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Butterflies in the Isotropic Plane

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ABSTRACT

A real affine plane A_2 is called an isotropic plane I_2 , if in A_2 a metric is induced by an absolute $\{f, F\}$, consisting of the line at infinity f of A_2 and a point $F \in f$. In this paper the well-known Butterfly theorem has been adapted for the isotropic plane. For the theorem that we will further-on call an Isotropic butterfly theorem, four proofs are given.

Key words: isotropic plane, butterfly theorem

MSC 2000: 51N025

Leptiri u izotropnoj ravnini

SAŽETAK

Realna afina ravnina A_2 se naziva izotropnom ravninom I_2 ako je metrika u A_2 inducirana apsolutnom figurom $\{f, F\}$, koja se sastoji od neizmjereno dalekog pravca f ravnine A_2 i točke $F \in f$. U ovom je radu poznati Leptirov teorem smješten u izotropnu ravninu. Za taj teorem, kojeg od sada nazivamo Izotropnim leptirovim teoremom, dana su četiri dokaza.

Ključne riječi: izotropna ravnina, leptirov teorem

1 Isotropic Plane

Let $P_2(\mathbf{R})$ be a real projective plane, f a real line in P_2 , and $A_2 = P_2 \setminus f$ the associated affine plane. The *isotropic plane* $I_2(\mathbf{R})$ is a real affine plane A_2 where the metric is introduced with a real line $f \subset P_2$ and a real point F incidental with it. The ordered pair $\{f, F\}$, $F \in f$ is called *absolute figure* of the isotropic plane $I_2(\mathbf{R})$ ([3], [5]). In the affine model, where

$$x = x_1/x_0, \quad y = x_2/x_0, \quad (1)$$

the absolute figure is determined by the *absolute line* $f \equiv x_0 = 0$, and the *absolute point* $F (0:0:1)$. All projective transformations that are keeping the absolute figure fixed form a 5-parametric group

$$G_5 \left\{ \begin{array}{l} \bar{x} = c_1 + c_4x \\ \bar{y} = c_2 + c_3x + c_5y \end{array} \right. , \quad \begin{array}{l} c_1, c_2, c_3, c_4, c_5 \in \mathbf{R} \\ \& \quad c_4c_5 \neq 0. \end{array} \quad (2)$$

We call it *the group of similarities* of isotropic plane.

Defining in I_2 the usual metric quantities such as the distance between two points, the angle between two lines etc., we look for the subgroup of G_5 for those quantities to be invariant. In such a way one obtains the fundamental group of transformations that are the mappings of the form:

$$G_3 \left\{ \begin{array}{l} \bar{x} = c_1 + x \\ \bar{y} = c_2 + c_3x + y \end{array} \right. \quad (3)$$

It is called *the motion group* of isotropic plane. Hence, the group of isotropic motions consists of translations and rotations, that is

$$\left\{ \begin{array}{l} \bar{x} = c_1 + x \\ \bar{y} = c_2 + y \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \bar{x} = x \\ \bar{y} = c_3x + y \end{array} \right. .$$

In the affine model, rotation is understood as stretching along the y-axis.

2 Terms of Elementary Geometry within I_2

We will first define some terms and point out some properties of triangles and circles in I_2 that are going to be used further on. The geometry of I_2 could be seen for example in Sachs [3], or Strubecker [5].

Isotropic straight line, parallel points, isotropic distance, isotropic span

All straight lines through the point F are called *isotropic straight lines* (isotropic lines). All the other straight lines are simply called *straight lines*. Two points A, B ($A \neq B$) are called *parallel* if they are incidental with the same isotropic line. For two no parallel points $A(a_1, a_2), B(b_1, b_2)$, the *isotropic distance* is defined by $d(A, B) := b_1 - a_1$. Note that the isotropic distance is directed. For two parallel points $A(a_1, a_2), B(b_1, b_2)$, $a_1 = b_1$, the quantity known as *isotropic span* is defined by $s(A, B) := b_2 - a_2$. A straight line p through two points A and B will be denoted by $p \equiv A \vee B$, or simply $p \equiv AB$.

