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Special sextics with a quadruple line

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Abstract. This paper deals with a special class of 6th order surfaces with a quadruple straight line in a three-dimensional Euclidean space. These surfaces, denoted by $\mathcal{P}_4^6$, are the pedal surfaces of one special 1st order 4th class congruence $\mathcal{C}_4^1$. The parametric and implicit equations of $\mathcal{P}_4^6$ are derived, some of their properties are proved and their visualizations are given. The singularities of $\mathcal{P}_4^6$ are classified according to the shapes of their tangent cones. The methods applied are analytic, synthetic and algebraic, supported by the program *Mathematica 6*.

AMS subject classifications: 51N20, 14J17

Key words: congruence of lines, pedal surface, multiple point, pinch-point, tangent cone

1. Introduction

A congruence $\mathcal{C}$ is the set of lines in a three-dimensional space (projective, affine or Euclidean) depending on two parameters. The line $l \in \mathcal{C}$ is said to be a *ray* of the congruence. The *order* of a congruence is the number of its rays which pass through an arbitrary point; the *class* of a congruence is the number of its rays which lie in an arbitrary plane. $\mathcal{C}_n^m$ denotes an mth *order* nth *class* congruence. A point is called a *singular point* of a congruence if ∞^1 rays pass through it. A plane is called a *singular plane* of a congruence if it contains ∞^1 rays (1-parametrically infinite lines).

According to [7, p. 64], [11, pp. 1184-1185], there are only two types of the first order congruences: the first one are nth class congruences and their rays are transversals of one straight line and one nth order space curve which cuts this straight line at $n-1$ points; the second type are only 3rd class congruences and their rays cut a twisted cubic twice. The properties of the first order congruences (the construction of their rays, singular points and planes, focal properties, etc.) can be found in [1].

In Euclidean space $\mathbb{E}^3$, the *pedal surface* of a congruence $\mathcal{C}$ with respect to a pole P is the locus of the feet of perpendiculars from a point P to the rays of a congruence $\mathcal{C}$. If $\mathcal{C}$ is an mth order nth class congruence, the order of its pedal surface is $2m+n$, [5].

In [2] the authors defined one transformation of a three-dimensional projective space, called the $(n+2)$-degree inversion, where corresponding points lie on the rays of the 1st order, nth class congruence $\mathcal{C}_n^1$ and are conjugate with respect to some proper quadric Ψ. This transformation maps a straight line onto an $(n+2)$-order space

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curve and a plane onto an $(n+2)$-order surface which contains an n-ple straight line. According to [2], the pedal surfaces of the first type congruence $\mathcal{C}_n^1$ with respect to the pole P is the image of the plane at infinity given by the $(n+2)$-degree inversion with respect to $\mathcal{C}_n^1$ and any sphere with the center P. Thus, it is an $(n+2)$-order surface with an n-ple straight line which contains the absolute conic.
In this paper, we investigate the pedal surfaces of special 1st order 4th class congruence.

2. Congruence $\mathcal{C}_4^1$

In Euclidean space $\mathbb{E}^3$, let the directing lines of a congruence $\mathcal{C}$ be the axis z and Viviani's curve c^4 (see Figure 1a) which is the intersection of the sphere

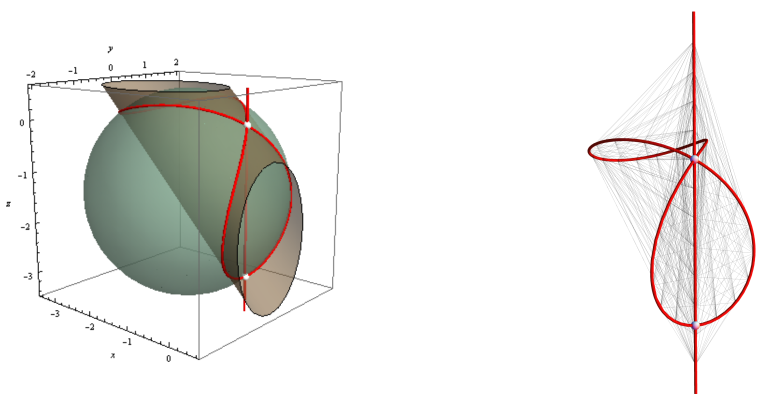
$$(x+\sqrt{2})^2 + y^2 + (z+\sqrt{2})^2 = 4 \tag{1}$$

and the cylinder

$$(x+z+\sqrt{2})^2 + 2y^2 = 2. \tag{2}$$

From equations (1) and (2), by using the substitution $y \to x\tan u$, we obtain the following parametrization of Viviani's curve:

$$\mathbf{r}(u) = 4\sqrt{2}\,\frac{1+3\cos 2u}{(3+\cos 2u)^2}\Big(-2(\cos u)^2, -\sin 2u, (\sin u)^2\Big), \qquad u \in [0,\pi). \tag{3}$$



a b

Figure 1.

The axis z cuts Viviani's curve at the points $S_1 = (0,0,0)$ and $S_2 = (0,0,-2\sqrt{2})$, where S_1 is the double point of Viviani's curve. Since Viviani's curve c^4 is the 4th order space curve, and the axis z cuts it in 3 points, then the transversals of z and c^4 form the 1st order and 4th class congruence $\mathcal{C}_4^1$. The directing lines and some rays of $\mathcal{C}_4^1$ are shown in Figure 1b. From eq. (1) and (2), for $z \to r$, (x,y) coordinates of the intersection points of c^4 and the plane $z = r$ are given by the following formulas:

$$\begin{aligned} x_{1,2} &= r - \sqrt{2} - \sqrt{2-4\sqrt{2}r}, \quad y_{1,2} = \pm\sqrt{2}\sqrt{-rx_{1,2}} \\ x_{3,4} &= r - \sqrt{2} + \sqrt{2-4\sqrt{2}r}, \quad y_{3,4} = \pm\sqrt{2}\sqrt{-rx_{3,4}}. \end{aligned} \tag{4}$$

• If $r \in (-\infty, -2\sqrt{2}) \cup (\sqrt{2}/4, +\infty)$, there are no real points of c^4 in the plane $z = r$. It follows from the inequalities $2 - 4\sqrt{2}r > 0$, $-rx_{1,2} < 0$, $-rx_{3,4} < 0$ or $2 - 4\sqrt{2}r < 0$.
• If $r \in [-2\sqrt{2}, 0)$, there are two real points of c^4 in the plane $z = r$. It follows from the inequalities $2 - 4\sqrt{2}r \geq 0$, $-rx_{1,2} \geq 0$, $-rx_{3,4} < 0$.
For $r = -2\sqrt{2}$, these points coincide.
• If $r \in [0, \sqrt{2}/4]$, there are four real points of c^4 in the plane $z = r$. It follows from the inequalities $2 - 4\sqrt{2}r \geq 0$, $-rx_{1,2} \geq 0$, $-rx_{3,4} \geq 0$.
For $r = 0, \sqrt{2}/4$, these points coincide in pairs.

