\text{-}Cl(f(Int(F)))$. Thus (5) $\Rightarrow$ (1). □

Next we investigate conditions under which weakly BR-closed functions are b-θ-closed.

Theorem 2. *Let $f : (X, \tau) \to (Y, \sigma)$ be weakly BR-closed. If for each closed subset F of X and each fiber $f^{-1}(y) \subset X - F$ there exists an open set U of X such that $f^{-1}(y) \subset U \subset Cl(U) \subset X - F$, then f is b-θ-closed.*

Proof. Let F be any closed subset of X and $y \in Y - f(F)$. Then $f^{-1}(y) \cap F = \phi$ and hence $f^{-1}(y) \subset X - F$. By hypothesis, there exists an open set U of X such that $f^{-1}(y) \subset U \subset Cl(U) \subset X - F$. Since f is weakly BR-closed by Theorem 1, there exists a b-regular V in Y with $y \in V$ and $f^{-1}(V) \subset Cl(U)$. Therefore , we obtain $f^{-1}(V) \cap F = \phi$ and hence $V \cap f(F) = \phi$. This shows that $y \notin b\text{-}\theta\text{-}Cl(f(F))$. Therefore, $f(F)$ is b-θ-closed in Y and f is a b-θ-closed function. □

Theorem 3. *If $f : (X, \tau) \to (Y, \sigma)$ is contra b-θ-open , then f is weakly BR-closed.*

Proof. Let F be a closed subset of X. Then, $b\text{-}\theta\text{-}Cl(f(Int(F))) = f(Int(F)) \subset f(F)$. □

Theorem 4. *If $f : (X, \tau) \to (Y, \sigma)$ is strongly continuous, then the following are equivalent:*
(1) f is weakly BR-closed.
(2) f is contra b-θ-open.

Proof. (1) $\Rightarrow$ (2): Let U be an open subset of X. By hypothesis and Theorem 1(2), we have $b\text{-}\theta\text{-}Cl(f(U)) \subset f(Cl(U)) \subset f(U)$. Hence $f(U)$ is b-θ-closed.
(2) $\Rightarrow$ (1): It follows from Theorem 3. □

Definition 3. *A function $f : (X, \tau) \to (Y, \sigma)$ is said to be b-θ-open if $f(U)$ is b-θ-open in Y for each open set U of X.*

Theorem 5. *Every weakly BR-closed strongly continuous bijection $f : (X, \tau) \to (Y, \sigma)$ is BR-open (and BR-closed).*

Proof. Let U be an open subset of X. Since f is weakly BR-closed b-θ-$Cl(f(Int(X - U))) \subset f(X - U)$. Hence, by hypothesis we obtain $f(U) \subset b\text{-}\theta\text{-}Int(f(Cl(U))) \subset b\text{-}\theta\text{-}In(f((U))) \subset f(U)$. Therefore $f(U)$ is b-θ-open. But, f is contra b-θ-open by Theorem 4, so $f(U)$ is also b-θ-closed. In particular, $f(U)$ is b-regular for each open $U \subset X$. □

Theorem 6. *If $f:(X,\tau)\to(Y,\sigma)$ is a weakly BR-closed bijection, then for every subset F in Y and every open set U in X with $f^{-1}(F)\subset U$, there exists a b-θ-closed set B in Y such that $F\subset B$ and $f^{-1}(B)\subset Cl(U)$.*

Proof. Let F be a subset of Y and U an open subset of X with $f^{-1}(F)\subset U$. Put $B=b\text{-}\theta\text{-}Cl(f(Int(Cl(U))))$, then B is a b-θ-closed set of Y such that $F\subset B$ since $F\subset f(U)\subset f(Int(Cl(U)))\subset b\text{-}\theta\text{-}Cl(f(Int(Cl(U))))=B$. And since f is weakly BR-closed, we have $f^{-1}(B)\subset Cl(U)$. □

Recall that a set F in a topological space X is θ-compact [12] if for each cover Ω of F by open sets U in X, there is a finite family $U_1,...,U_n$ in Ω such that $F\subset Int(\cup\{Cl(U_i):i=1,2,...,n\})$.