Invariants of a pair of straight lines

Each no isotropic straight line $g \subset I_2$ can be written in the normal form $y = ux + v$, that is, in line coordinates, $g(u, v)$. For two straight lines $g_1(u_1, v_1)$, $g_2(u_2, v_2)$ the *isotropic angle* is defined by $\varphi = \angle(g_1, g_2) := u_2 - u_1$. Note that the isotropic angle is directed as well. The Euclidean meaning of the isotropic angle can be understood from the affine model that is given in figure 1.

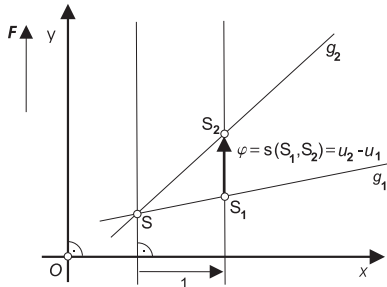


Fig. 1

For two parallel straight lines $g_1(u_1, v_1)$, $g_2(u_1, v_2)$ there exists an isotropic invariant defined by $\varphi^*(g_1, g_2) := v_2 - v_1$ (see fig. 2).

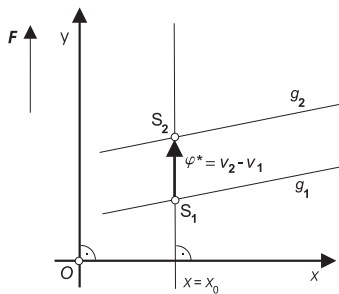


Fig. 2

Isotropic normal

An *isotropic normal* to the straight line $g(u, v)$ in the point $P(p_1, p_2)$, $P \notin g$ is an isotropic line through P . Inversely holds as well, i.e. each straight line $g \subset I_2$ is a normal for each isotropic straight line. Denoting by S the point of intersection of the isotropic normal in the point P with the straight line g , the isotropic distance of the point P from the line g is given by $d(P, g) := s(S, P) = p_2 - s_2 = p_2 - up_1 - v$ (see fig. 3).

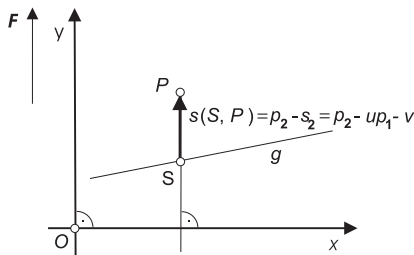


Fig. 3

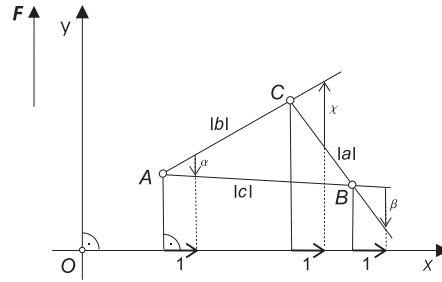


Fig. 4

Triangles and circles

Under a *triangle* in I_2 an ordered set of three no collinear points $\{A, B, C\}$ is understood. A, B, C are called *vertices*, and $a := B \vee C$, $b := C \vee A$, $c := A \vee B$ *sides* of a triangle. A triangle is called *allowable* if no one of its sides is isotropic. In a allowable triangle the *lengths* of the sides are defined by $|a| := d(B, C)$, $|b| := d(C, A)$, $|c| := d(A, B)$, with $|a| \neq 0$, $|b| \neq 0$, $|c| \neq 0$. For the directed angles we have $\alpha := \angle(b, c) \neq 0$, $\beta := \angle(c, a) \neq 0$, $\gamma := \angle(a, b) \neq 0$ (see figure 4).

Isotropic altitudes h_a, h_b, h_c associated with sides a, b , and c are isotropic straight lines passing through the vertices A, B, C , i.e. normals to the sides a, b , and c . Their lengths are defined by $|h_a| := s(L(A), A)$, where $L(A) = a \cap h_a$, etc. The Euclidian meaning is given in figure 5.