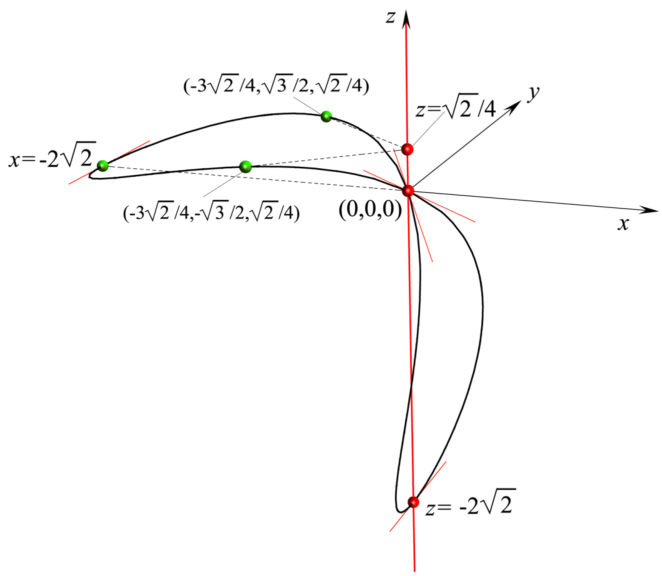


Figure 2.

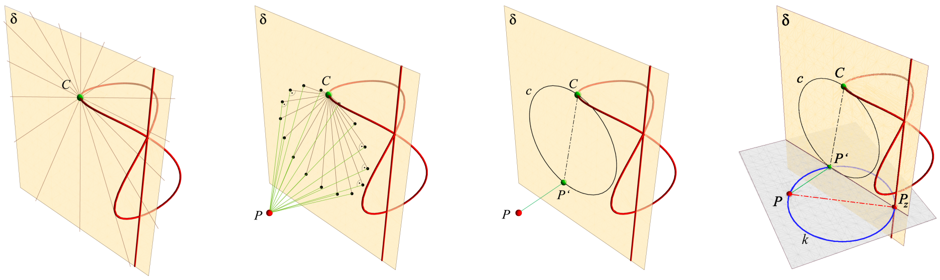
The tangent lines of c^4 at its node $(0, 0, 0)$ are given by the following equations:

$$y = \pm\sqrt{2}x, \quad z = -x. \tag{5}$$

3. Pedal surfaces of $\mathcal{C}_4^1$

The pedal surface of $\mathcal{C}_4^1$ is a sextic with the quadruple line z and it contains the absolute conic [2]. It is denoted by $\mathcal{P}_4^6$. It is clear that any plane through an n-ple line of an $n+2$-order surface cuts this surface in its n-ple line and one conic. If the surface contains the absolute conic, this conic is a circle.
In the plane δ through the axis z, the rays of $\mathcal{C}_4^1$ form a pencil of lines (C), where $C \notin z$ is the intersection point of δ and Viviani's curve c^4 [1], see Figure 3a. In three planes, determined by the tangent lines of c^4 at S_1 and S_2, the point C lies on the axis z and coincides with S_1 and S_2, respectively. If the pole P is in the general position to the directing lines of $\mathcal{C}_4^1$, the feet of perpendiculars from P to the rays of the pencil (C) (see Figure 3b) form the circle c with the diameter $\overline{CP'}$, where P' is the orthogonal projection of P to δ (see Figure 3c). For given pole P, the path

of the point P' is the circle k which lies in the plane through P perpendicular to the axis z. The diameter of k is $\overline{PP_z}$, where P_z is the normal projection of P to z (see Figure 3d). Thus, we can regard the surface $\mathcal{P}_4^6$ as the system of circles in the planes through quadruple line z with the end points of diameters on Viviani's curve c^4 and circle k.



a b c d

Figure 3.

3.1. Parametric equations of $\mathcal{P}_4^6$ and *Mathematica* visualizations

Let $(p, q, r) \in \mathbb{R}^3$ be the coordinates of a pole P and let $\mathbf{r}(u) = (x_{c^4}(u), y_{c^4}(u), z_{c^4}(u))$, where functions $x_{c^4}, y_{c^4}, z_{c^4} : [0, \pi) \to \mathbb{R}$ are given by (3), be the radi-vector of the point on Viviani's curve c^4. Let (t, z), where $|t| = \sqrt{x^2 + y^2}$, be the coordinates of the points in the plane $\delta(u)$ which is given by equation $y = x \tan u$ if $u \in [0, \pi)$, $u \neq \pi/2$, and $x = 0$ if $u = \pi/2$, see Figure 4.

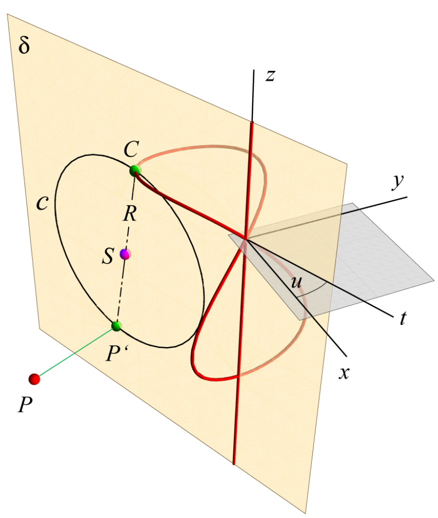


Figure 4.

The coordinates of the points $C, P' \in \delta(u)$ are

$$
\begin{aligned}
t_C(u) &= \sqrt{x_{c^4}(u)^2 + y_{c^4}(u)^2} = \frac{x_{c^4}(u)}{\cos u} = -8\sqrt{2}\,\frac{(1+3\cos 2u)\cos u}{(3+\cos 2u)^2} \\
z_C(u) &= z_{c^4}(u) \\
t_{P'}(u) &= p\cos u + q\sin u, \quad z_{P'}(u) = r. \qquad (6)
\end{aligned}
$$

$R(u)$ is the radius and $S(t_S(u), z_S(u))$ is the center of the circle c in the plane $\delta(u)$:

$$
\begin{aligned}
R(u) &= \frac{\sqrt{(t_C(u) - t_{P'}(u))^2 + (z_C(u) - r)^2}}{2} \\
t_S(u) &= \frac{t_C(u) + t_{P'}(u)}{2}, \quad z_S(u) = \frac{z_C(u) + r}{2}. \qquad (7)
\end{aligned}
$$

