Review
Received: 27-01-2009
Accepted: 01-04-2009

Origins of Classic Trigonometric Networks

Miljenko SOLARIĆ, Nikola SOLARIĆ

University of Zagreb, Faculty of Geodesy, Zagreb, Croatia
e-mail: miljenko.solaric@geof.hr, nikola.solaric@geof.hr

Abstract: *The paper is a historical overview of beginnings of establishing trigonometric networks to determine the Earth's size and produce accurate maps. First, the book "Five Books About All Kinds of Triangles" from 1464 by Johannes Regiomontanus about existing trigonometry knowledge is described. This is followed by mention of Petrus Apianus and his book "Cosmographicus Liber" in which Reigner Gemma Frisius published his work in a supplement to a subsequent edition in 1533. In it, he proposed using trigonometric networks (triangulation) as an accurate method for determining position of points on the Earth. It is little-known that the famous astronomer Tycho Brahe established a trigonometric network in Öresund, intending for it to become the foundation for producing a map of the entire Danish Empire. Unfortunately, he surveyed all required values in the trigonometric network, but did not calculate the trigonometric points' coordinates. Snellius (Willebrord Snel van Royen) was the first to establish a trigonometric network and calculate point coordinates and determined the Earth's size with the greatest accuracy of the time.*

Key words*: azimuth, astronomic measurements, latitude and longitude, instruments, meridian, distance, trigonometry, trigonometric network, trigonometric points, trigonometric tables*

1 Introduction

At the beginning of antiquity, “maps” of a certain area were drawn by eye and freeform, which is nowadays referred to as *crocquis* (French). However, *Eratosthenes of Cyrene* (276 – 194 B.C.) marked latitudes and longitudes of places on his maps in Alexandria. Nevertheless, people had difficulty surveying the longitude of a particular point (place) on the Earth astronomically. This was the case until 1735, when *John Harrison* successfully constructed an accurate portable clock (chronometer).

At the end of the middle age and at the beginning of the new age, there was a need to produce more accurate maps, and astronomic measurements were not accurate enough for their production. Thus a new approach had to be found in order to produce accurate geographic maps with trigonometric networks. Trigonometry knowledge needed to be advanced in order to be able to draw accurate maps with trigonometric networks.

Origins of trigonometry can be found in work by famous Alexandrian astronomer and scientist *Aristarchus of Samos* (310 – 230 B.C.). After being advanced in Indian and Arabic mathematics, trigonometry was brought to Europe in the 15th century (Gusić 1995, p. 239).

The German mathematician and astronomer *Johann Müller Königsberger*, noted by his alias *Johannes Regiomontanus* (1436 – 1476) (Fig.1) (URL 1) wrote *Five*

Pregledni rad
Primljeno: 27-01-2009.
Prihvaćeno: 01-04-2009.

Prvi začeci klasičnih trigonometrijskih mreža

Miljenko SOLARIĆ, Nikola SOLARIĆ

Geodetski fakultet Sveučilišta u Zagrebu, Zagreb

e-pošta: miljenko.solaric@geof.hr, nikola.solaric@geof.hr

Sažetak: *U članku je dan povijesni pregled od prvih početaka uspostava trigonometrijskih mreža za određivanje dimenzija Zemlje i izradu točnih karata. Na početku je prikaz knjige Johannesa Regiomontana "Pet knjiga o trokutima svih vrsta" iz 1464. godine u kojoj je prenio sve dotadašnje znanje iz trigonometrije. Zatim se spominje Petrus Apianus i njegova knjiga "Cosmographicus Liber", u kojoj je poslije, u dodatku u jednom kasnijem izdanju, 1533. godine Reigner Gemma Frisius objavio svoj rad. U njemu je dao svoj prijedlog uporabe trigonometrijskih mreža (triangulacije) kao točne metode određivanja položaja točaka na Zemlji. Malo je poznato da je slavni astronom Tycho Brahe uspostavio trigonometrijsku mrežu u Öresundu s namjerom da to postane baza za izradu karte cijeloga Danskoga kraljevstva. On je izmjerio sve potrebne veličine u trigonometrijskoj mreži, ali nažalost nije izračunao koordinate položaja trigonometrijskih točaka. Snellius (Willebrord Snel van Royen) prvi je uspostavio trigonometrijsku mrežu i izračunao koordinate točaka te odredio dimenzije Zemlje tada s najvećom točnosti.*