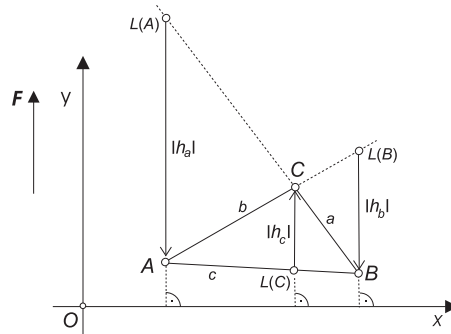


Fig. 5

An *isotropic circle* (*parabolic circle*) is a regular 2^{nd} order curve in $P_2(\mathbf{R})$ which touches the absolute line f in the absolute point F . According to the group G_3 of motions of the isotropic plane there exists in I_2 a three parametric family of isotropic circles, given by $y = Rx^2 + \alpha x + \beta$, $R \neq 0$, $\alpha, \beta \in \mathbf{R}$. Using transformations from G_3 , each isotropic circle can be reduced in the normal form $y = Rx^2$, $R \neq 0$. R is a G_3 invariant and it is called the *isotropic radius* of the parabolic circle.

3 The Isotropic Butterfly Theorem

Theorem 1 (Euclidean version) Let M be the midpoint of a chord PQ of the circle, through which two other chords AB and CD are drawn; AD cuts PQ at X and BC cuts PQ at Y . M is also the midpoint of XY .

This theorem has been proved in a series of books and papers (e.g. [1], [2], [4]).

Theorem 2 (Isotropic version) Let M be the midpoint of a chord \overrightarrow{PQ} of the parabolic circle, through which two other chords \overrightarrow{AB} and \overrightarrow{CD} are drawn; \overrightarrow{AD} cuts \overrightarrow{PQ} at X and \overrightarrow{BC} cuts \overrightarrow{PQ} at Y . M is also the midpoint of \overrightarrow{XY} .

Proof 1

The point coordinates are: $P(p_1, p_2)$, $Q(q_1, q_2)$, $M(m_1, m_2)$, $X(x_1, x_2)$, $Y(y_1, y_2)$, with $p_1 \neq q_1$, since \overrightarrow{PQ} is a chord and as such a no isotropic line, wherefrom we derive that $x_1 \neq y_1 \neq m_1$ must be fulfilled as well. Let us drop perpendiculars h_1, h_2 from X , and g_1, g_2 from Y on AB and CD . Let's also denote

$$\begin{aligned} d(P, M) = d(M, Q) &= |s|, \\ d(X, M) = |x|, \quad d(M, Y) &= |y|, \end{aligned} \tag{4}$$

$$\begin{aligned} H_1 = h_1 \cap AM, \quad H_2 = h_2 \cap DM, \\ G_1 = g_1 \cap MB, \quad G_2 = g_2 \cap MC. \end{aligned} \tag{5}$$

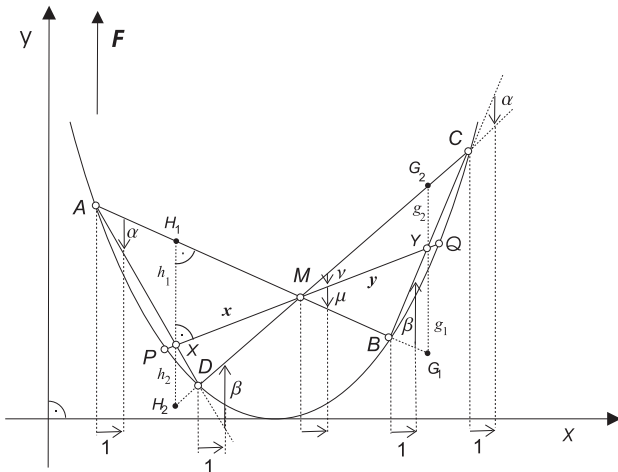


Fig. 6: The Isotropic butterfly theorem in the affine model

As first we need the following:

Lemma 1 Let $P, Q, P \neq Q$, be two points on a parabolic circle k , and $A \neq P, A \neq Q$, any other point on the same circle k . The isotropic angle $\varphi = \angle(\overrightarrow{PA}, \overrightarrow{QA})$ does not depend on the position of point A .

The proof is given in [3, p. 32].

Lemma 2 The relations

$$\frac{|a|}{\alpha} = \frac{|b|}{\beta} = \frac{|c|}{\chi}, \quad |h_a| = |c|\beta, \quad |h_b| = |a|\chi, \quad |h_c| = |b|\alpha$$

hold for every allowable triangle.

The proof is given in [3, p. 28].