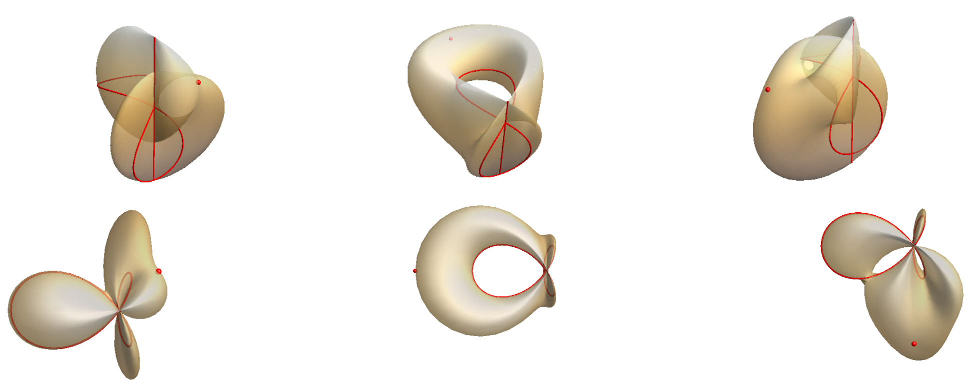
Since the parametric equations of the circle c in the plane $\delta(u)$ are

$$
\begin{aligned}
t(v) &= R(u)\sin v + t_S(u) \\
z(v) &= R(u)\cos v + z_S(u), \quad v \in [0, 2\pi), \qquad (8)
\end{aligned}
$$

the parametric equations of the surface $\mathcal{P}_4^6$ are the following:

$$
\begin{aligned}
x(u,v) &= \cos u\,(R(u)\sin v + t_S(u)) \\
y(u,v) &= \sin u\,(R(u)\sin v + t_S(u)) \\
z(u,v) &= R(u)\cos v + z_S(u), \qquad u \in [0,\pi),\ v \in [0, 2\pi). \qquad (9)
\end{aligned}
$$

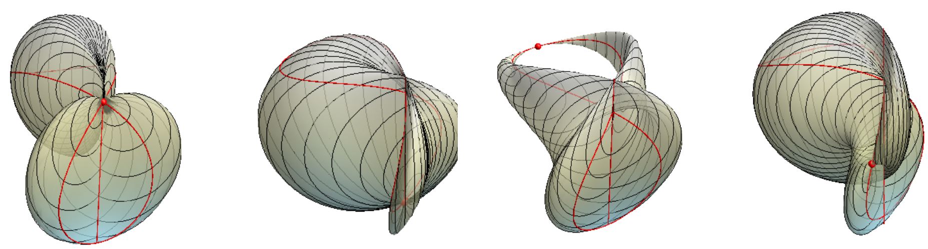
Equations (9) allow for *Mathematica* visualizations of surfaces $\mathcal{P}_4^6$. Three pedal surfaces of $\mathcal{C}_4^1$ with respect to the poles $(1,1,1)$, $(-5,0,0)$ and $(0,-3,0)$ are shown in Figure 5a, 5b and 5c, respectively. The directing lines of $\mathcal{C}_4^1$ and the pole are pointed out. Each surface is viewed from two different viewpoints.



a b c

Figure 5.

Equations (9) are valid for every position of a pole P. Four examples of the pedal surfaces $\mathcal{P}_4^6$ with respect to the poles which lie on the directing lines of $\mathcal{C}_4^1$ are shown in Figure 7. If a pole lies on z, all circles c pass through it and, as we will see in the following, it is the quintuple point of $\mathcal{P}_4^6$. These are the cases in Figure 6a and 6b where $P = (0, 0, 0)$ and $P = (0, 0, -2\sqrt{2})$, respectively. If a pole lies on c^4, the circle c through it splits into isotropic lines in the plane δ through P and P is a double point of $\mathcal{P}_4^6$. These are the cases in Figure 6c and 6d where P is given by vectors $\mathbf{r}(0°)$ and $\mathbf{r}(110°)$, respectively.



a b c d

Figure 6.

3.2. Implicit equation of $\mathcal{P}_4^6$

In the plane $\delta(u)$ through z, in the coordinates (t, z) (see Figure 3), the equation of the circle c is

$$(t - t_S(u))^2 + (z - z_S(u))^2 = R(u)^2, \quad u \in [0, \pi). \tag{10}$$

From eq. (6), by using the substitutions $\cos u = \frac{x}{\sqrt{x^2+y^2}}$, $\sin u = \frac{y}{\sqrt{x^2+y^2}}$, we obtain the following

$$\begin{aligned} t_C(u) &= -\frac{4\sqrt{2}\, x(2x^2 - y^2)\sqrt{x^2+y^2}}{(2x^2+y^2)^2} \\ z_C(u) &= \frac{2\sqrt{2}\, y^2(2x^2 - y^2)}{(2x^2+y^2)^2} \\ t_{P'}(u) &= \frac{p\, x + qy}{\sqrt{x^2+y^2}}\,. \end{aligned} \tag{11}$$

Now, we can express $t_S(u)$, $z_S(u)$, $R(u)$, given by formulas (7), as the functions of x and y. If we put these functions and $t = \sqrt{x^2+y^2}$ into eq. (10) and multiply it by $(2x^2+y^2)^2$, we obtain the implicit equation of $\mathcal{P}_4^6$ which can be written in the following form

$$(2x^2+y^2)^2(x^2+y^2+z^2) + H^5(x, y) + H_1^4(x, y)z + H_2^4(x, y) = 0, \tag{12}$$

where $H^i(x,y)$ are homogeneous polynomials in x and y of degree i, given by the formulas:

$$\begin{aligned}
H^5(x,y) &= (8\sqrt{2}-4p)x^5 - 4qx^4y + (4\sqrt{2}-4p)x^3y^2 - 4qx^2y^3 + (-4\sqrt{2}-p)xy^4 - qy^5\\
H_1^4(x,y) &= -4rx^4 + (-4\sqrt{2}-4r)x^2y^2 + (2\sqrt{2}-r)y^4\\
H_2^4(x,y) &= -2\sqrt{2}(2x^2-y^2)(2px^2+2qxy-ry^2). \qquad (13)
\end{aligned}$$

3.3. Properties of $\mathcal{P}_4^6$

Proposition 1. *The plane at infinity cuts the surface $\mathcal{P}_4^6$ at the absolute conic of $\mathbb{E}^3$ and the rays of the congruence $\mathcal{C}_4^1$.*

Proof. In the Cartesian homogeneous coordinates $(x:y:z:w)$, where $w=0$ means that the point lies in the plane at infinity, the equation of the surface $\mathcal{P}_4^6$ takes the following form:

$$(2x^2+y^2)^2(x^2+y^2+z^2) + H^5(x,y)w + H_1^4(x,y)zw + H_2^4(x,y)w^2 = 0. \qquad (14)$$

Therefore, the intersection of $\mathcal{P}_4^6$ and the plane at infinity splits into the absolute conic, given by equations

$$x^2+y^2+z^2=0, \quad w=0, \qquad (15)$$

and the pair of imaginary lines through the point $(0:0:1:0)$, counted twice, which are given by equations

$$(2x^2+y^2)^2=0, \quad w=0. \qquad (16)$$

The intersection points of Viviani's curve, given by equations (1) and (2), and the plane at infinity are given by equations

$$x^2+y^2+z^2=0, \quad x^2+2zx+2y^2+z^2=0, \quad w=0. \qquad (17)$$

If we eliminate z in (17), we obtain (16) which present the four rays of the congruence $\mathcal{C}_4^1$ in the plane at infinity. □

3.3.1. Singularities on axis z

Proposition 2. *The axis z is the quadruple line of the surface $\mathcal{P}_4^6$.*

Proof. According to [4, p. 251], if an nth order surface in $\mathbb{E}^3$ which passes through the origin is given by equation $F(x,z,y) = f_m(x,y,z) + f_{m+1}(x,y,z) + \cdots + f_n(x,y,z) = 0$, where $f_k(x,y,z)$ $(1 \le k \le n)$ is a homogeneous polynomial of degree k, then the tangent cone at the point $(0,0,0)$ is given by equation $f_m(x,y,z)=0$.