Theorem 7. *If $f:(X,\tau)\to(Y,\sigma)$ is weakly BR-closed with all fibers θ-closed in X, then $f(F)$ is b-θ-closed for each θ-compact set F in X.*

Proof. Let F be θ-compact and $y\in Y-f(F)$. Then $f^{-1}(y)\cap F=\phi$ and for each $x\in F$ there is an open U_x in X containing x such that $Cl(U_x)\cap f^{-1}(y)=\phi$. Clearly $\Omega=\{U_x:x\in F\}$ is an open cover of F and since F is θ-compact, there is a finite family $\{U_{x_1},...,U_{x_n}\}$ in Ω such that $F\subset Int(A)$, where $A=\cup\{Cl(U_{x_i}):i=1,...,n\}$. Since f is weakly BR-closed by Theorem 1, there exists a b-regular B in Y with $f^{-1}(y)\subset f^{-1}(B)\subset Cl(X-A)=X-Int(A)\subset X-F$. Therefore $y\in B$ and $B\cap f(F)=\phi$. Thus $y\in Y-b\text{-}\theta\text{-}Cl(f(F))$. This shows that $f(F)$ is b-θ-closed. □

Two non-empty subsets A and B in X are strongly separated [12], if there exist open sets U and V in X with $A\subset U$ and $B\subset V$ such that $Cl(U)\cap Cl(V)=\phi$. If A and B are singleton sets we may speak of points being strongly separated. We will use the fact (see [3]) that in a normal space, disjoint closed sets are strongly separated.

Recall that a space X is said to be br-Hausdorff (briefly br-T_2) [4] if for every pair of distinct points x and y, there exist two b-regular sets U and V such that $x\in U$ and $y\in V$ and $U\cap V=\phi$.

Theorem 8. *If $f:(X,\tau)\to(Y,\sigma)$ is a weakly BR-closed surjection and all pairs of disjoint fibers are strongly separated, then Y is br-T_2.*

Proof. Let y and z be two distinct points in Y. Let U and V be open sets in X such that $f^{-1}(y)\in U$ and $f^{-1}(z)\in V$, respectively, with $Cl(U)\cap Cl(V)=\phi$. By weak BR-closedness (Theorem 1 (5)), there are b-regular sets F and B in Y such that $y\in F$ and $z\in B$, $f^{-1}(F)\subset Cl(U)$ and $f^{-1}(B)\subset Cl(V)$. Therefore $F\cap B=\phi$, since $Cl(U)\cap Cl(V)=\phi$ and f is surjective. Then Y is br-T_2. □

Corollary 1. *If $f:(X,\tau)\to(Y,\sigma)$ is a weakly BR-closed surjection with all fibers closed and X is normal, then Y is br-T_2.*

Corollary 2. *If $f:(X,\tau)\to(Y,\sigma)$ is a continuous weakly BR-closed surjection with X a compact T_2 space and Y a T_1 space, then Y is a compact br-T_2 space.*

Proof. Since f is a continuous surjection and Y is a T_1 space, Y is compact and all fibers are closed. Since X is normal, Y is also br-T_2. □

Definition 4. *A topological space X is said to be:*

(i) quasi H-closed [10] if every open cover of X has a finite subfamily whose closures cover X. A subset A of a space X is quasi H-closed relative to X if every cover of A by open sets of X has a finite subfamily whose closures cover A.

(ii) almost BR-compact space if every cover of X with b-regular sets has a finite subfamily of members whose closures cover X. And a subset A of a space X is almost BR-compact relative to X if every cover of A with b-regular subsets has a finite subfamily of members whose closures cover A.

Lemma 2 (see [6])**.** *A function $f : (X, \tau) \to (Y, \sigma)$ is open if and only if for each $B \subset Y$, $f^{-1}(Cl(B)) \subset Cl(f^{-1}(B))$.*

Recall that a topological space is extremelly disconnected if the closure of every open set is open.

Theorem 9. *Let (X, τ) be an extremelly disconnected space and let $f : (X, \tau) \to (Y, \sigma)$ be an open and weakly BR-closed function with quasi H-closed fibers. Then $f^{-1}(G)$ is quasi H-closed for each almost BR-compact set $G \subset Y$.*

Proof. Let $\{V_\beta : \beta \in I\}$ be an open cover of $f^{-1}(G)$. Then for each $y \in G$, $f^{-1}(y) \subset \cup\{Cl(V_\beta) : \beta \in I(y)\} = H_y$ for some finite $I(y) \subset I$. Then H_y is closed and open since X is extremelly disconnected. So, by Theorem 1(5), there exists a b-regular set U_y containing y such that $f^{-1}(U_y) \subset Cl(H_y) = H_y$. Then, $\{U_y : y \in G\}$ is a cover of G by b-regular sets and $G \subset \cup\{Cl(U_y) : y \in K\}$ for some finite subset K of G. Hence, by Lemma 2, $f^{-1}(G) \subset \cup\{Cl(f^{-1}(U_y)) : y \in K\} \subset \cup\{H_y : y \in K\}$. Thus $f^{-1}(G) \subset \cup\{Cl(V_\beta) : \beta \in I(y)$ and $y \in K\}$. Therefore $f^{-1}(G)$ is quasi H-closed. □

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