Lemma 3 Let k be a parabolic circle in I_2 , a point $P \in I_2, P \notin k$, and S_1, S_2 two points of intersection of a no isotropic straight line g through P with k . The product $f(P) := d(P, S_1) \cdot d(P, S_2)$ doesn't depend of the line g , but only of k and P .

The proof is given in [3, p. 38].

Let's now continue the proof of the isotropic Butterfly theorem.

According to lemma 1,

$$\alpha = \angle(\overrightarrow{AB}, \overrightarrow{AD}) = \alpha' = \angle(\overrightarrow{CB}, \overrightarrow{CD}),$$

and

$$\beta = \angle(\overrightarrow{DA}, \overrightarrow{DC}) = \beta' = \angle(\overrightarrow{BA}, \overrightarrow{BC}). \tag{6}$$

We will also need

$$\mu = \angle(\overrightarrow{XM}, \overrightarrow{MA}) = \mu' = \angle(\overrightarrow{YM}, \overrightarrow{MB}),$$

and

$$v = \angle(\overrightarrow{DM}, \overrightarrow{MX}) = v' = \angle(\overrightarrow{CM}, \overrightarrow{MY}). \tag{7}$$

Let's apply furthermore lemma 2 on the following pairs of allowable triangles:

- 1st) $\triangle AXM$ & $\triangle MBY$, 2nd) $\triangle XDM$ & $\triangle MYC$,
 - 3rd) $\triangle AXM$ & $\triangle MYC$, 4th) $\triangle XDM$ & $\triangle MBY$,
- marking sides, angles and altitudes as given in figure 7.

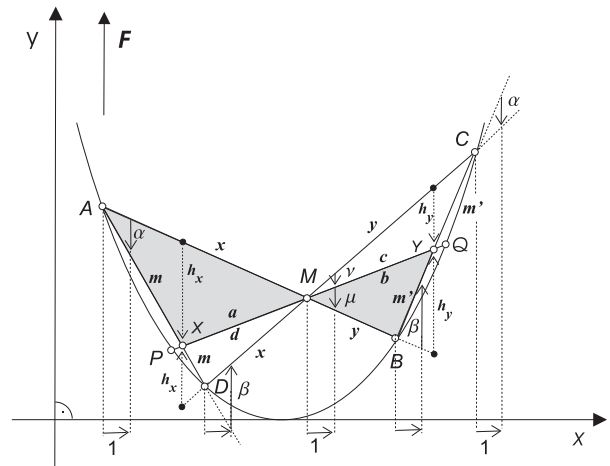


Fig. 7

$$\begin{aligned}
 \text{1st) } \triangle AXM &\Rightarrow \frac{|x|}{\angle(\overrightarrow{AX}, \overrightarrow{XM})} = \frac{|a|}{\alpha} = \frac{|m|}{\mu}, \\
 |h_x| &= |a| \cdot \mu; \\
 \triangle MBY &\Rightarrow \frac{|y|}{\angle(\overrightarrow{BY}, \overrightarrow{YM})} = \frac{|m'|}{\mu} = \frac{|b|}{\beta}, \\
 |h_y| &= |b| \cdot \mu;
 \end{aligned}$$

$\Rightarrow \frac{|h_x|}{|h_y|} = \frac{|a|}{|b|}$, and using marks from fig. 6 we get

$$\frac{|x|}{|y|} = \frac{|h_1|}{|g_1|}. \tag{8}$$

$$\begin{aligned}
 \text{2nd) } \triangle XDM &\Rightarrow \frac{|x|}{\angle(\overrightarrow{MX}, \overrightarrow{XD})} = \frac{|d|}{\beta} = \frac{|m|}{\nu}, \\
 |h_y| &= |m| \cdot \beta = |d| \cdot \nu; \\
 \triangle MYC &\Rightarrow \frac{|y|}{\angle(\overrightarrow{MY}, \overrightarrow{YC})} = \frac{|c|}{\alpha} = \frac{|m'|}{\nu}, \\
 |h_y| &= |m'| \cdot \alpha = |c| \cdot \nu;
 \end{aligned}$$