If we translate the origin to any point $Z_0=(0,0,z_0)$ on the axis z, eq. (12) takes the form

$$(2x^2+y^2)^2(x^2+y^2+(z+z_0)^2) + H^5(x,y) + H_1^4(x,y)(z+z_0) + H_2^4(x,y) = 0. \qquad (18)$$

Thus, the tangent cone $\mathcal{T}_{Z_0}$ of $\mathcal{P}_4^6$ at the point Z_0, in the coordinate system with the origin Z_0, is given by the following equation

$$(2x^2+y^2)^2z_0^2+H_1^4(x,y)z_0+H_2^4(x,y)=0. \tag{19}$$

Since this equation is 4th degree homogeneous in x and y, in the general case $\mathcal{T}_{Z_0}$ always splits into the four planes through the axis z. □

Proposition 3. *The surface $\mathcal{P}_4^6$ has a quintuple point on the axis z iff a pole P lies on the axis z. In this case, P is a unique quintuple point of $\mathcal{P}_4^6$. For the tangent cone $\mathcal{T}_P^5$ of $\mathcal{P}_4^6$ at P, the following statements are valid:*

1. *If $r\in(-\infty,-2\sqrt{2})\cup(\sqrt{2}/4,+\infty)$, the axis z is the isolated quadruple line of $\mathcal{T}_P^5$.*
2. *If $r=-2\sqrt{2}$, $\mathcal{T}_P^5$ splits into one plane through z and the 4th degree cone. The axis z is the triple line of this 4th degree cone with one real and one pair of imaginary tangent planes through it.*
3. *If $r\in(-2\sqrt{2},0)$, the axis z is the quadruple line of $\mathcal{T}_P^5$ with one pair of real and different, and one pair of imaginary tangent planes through it.*
4. *If $r=0$, $\mathcal{T}_P^5$ splits into two planes through z and the 3rd degree cone. The axis z is the cuspidal line of this 3rd degree cone.*
5. *If $r\in(0,\sqrt{2}/4)$, the axis z is the quadruple line of $\mathcal{T}_P^5$ with four real and different tangent planes through it.*
6. *If $r=\sqrt{2}/4$, the axis z is the double cuspidal line of $\mathcal{T}_P^5$.*

Proof. The tangent cone of $\mathcal{P}_4^6$ at its point $Z_0=(0,0,z_0)$, in the coordinate system with the origin Z_0, is given by eq. (19). The expanded form of this equation is the following:

$$\begin{aligned}&-4\left(2\sqrt{2}p+(r-z_0)z_0\right)x^4-8\sqrt{2}qx^3y+4\left(\sqrt{2}p+\left(\sqrt{2}-z_0\right)(r-z_0)\right)x^2y^2\\&+4\sqrt{2}qxy^3-(r-z_0)\left(z_0+2\sqrt{2}\right)y^4=0.\end{aligned} \tag{20}$$

According to [4, p. 251], the point Z_0 is the quintuple point of $\mathcal{P}_4^6$, iff all coefficients in (20) vanish and the 5th degree homogeneous polynomial in (18) does not vanish. It is easy to show that all coefficients in eq. (20) vanish only in the case: $p=0$, $q=0$, $r=z_0$, i.e. if a pole P lies on the axis z and $Z_0=P$. In this case P is the quintuple point of $\mathcal{P}_4^6$ with the tangent cone, in the coordinate system with origin P, given by the following equation

$$8\sqrt{2}x^5+4\sqrt{2}x^3y^2-4\sqrt{2}xy^4+\left(4rx^4+4\left(r-\sqrt{2}\right)x^2y^2+\left(r+2\sqrt{2}\right)y^4\right)z, \tag{21}$$

which represents the 5th degree cone [6, p. 56].
If P lies on the axis z, the tangent cone at the point $(0,0,z_0)$, $z_0\neq r$, is given by the following equation

$$-4(r-z_0)z_0\,x^4+4\left(\sqrt{2}-z_0\right)(r-z_0)\,x^2y^2-(r-z_0)\left(z_0+2\sqrt{2}\right)y^4=0 \tag{22}$$

and therefore, all other points on the axis z are the quadruple points of $\mathcal{P}_4^6$.

If $r = 0$, eq. (21) takes the form:

$$\left(2x^2 - y^2\right)\left(2x^3 + 2y^2x - y^2z\right) = 0. \tag{23}$$

Thus, for $P=(0,0,0)$ the tangent cone $\mathcal{T}_P^5$ splits into two planes $y=\pm\sqrt{2}x$ (planes through z and two tangent lines of c^4 at its node, see eq. (5)) and the 3rd degree cone $2x^3+2y^2x-y^2z=0$ with a cuspidal line on the axis z where coinciding tangent planes are given by equation $y=0$. It proves statement 4 from the proposition.

If $r = -2\sqrt{2}$, eq. (21) takes the form:

$$x\left(2x^4 + y^2x^2 - y^4 - x\left(2x^2 + 3y^2\right)z\right). \tag{24}$$

Thus, for $P=(0,0,-2\sqrt{2})$ the tangent cone $\mathcal{T}_P^5$ splits into the plane $x=0$ and the 4th degree cone $2x^4 + y^2x^2 - y^4 - x\left(2x^2+3y^2\right)z = 0$.
The axis z is the triple line of this 4th degree cone with one real tangent plane $x=0$ and the pair of imaginary tangent planes given by equation $2x^2+3y^2=0$. It proves statement 2 from the proposition.

If $r = \sqrt{2}/4$, eq. (21) takes the form:

$$\left(32x^5 + 16y^2x^3 - 16y^4x + \left(2x^2 - 3y^2\right)^2 z\right) = 0. \tag{25}$$

Thus, for $P=(0,0,\sqrt{2}/4)$ the axis z is the double cuspidal line of $\mathcal{T}_P^5$ with two pairs of coinciding tangent planes given by equations $\sqrt{2}x\pm\sqrt{3}y=0$. It proves statement 6 from the proposition.

If P lies on the axis z, every circle c passes through it and the generators of the cone $\mathcal{T}_P^5$ are the tangent lines of the circles c at point P.
If the axis z touches c, the generator coincides with z, and the plane of this circle c is the tangent plane of $\mathcal{T}_P^5$ through the axis z.

Since the circle c touches the axis z iff it passes though the intersection point of Viviani's curve c^4 and the plane $z=r$, we can conclude, according to formulas (4) and the discussion which follows them, that statements 1, 3 and 5 are valid. □

In Figure 7 the pedal surfaces with the tangent cones at their quintuple points are shown for seven positions of a pole P. The coordinates of the pole P are: a $-(0,0,\sqrt{2})$, b $-(0,0,\sqrt{2}/4)$, c $-(0,0,\sqrt{2}/6)$, d $-(0,0,0)$, e $-(0,0,-\sqrt{2})$, f $-(0,0,-2\sqrt{2})$ and g $-(0,0,-4)$.

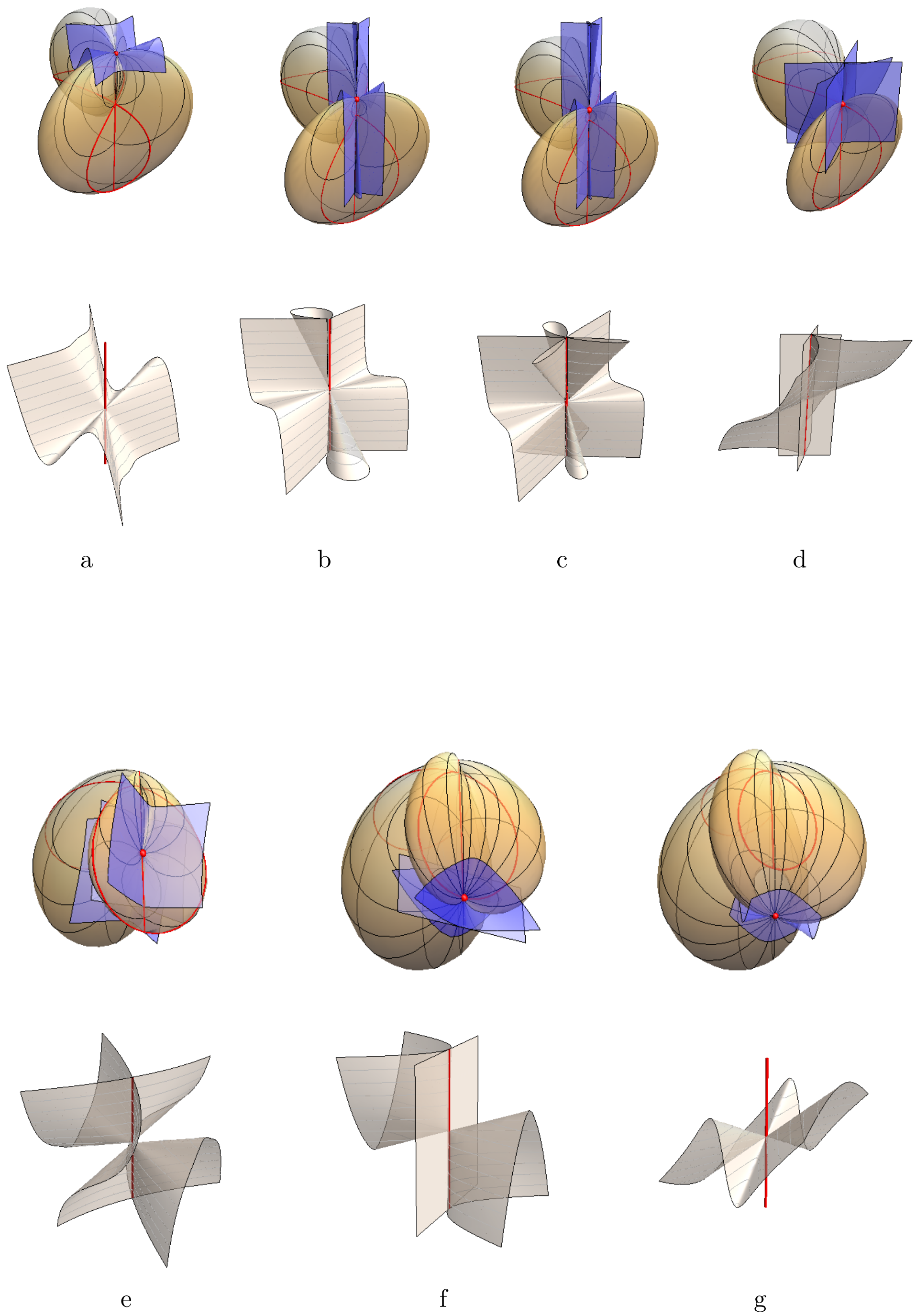


Figure 7.

The points $Z_0(0,0,z_0)$ on the axis z are the quadriplanar points of surface $\mathcal{P}_4^6$. Their tangent cones $\mathcal{T}_{Z_0}$, given by eq. (20), split into four planes through the axis z. We distinguish nine types as follows:

Type 1: $\mathcal{T}_{Z_0}-$ four real and different planes.
Type 2: $\mathcal{T}_{Z_0}-$ two real and different planes and a pair of imaginary planes.
Type 3: $\mathcal{T}_{Z_0}-$ two different pairs of imaginary planes.
Type 4: $\mathcal{T}_{Z_0}-$ one double plane and two different single real planes.
Type 5: $\mathcal{T}_{Z_0}-$ one double plane and a pair of imaginary planes.
Type 6: $\mathcal{T}_{Z_0}-$ a pair of double real planes.
Type 7: $\mathcal{T}_{Z_0}-$ a double pair of imaginary planes.
Type 8: $\mathcal{T}_{Z_0}-$ one triple plane and one single plane.
Type 9: $\mathcal{T}_{Z_0}-$ one quadruple plane.

On the axis z the intervals with quadriplanar points of types $1-3$ are bounded by points of types $4-9$ which are the pinch-points of $\mathcal{P}_4^6$.

Proposition 4. *The surface $\mathcal{P}_4^6$ has twelve pinch-points on the quadruple line z (real or complex). Among them, one is always the point at infinity and it is the pinch-point of type 7.*

Proof. The proof that an nth order surface with an $(n-2)$-ple line always possesses $4(n-3)$ pinch-points is given in [8, p. 317]. We give here only its interpretation on this 6th order case: every plane δ through the axis z cuts $\mathcal{P}_4^6$ into a quadruple line and one conic c which cuts quadruple line in two points - touching points of the plane δ and $\mathcal{P}_4^6$. The correspondence on the pencil of planes $[z]$, where corresponding planes have the same touching point, is the involution of order 6 since through each touching point of δ another 3 tangent planes pass. This involution has $2\cdot 6$ double elements which are the coinciding tangent planes through the points on the quadruple line and their touching points are the pinch-points of $\mathcal{P}_4^6$.
According to eq. (14), the tangent cone at the point $Z_0^\infty(0{:}0{:}1{:}0)$ is given by equation $(2x^2+y^2)^2=0$, and Z_0^∞ is the pinch-point of type 7. $\square$

The above proposition includes complex points, but below we will refer only to the real pinch-points. The type of quadriplanar point Z_0 depends on factorization of the homogeneous 4th degree polynomial in x and y which represents $\mathcal{T}_{Z_0}$. If we use the substitutions $y=k\,x$, $x=h\,y$, the polynomial from eq. (20) takes the forms:

$$\begin{aligned} &Ak^4+Bk^3+Ck^2+Dk+E=0,\\ &Eh^4+Dh^3+Ch^2+Bh+A=0, \end{aligned} \tag{26}$$

where

$$\begin{aligned} &A=-4\left(2\sqrt{2}p+(r-z_0)z_0\right), \quad B=-8\sqrt{2}q\\ &C=4\left(\sqrt{2}p+\left(\sqrt{2}-z_0\right)(r-z_0)\right)\\ &D=4\sqrt{2}q, \quad E=-(r-z_0)\left(z_0+2\sqrt{2}\right). \end{aligned} \tag{27}$$

For the given Z_0, the roots of polynomials (26) are the tangent and cotangent of the angles between the planes of $\mathcal{T}_{Z_0}$ and the plane $y = 0$. If polynomials (26) have a multiple root for z_0, Z_0 is the pinch-point of $\mathcal{P}_4^6$.