$\Rightarrow \frac{|h_x|}{|h_y|} = \frac{|d|}{|c|}$, and using marks from fig. 6 we have

$$\frac{|x|}{|y|} = \frac{|h_2|}{|g_2|}. \tag{9}$$

Analogously, for the third pair of triangles we get

$$\frac{|h_1|}{|g_2|} = \frac{d(A, X)}{d(Y, C)}. \tag{10}$$

Finally, for the fourth pair of triangles we have

$$\frac{|h_2|}{|g_1|} = \frac{d(X, D)}{d(B, Y)}. \tag{11}$$

From (4), (8), (9), (10), (11), and lemma 3 one computes

$$\begin{aligned}
 \frac{|x|^2}{|y|^2} &= \frac{|h_1|}{|g_1|} \cdot \frac{|h_2|}{|g_2|} = \frac{|h_1|}{|g_2|} \cdot \frac{|h_2|}{|g_1|} = \\
 &= \frac{d(A, X)}{d(Y, C)} \cdot \frac{d(X, D)}{d(B, Y)} = \frac{-d(X, A) \cdot d(X, D)}{-d(Y, C) \cdot d(Y, B)} = \\
 &= \frac{d(X, P) \cdot d(X, Q)}{d(Y, P) \cdot d(Y, Q)} = \frac{(p_1 - x_1)(q_1 - x_1)}{(p_1 - y_1)(q_1 - y_1)} = \\
 &= \frac{(p_1 - m_1 + m_1 - x_1)(q_1 - m_1 + m_1 - x_1)}{(p_1 - m_1 + m_1 - y_1)(q_1 - m_1 + m_1 - y_1)} = \\
 &= \frac{-(|s| - |x|)(|s| + |x|)}{-(|s| + |y|)(|s| - |y|)} = \frac{|s|^2 - |x|^2}{|s|^2 - |y|^2}. \tag{12}
 \end{aligned}$$

$$\frac{|x|^2}{|y|^2} = \frac{|s|^2 - |x|^2}{|s|^2 - |y|^2} \Rightarrow |x|^2 = |y|^2 \Rightarrow |x| = \pm |y|$$

The solution $|x| = -|y| \Rightarrow d(X, M) = -d(M, Y) = d(Y, M)$, wherefrom it follows that points X and Y are parallel points, which has been excluded earlier. So, $|x| = |y| \Rightarrow d(X, M) = d(M, Y)$. \square

Proof 2

Let's use the notation given in (4), that is, $d(P, M) = d(M, Q) = |s|$, $d(X, M) = |x|$, $d(M, Y) = |y|$, as well as (6) and (7) for the observed angles.

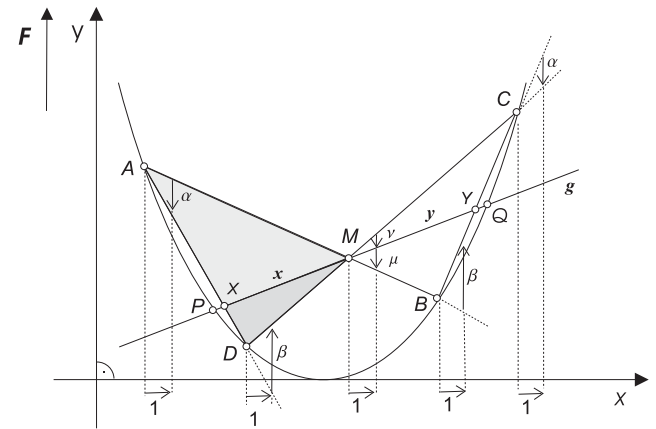


Fig. 8

From lemma 3, as shown in (12), we have

$$d(X, A) \cdot d(X, D) = d(X, P) \cdot d(X, Q),$$

$$d(X, P) \cdot d(X, Q) = -(|s| - |x|)(|s| + |x|) = |x|^2 - |s|^2. \tag{13}$$

Lemma 2 applied on the allowable triangles $\triangle DMX$ and $\triangle AXM$ yields

$$\begin{aligned}
 \triangle DMX &\Rightarrow \frac{d(X, D)}{\nu} = \frac{d(D, M)}{\angle(\overrightarrow{MX}, \overrightarrow{XD})} = \frac{d(M, X)}{\beta} \\
 &\Rightarrow \frac{d(X, D)}{\nu} = \frac{d(M, X)}{\beta} \tag{14}
 \end{aligned}$$

$$\begin{aligned}
 \triangle AXM &\Rightarrow \frac{d(A, X)}{\mu} = \frac{d(X, M)}{\alpha} = \frac{d(M, A)}{\angle(\overrightarrow{AX}, \overrightarrow{XM})} \\
 &\Rightarrow \frac{d(A, X)}{\mu} = \frac{d(X, M)}{\alpha}. \tag{15}
 \end{aligned}$$

Lemma 4 *The sum of the directed sides of an allowable triangle in I_2 equals zero; the sum of the directed angles of an allowable triangle in I_2 equals zero as well.*

The proof is given in [3, p. 22].