In [10], for the depressed quartic polynomial

$$P_4(x) = x^4 + a_2x^2 + a_1x + a_0, \quad a_i \in \mathbb{R}, \tag{28}$$

the author gives the following relations between its coefficients and multiple roots:

$$\begin{aligned}
&P_4(x) \text{ has three different real roots and one of them is a double root} \Longleftrightarrow \\
&\qquad a_2 < 0,\ a_2^2 - 4a_0 > 0,\ a_2^2 + 12a_0 > 0, \\
&\qquad 4(a_2^2 + 12a_0)^3 = (2a_2^3 - 72a_2a_0 + 27a_1^2)^2. \\
&P_4(x) \text{ has one double real root and a pair of complex roots} \Longleftrightarrow \\
&\qquad a_2^2 + 12a_0 > 0,\ -2a_2 < \sqrt{a_2^2 + 12a_0}, \\
&\qquad 2(a_2^2 + 12a_0)^{\frac{3}{2}} = 2a_2^3 - 72a_2a_0 + 27a_1^2. \\
&P_4(x) \text{ has two double real roots} \Longleftrightarrow a_1 = 0,\ a_2^2 - 4a_0 = 0,\ a_2 < 0. \\
&P_4(x) \text{ has two double complex roots} \Longleftrightarrow a_1 = 0,\ a_2^2 - 4a_0 = 0,\ a_2 > 0. \\
&P_4(x) \text{ has two different real roots and one of them is a triple root} \Longleftrightarrow \\
&\qquad a_2^2 + 12a_0 = 0,\ 8a_2^3 + 27a_1^2 = 0,\ a_2 < 0. \\
&P_4(x) \text{ has one quadruple real root} \Longleftrightarrow a_0 = a_1 = a_2 = 0.
\end{aligned} \tag{29}$$

By using the substitutions $k = t - B/4A$ and $h = s - D/4E$, polynomials (26) take the depressed forms. On this basis and based on conditions (29), we made a program in *Mathematica 6* (available online: www.grad.hr/sgorjanc/pinch_points.nb) which calculates coordinates z_0 for the pinch-points of $\mathcal{P}_4^6$ for every choice of a pole P, i.e. for the given (p, q, r). Here is one example:

$P(1, 0, 1)$ − 12 real pinch-points, see Figure 8.

- two single pinch-points of type 4
  $z_0 = \frac{1}{2}\left(1 - \sqrt{1 + 8\sqrt{2}}\right)$ and $z_0 = 1$ with $\mathcal{T}_{Z_0}$ given by
  $y^2\left(21.0182x^2 - 2.50909y^2\right) = 0$ and $x^2\left(2x^2 - y^2\right) = 0$, respectively.

- two single pinch-points of type 5
  $z_0 = -2\sqrt{2}$ and $z_0 = \frac{1}{2}\left(1 + \sqrt{1 + 8\sqrt{2}}\right)$ with $\mathcal{T}_{Z_0}$ given by
  $x^2\left(2x^2 + 4.41421y^2\right) = 0$ and $y^2\left(6.98182x^2 + 4.50909y^2\right) = 0$, respectively.

- two double pinch-points of type 6
  $z_0 = 0$ and $z_0 = \frac{1}{8}\left(12 + \sqrt{2} - \sqrt{6\left(3 + 4\sqrt{2}\right)}\right)$ with $\mathcal{T}_{Z_0}$ given by
  $\left(2x^2 - y^2\right)^2 = 0$ and $\left(y^2 - 3.85588x^2\right)^2 = 0$, respectively.

- two double pinch-points of type 7

$z_0 = \frac{1}{8}\left(12+\sqrt{2}+\sqrt{6\left(3+4\sqrt{2}\right)}\right)$ and $Z_0(0{:}0{:}1{:}0)$ with $\mathcal{T}_{Z_0}$ given by $\left(0.762047x^2+y^2\right)^2=0$ and $(2x^2+y^2)^2=0$, respectively.

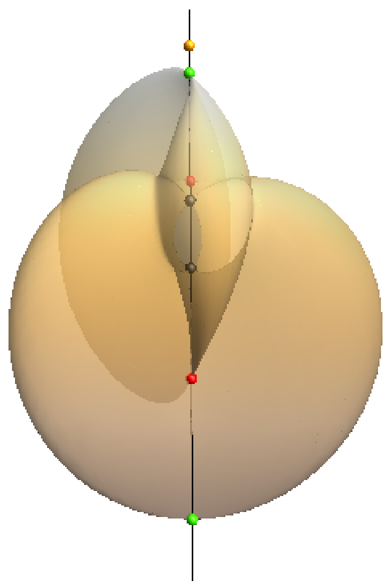


Figure 8.

Proposition 5. *On the surfaces $\mathcal{P}_4^6$, all types of pinch-points (type 4-9) exist.*

Proof. In the previous example the tangent cones and the coordinates of the pinch-points of types 4, 5, 6 and 7 are given.
For $P(\frac{29}{64}, 1, -\frac{79}{64}+\frac{2\sqrt{2}}{79})$ the tangent cone $\mathcal{T}_{Z_0}$ at the point $Z_0(0,0,\frac{2\sqrt{2}}{79})$ is given by equation $(x+y)^3(7x-5y)=0$. Thus, $Z_0(0,0,\frac{2\sqrt{2}}{79})$ is the pinch-point of type 8.
For $P(-3,0,1-2\sqrt{2})$ the tangent cone $\mathcal{T}_{Z_0}$ at the point $Z_0(0,0,-2\sqrt{2})$ is given by equation $x^4=0$. Thus, $Z_0(0,0,-2\sqrt{2})$ is the pinch-point of type 9. □

3.3.2. Real singularities outside the axis z

Except for the points on the quadruple line z, the highest singularity $\mathcal{P}_4^6$ can possess is a double point. Namely, if $\mathcal{P}_4^6$ had a higher multiple point out of z, the line through that point which cuts z would cut $\mathcal{P}_4^6$ in more than 6 points, which is impossible.

If D is the double point of $\mathcal{P}_4^6$, it is the double point of every section of $\mathcal{P}_4^6$ through D. Thus, the circle c in the plane through D and the axis z splits into a pair of isotropic lines through D. It is the case when the end points of the diameter $\overline{CP'}$ coincide, i.e. circle k intersects Viviani's curve c^4 at point D, see Figure 9.

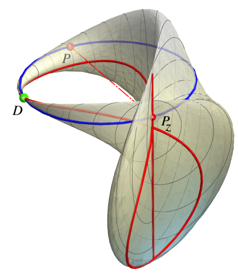
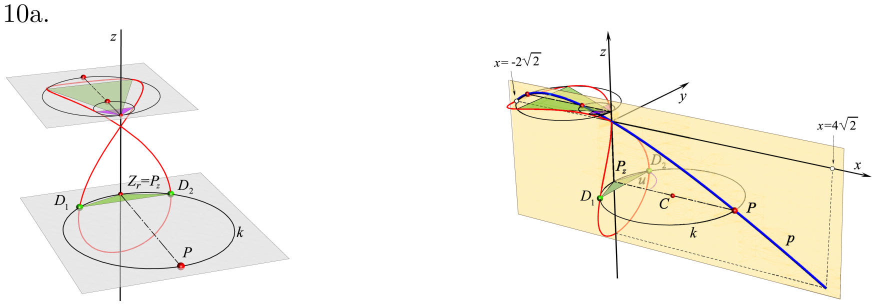


Figure 9.

Proposition 6. *The surface $\mathcal{P}_4^6$ has exactly two real double points out of the axis z iff a pole P lies on the part of one parabola given by the following relations:*

$$x^2 + 2\left(z + \sqrt{2}\right)x + z\left(z + 6\sqrt{2}\right) = 0, \quad y = 0,$$
$$x \in (-2\sqrt{2}, 4\sqrt{2}) \setminus \{0\}. \tag{30}$$

Proof. According to (4), in the planes $z = r$, $r \in (-2\sqrt{2}, 0) \cup (0, \sqrt{2}/4)$ the curve c^4 has at least two real points D_1, D_2 which are symmetrical with respect to the plane $y = 0$. If the circle k passes through these points, they are the double points of the surface $\mathcal{P}_4^6$. In this case k is the circumcircle of $\triangle D_1D_2Z_r$ (where Z_r is the intersection point of the plane $z = r$ and the axis z) and the pole P is the end point of its diameter through Z_r, i.e. $P_z = Z_r$. For $r \in (0, \sqrt{2}/4)$, there are two such circles k in the plane $z = r$, and for $r \in (-2\sqrt{2}, 0)$ only one such circle k exists, see Figure 10a.



a b

Figure 10.

It is clear that the locus of points P is the part of a curve in the plane $y = 0$. According to parametrization of Viviani's curve (3) and the sine formula, the circumradius of $\triangle D_1D_2P_z$ is $|y_{c^4}(u)/\sin 2u|$, and the (x, z) coordinates of the circumcenter C (see Figure 10b) are $(y_{c^4}(u)/\sin 2u, z_{c^4}(u))$, $u \in [0, \pi/2]$. Thus, the parametric equations of the curve which contains the path of the point P are the following:

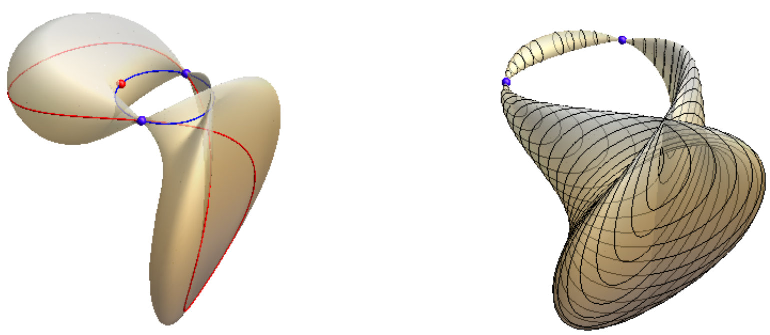
$$\begin{aligned} x(u) &= -\frac{8\sqrt{2}(3\cos(2u)+1)}{(\cos(2u)+3)^2} \\ y(u) &= 0 \\ z(u) &= \frac{4\sqrt{2}(3\cos(2u)+1)\sin^2(u)}{(\cos(2u)+3)^2}, \qquad u \in [0, \pi/2]. \end{aligned} \tag{31}$$

If we substitute $\sin^2 u \to (1-\cos 2u)/2$ in eq. (31) and then eliminate $\cos 2u$, we obtain the following equations:

$$x^2 + 2xz + 2 + z^2\sqrt{2}x + 6\sqrt{2}z = 0, \quad y = 0. \tag{32}$$

They are the equations of one parabola p and if P lies on it and belongs to the region $x \in (-2\sqrt{2}, 4\sqrt{2}) \setminus \{0\}$, the circle k intersects c^4 in two different real points, see Figure 10b. □

In Figure 11 two examples of $\mathcal{P}_4^6$ with two double points are shown.



a b

Figure 11.

Proposition 7. *The surface $\mathcal{P}_4^6$ has at least one real double point out of the axis z iff a pole P lies on one 5th degree ruled surface.*

Proof. Every plane $z = r$, $r \in (-2\sqrt{2}, \sqrt{2}/4]$, cuts the axis z at the point P_z and Viviani's curve at the real point $D \notin z$. If the circle k belongs to the pencil of circles (D, P_z), the point D is the real double point of $\mathcal{P}_4^6$. The end points of diameters through P_z of circles of the pencil (P_zD) lie on the line l which is perpendicular to P_zD and passes through D. Thus, if the pole P lies on l, D is the double point of $\mathcal{P}_4^6$.

a b

Figure 12.

It is always the unique double point of $\mathcal{P}_4^6$ except in the case when P is the intersection point of the plane $z=r$ and the part of parabola p for which $-2\sqrt{2} \leq x < 4\sqrt{2}$. This is denoted by P_p and in this case, as shown in proposition 6, the surface $\mathcal{P}_4^6$ has two double points (see Figure 12a). The lines l are the rulings of one ruled surface $\mathcal{R}$ (see Figure 12b) which is part of the ruled surface directed by lines: Viviani's curve c^4, parabola p and the line at infinity l^∞ in the plane $z=0$. Below we will derive the implicit equation of the surface $\mathcal{R}$.
According to formulas (4) and the corresponding relations (see Figure 2), in the plane $z=r$ there are 2 or 4 real lines l (which are in pairs symmetrical with respect to the plane $y=0$), if $r \in (-2\sqrt{2}, 0)$ or $r \in (0, \sqrt{2}/4)$, respectively. Especially, if $r \in \{-2\sqrt{2}, 0, \sqrt{2}/4\}$, two lines l coincide and they are the torsal lines of the surface $\mathcal{R}$. In the planes $z=-2\sqrt{2}$ and $z=0$ one torsal line exists and in the plane $z=\sqrt{2}/4$ two torsal lines exist (see Figure 13).