For the allowable triangle $\triangle ADM$, from lemma 4,

$$v + \mu + \alpha + \beta = 0 \Rightarrow \beta = -(v + \mu + \alpha). \quad (16)$$

Using (13)-(16) together, we obtain

$$d(X,A) \cdot d(X,D) = -d(X,M) \cdot \frac{\mu}{\alpha} \cdot d(M,X) \cdot \frac{v}{\beta} =$$

$$= |x|^2 \frac{v\mu}{-\alpha(v + \mu + \alpha)} = |x|^2 - |s|^2$$

$$\Rightarrow |x|^2 \left(1 + \frac{v\mu}{\alpha(v + \mu + \alpha)} \right) = |s|^2$$

$$\Rightarrow |x|^2 = \frac{|s|^2 [\alpha(v + \mu + \alpha)]}{v\mu + \alpha(v + \mu + \alpha)}. \quad (17)$$

Following the same procedure ((13)-(16)) for the segment $|y| = d(M,Y)$, due to the symmetry in v and μ in the latter expression, we'll get exactly same result. So, $|x|^2 = |y|^2$, that is $|x| = \pm |y|$, and following the conclusion from proof 1, $|x| = |y| \Rightarrow d(X,M) = d(M,Y)$. \square

Proof 3

The proof is based on the following:

Lemma 5 *If in two allowable triangles in I_2 a directed angle of one is equal to a directed angle of the other, then the areas of the triangles are in the same ratio as the products of the sides composing the equal angles.*

Proof According [3, p. 26] the isotropic area of an allowable triangle ABC , $A(a_1, a_2)$, $B(b_1, b_2)$, and $C(c_1, c_2)$ is given by

$$F_{ABC} = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}.$$

Let's mark the directed angles as given before in (6) and (7) (see figure 6), and let's observe the allowable triangles AXM and MYC (figure 9).

Lemma 1 yields that $\alpha = \angle(\overrightarrow{MA}, \overrightarrow{AX}) = \alpha' = \angle(\overrightarrow{YC}, \overrightarrow{CM})$, hence, we have to proof the equality:

$$\frac{F_{AXM}}{F_{MYC}} = \frac{d(M,A) \cdot d(A,X)}{d(Y,C) \cdot d(C,M)}. \quad (18)$$

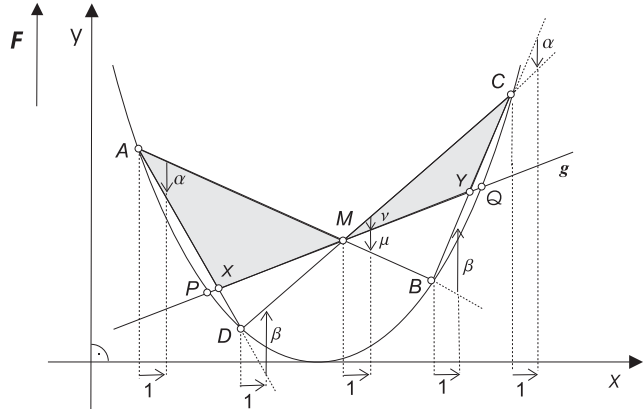


Fig. 9

For the points $A(a_1, a_2)$, $C(c_1, c_2)$, $M(m_1, m_2)$, $X(x_1, x_2)$ and $Y(y_1, y_2)$, the isotropic areas of the triangles are given by

$$F_{AXM} = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ a_1 & x_1 & m_1 \\ a_2 & x_2 & m_2 \end{vmatrix},$$

and

$$F_{MYC} = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ m_1 & y_1 & c_1 \\ m_2 & y_2 & c_2 \end{vmatrix}.$$

The sides composing the equal angles are $d(M,A) = (a_1 - m_1)$, $d(A,X) = (x_1 - a_1)$, $d(Y,C) = (c_1 - y_1)$, and $d(C,M) = (m_1 - c_1)$. For the directed angles α and α' we have

$$\alpha = \angle(\overrightarrow{MA}, \overrightarrow{AX}) = \frac{x_2 - a_2}{x_1 - a_1} - \frac{a_2 - m_2}{a_1 - m_1}$$

$$\alpha' = \angle(\overrightarrow{YC}, \overrightarrow{CM}) = \frac{m_2 - c_2}{m_1 - c_1} - \frac{c_2 - y_2}{c_1 - y_1}$$

$$\begin{aligned} \alpha = \alpha' &\Rightarrow \frac{x_2 - a_2}{x_1 - a_1} - \frac{a_2 - m_2}{a_1 - m_1} = \frac{m_2 - c_2}{m_1 - c_1} - \frac{c_2 - y_2}{c_1 - y_1} \\ &\Rightarrow \frac{x_1 m_2 - x_2 m_1 - a_1 m_2 + a_2 m_1 + a_1 x_2 - a_2 x_1}{y_1 c_2 - y_2 c_1 - m_1 c_2 + m_2 c_1 + m_1 y_2 - m_2 y_1} = \\ &= \frac{a_1 x_1 - x_1 m_1 + m_1 a_1 - a_1^2}{m_1 c_1 - m_1 y_1 + c_1 y_1 - c_1^2}. \end{aligned}$$

The latter equation can be reach writing extensively equation (18). \square

Let's apply now lemma 5 on the following pairs of allowable triangles:

$\triangle MAX$ and $\triangle YCM \Rightarrow$

$$\frac{F_{MAX}}{F_{YCM}} = \frac{d(M,A) \cdot d(A,X)}{d(Y,C) \cdot d(C,M)}, \quad (19)$$

$\triangle CMY$ and $\triangle DMX \Rightarrow$

$$\frac{F_{CMY}}{F_{DMX}} = \frac{d(C,M) \cdot d(M,Y)}{d(D,M) \cdot d(M,X)}, \quad (20)$$

$\triangle XDM$ and $\triangle MBY \Rightarrow$

$$\frac{F_{XDM}}{F_{MBY}} = \frac{d(X,D) \cdot d(D,M)}{d(M,B) \cdot d(B,Y)}, \quad (21)$$

$\triangle YMB$ and $\triangle XMA \Rightarrow$

$$\frac{F_{YMB}}{F_{XMA}} = \frac{d(Y,M) \cdot d(M,B)}{d(X,M) \cdot d(M,A)}. \quad (22)$$

$$(19) \cdot (20) \cdot (21) \cdot (22) = \frac{F_{MAX}}{F_{YCM}} \cdot \frac{F_{CMY}}{F_{DMX}} \cdot \frac{F_{XDM}}{F_{MBY}} \cdot \frac{F_{YMB}}{F_{XMA}} = 1$$

$$\begin{aligned} &\Rightarrow \frac{d(A,X) \cdot d(M,Y)}{d(Y,C) \cdot d(M,X)} \cdot \frac{d(X,D) \cdot d(Y,M)}{d(B,Y) \cdot d(X,M)} = 1 \\ &\Rightarrow \frac{d(A,X) \cdot d(X,D)}{d(B,Y) \cdot d(Y,C)} = \frac{d(M,X) \cdot d(X,M)}{d(M,Y) \cdot d(Y,M)}. \quad (23) \end{aligned}$$

According lemma 3, and using the notation given in (4), we have

$$d(A,X) \cdot d(X,D) = d(P,X) \cdot d(X,Q) = |s|^2 - |x|^2, \quad (24)$$

and

$$d(B,Y) \cdot d(Y,C) = d(P,Y) \cdot d(Y,Q) = |s|^2 - |y|^2. \quad (25)$$

Inserting (24) and (25) in (23) we obtain

$$\frac{|s|^2 - |x|^2}{|s|^2 - |y|^2} = \frac{-|x|^2}{-|y|^2} \Rightarrow |x|^2 = |y|^2 \Rightarrow |x| = \pm |y|,$$

and finally, as it has been shown before,

$$|x| = |y| \Rightarrow d(X,M) = d(M,Y). \square$$

Proof 4

Let k be a parabolic circle in I_2 , and let M be the midpoint of the chord \overrightarrow{PQ} of k . Let's choose the coordinate system as shown (in the affine model) in figure 10, i.e, the tangent on the circle k parallel to the chord \overrightarrow{PQ} as the x -axis, and the isotropic straight line through M as the y -axis.