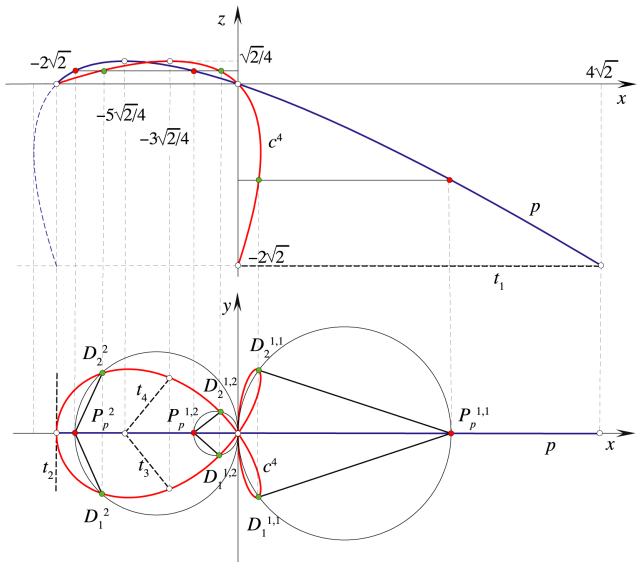


Figure 13.

If we solve equations (1) and (2) for variables x and y, we can express curve c^4 by the following parametrization:

$$\begin{aligned} x_{c^4}^1(z) &= z - \sqrt{2} + \sqrt{2 - 4\sqrt{2}z} & x_{c^4}^2(z) &= z - \sqrt{2} - \sqrt{2 - 4\sqrt{2}z} \\ y_{c^4}^1(z) &= \pm\sqrt{2}\sqrt{-zx_{c^4}^1(z)} & y_{c^4}^2(z) &= \pm\sqrt{2}\sqrt{-zx_{c^4}^2(z)} \\ z_{c^4}^1(z) &= z, \quad z \in [-2\sqrt{2}, \sqrt{2}/4] & z_{c^4}^2(z) &= z, \quad z \in [0, \sqrt{2}/4]. \end{aligned} \tag{33}$$

The corresponding parts of the parabola p, given by eq. (32), can be parametrized as follows:

$$\begin{aligned} x_p^1(z) &= -z - \sqrt{2} + \sqrt{2 - 4\sqrt{2}z} & x_p^2(z) &= -z - \sqrt{2} - \sqrt{2 - 4\sqrt{2}z} \\ y_p^1(z) &= 0 & y_p^2(z) &= 0 \\ z_p^1(z) &= z, \quad z \in [-2\sqrt{2}, \sqrt{2}/4] & z_p^2(z) &= z, \quad z \in [0, \sqrt{2}/4]. \end{aligned} \tag{34}$$

In the planes $z = z_0$, $z_0 \in [-2\sqrt{2}, \sqrt{2}/4]$, the rulings of the surface $\mathcal{R}$ are:

1. the lines which join the point $(x_p^1(z_0), 0)$ with the points $(x_{c^4}^1(z_0), y_{c^4}^1(z_0))$, for $z_0 \in [-2\sqrt{2}, \sqrt{2}/4]$;
2. the lines which join the points $(x_p^2(z_0), 0)$ with the corresponding points $(x_{c^4}^2(z_0), y_{c^4}^2(z_0))$ for $z_0 \in [0, \sqrt{2}/4]$.

From the equations of the rulings, by substitution $z_0 \to z$, multiplying by z and squaring, we obtain the following equations for the parts of $\mathcal{R}$:

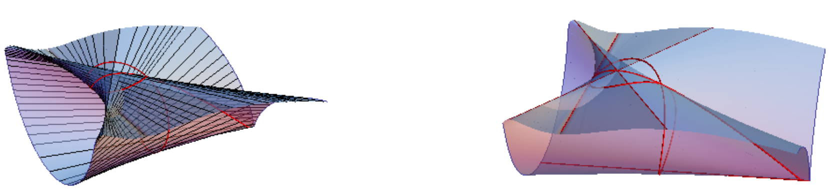
$$1.\ 2y^2 z + \left(z + \sqrt{2 - 4\sqrt{2}z} - \sqrt{2}\right)\left(x + z - \sqrt{2 - 4\sqrt{2}z} + \sqrt{2}\right)^2 = 0,\quad z \in [-2\sqrt{2}, \sqrt{2}/4], \tag{35}$$

$$2.\ 2y^2 z + \left(z - \sqrt{2 - 4\sqrt{2}z} - \sqrt{2}\right)\left(x + z + \sqrt{2 - 4\sqrt{2}z} + \sqrt{2}\right)^2 = 0,\quad z \in [0, \sqrt{2}/4]. \tag{36}$$

Eqs. (35) and (36), after the elimination of roots and dividing by z, give the same equation of $\mathcal{R}$ as follows:

$$\begin{aligned} z^5 &+2\left(2x + 7\sqrt{2}\right)z^4 + 2\left(3x^2 + 18\sqrt{2}x + 2y^2 + 60\right)z^3 \\ &+4\left(x^3 + 8\sqrt{2}x^2 + 2y^2 x + 40x + 5\sqrt{2}y^2 + 36\sqrt{2}\right)z^2 \\ &+\left(x^4 + 12\sqrt{2}x^3 + 4y^2x^2 + 88x^2 + 32\sqrt{2}y^2x + 96\sqrt{2}x + 4y^4 + 80y^2\right)z \\ &+2\left(\sqrt{2}x^2 + 8x + 8\sqrt{2}\right)\left(x^2 - 2y^2\right) = 0. \end{aligned} \tag{37}$$

From the previous analysis we can conclude that iff a pole P lies on the surface given by eq. (37) and $r \neq -2\sqrt{2}$, the surface $\mathcal{P}_4^6$ has at least one real double point which does not lies on the axis z. We excluded the value $r = -2\sqrt{2}$ because in this case the double point D coincides with the quadruple point on the axis z. □



a b

Figure 14.

As is clear from eq. (37), $\mathcal{R}$ is a 5th degree ruled surface. In Figure 14 this surface is viewed from two different viewpoints and in Figure 14b its torsal lines are indicated.

The degree of the complete ruled surface with directing lines c^4, p and l^∞ in the plane $z=0$ is $2\cdot 4\cdot 2\cdot 1 - 3\cdot 1 = 13$, [6, p. 90]. The residual surface $\mathcal{S}$, which is obtained by joining the intersection points of c^4 and p with the planes parallel to $z=0$ in a different way, is an 8th degree ruled surface. Although the construction of this residual is the same as the construction of $\mathcal{R}$ (2 or 4 real lines in the planes $z=z_0$, $z_0\in[-2\sqrt{2},\sqrt{2}/4]$), the differences in their degrees is a result of the fact that the line l^∞ is the quadruple line of $\mathcal{S}$ when it is a simple line of $\mathcal{R}$. Namely, the line l^∞ is the quintuple line of the complete ruled surface [6, p. 91]: it is clear from eq. (37) that it is the simple line of $\mathcal{R}$ and it can be shown that it is the quadruple line of $\mathcal{S}$, but proving this here is beyond the concept of this paper.

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