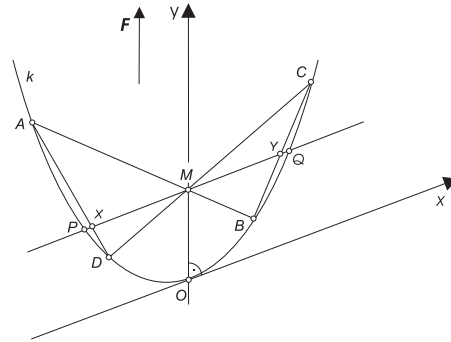


Fig. 10

Let $A(a_1, Ra_1^2)$, $B(b_1, Rb_1^2)$, $A \neq B \Rightarrow a_1 \neq b_1$, and $C(c_1, Rc_1^2)$, $D(d_1, Rd_1^2)$, $C \neq D \Rightarrow c_1 \neq d_1$, be four points on the parabolic circle k . Choosing $M(0, m)$, for the chord \overrightarrow{PQ} we have $\overrightarrow{PQ} \equiv y = m$. Besides, for \overrightarrow{AB} being a chord through M , the following relations are obtained:

M, A, B collinear points \Leftrightarrow

$$\begin{vmatrix} 0 & m & 1 \\ a_1 & Ra_1^2 & 1 \\ b_1 & Rb_1^2 & 1 \end{vmatrix} = 0 \Leftrightarrow a_1 b_1 = -\frac{m}{R}. \quad (26)$$

Analogously, for \overrightarrow{CD} being a chord through M , we have:

M, C, D collinear points \Leftrightarrow

$$\begin{vmatrix} 0 & m & 1 \\ c_1 & Rc_1^2 & 1 \\ d_1 & Rd_1^2 & 1 \end{vmatrix} = 0 \Leftrightarrow c_1 d_1 = -\frac{m}{R}. \quad (27)$$

Let's denote further on $X(x_1, m)$ and $Y(y_1, m)$.

One obtains the following:

A, D, X collinear points \Leftrightarrow

$$\begin{vmatrix} x_1 & m & 1 \\ a_1 & Ra_1^2 & 1 \\ d_1 & Rd_1^2 & 1 \end{vmatrix} = 0 \Leftrightarrow Rx_1(a_1 + d_1) = m + Ra_1 d_1. \quad (28)$$

C, B, Y collinear points \Leftrightarrow

$$\begin{vmatrix} y_1 & m & 1 \\ b_1 & Rb_1^2 & 1 \\ c_1 & Rc_1^2 & 1 \end{vmatrix} = 0 \Leftrightarrow Ry_1(b_1 + c_1) = m + Rb_1 c_1. \quad (29)$$

Finally, using (26), (27), (28), and (29) it follows:

$$\begin{aligned} x_1 + y_1 &= \frac{m + Ra_1d_1}{R(a_1 + d_1)} + \frac{m + Rb_1c_1}{R(b_1 + c_1)} = \\ &= \frac{(m + Ra_1d_1)(b_1 + c_1) + (m + Rb_1c_1)(a_1 + d_1)}{R(a_1 + d_1)(b_1 + c_1)} = \\ &= \frac{R(a_1b_1d_1 + a_1c_1d_1 + a_1b_1c_1 + b_1c_1d_1) + m(a_1 + b_1 + c_1 + d_1)}{R(a_1 + d_1)(b_1 + c_1)} = \\ &= \frac{R(-\frac{m}{R}d_1 - \frac{m}{R}a_1 - \frac{m}{R}c_1 - \frac{m}{R}b_1) + m(a_1 + b_1 + c_1 + d_1)}{R(a_1 + d_1)(b_1 + c_1)} = 0 \end{aligned}$$

$\Rightarrow M$ is the midpoint of \overrightarrow{XY} . \square

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