Ključne riječi: *azimut, astronomska mjerenja, geografska širina i dužina, instrumenti, meridijan, udaljenost, trigonometrija, trigonometrijske mreže, trigonometrijske točke, trigonometrijske tablice*

1. Uvod

Na početku starog vijeka "karte" nekog predjela crtane su odoka, prostoručno, a danas se taj tip karte naziva *kroki* (prema francuskom *croquis*). Međutim, već u starom vijeku u Aleksandriji je *Eratosten iz Kirene* (276–194. pr. Kr.) na svojim kartama označio geografske širine i dužine mjesta, iako je u to doba bio problem izmjeriti geografske dužine neke točke (mjesta) na Zemlji. Tako je bilo sve do 1735., kada je *John Harrison* uspješno konstruirao točni prijenosni sat (kronometar).

Krajem srednjeg i početkom novoga vijeka pojavila se potreba za izradom sve točnijih karata, a za njihovu izradu astronomska mjerenja nisu bila dovoljno točna. Zato se morao pronaći novi pristup za izradu geografskih karata s pomoću trigonometrijskih mreža. Da bi se moglo prijeći na iscrtavanje karata s pomoću trigonometrijskih mreža, trebalo je unaprijediti znanje iz trigonometrije.

Prvi začeci trigonometrije javljaju se u radovima slavnog aleksandrijskog astronoma znanstvenika *Aristarha sa Samosa* (310–230. pr. Kr.). Nakon njezina unapređivanja u indijskoj i arapskoj matematici, trigonometrija je u 15. stoljeću prenesena u Europu (Gusić 1995, str. 239).

Njemački matematičar i astronom *Johann Müller Königsberger*, poznatiji pod pseudonimom *Johannes Regiomontan* (latinizirano *Regiomontanus*) (1436–1476.,

Fig. 1. (left) German mathematician and astronomer Johannes Regiomontanus (1436- 1476), woodcut (URL 1)
Fig. 2. (right) German Petrus Apianus (1495-1552) (URL 2)

Slika 1. (lijevo) Njemački matematičar i astronom Johannes Regiomontan (1436-1476), drvorez (URL 1)
Slika 2. (desno) Nijemac Petrus Apianus (1495-1552, URL 2)

Books About All Kinds of Triangles (De triangulis omnimodis libri quinque) in 1464, which was published in 1533. The first book contained all existing trigonometry knowledge, which affected the development of that kind of science in the whole Europe. Thus the book contributed the development of technical knowledge. Trigonometric science has been independent of astronomy since then. From 1468 to 1471, Regiomontanus also composed astronomic tables, which he called ephemeris, and they were *the first printed tables* in Nürnberg titled *Ephemerides* (Šentija 1981, vol. 7, p. 42) in 1474.

Regiomontanus made a lot of effort to calculate trigonometric tables, which were published in 1490, i.e. after he passed away. He composed sine tables with a 1' (minute) step and tan with a larger step.

Going by Arab model, Regiomontanus produced several measuring instruments, such as planispheric astrolabe and torquetum or turquetom. In addition, he also produced a sun dial with a magnetic needle and made sun dials applicable for all latitudes and could thus also be used on voyages.

He also travelled to Constantinople (Istanbul), where he found numerous old Greek manuscripts, which he brought to Italy and saved from oblivion.

In sum, it can be said Regiomontanus made a great contribution to the development of science in the otherwise dark middle age.

The German *Petrus Apianus*, also known as Peter Apian, Peter Bennewitz and Peter Bienewitz (1495–1552) (Fig. 2, URL 2 and URL 3) was a famous expert of astronomy, mathematics, geography, cartography, surveying and sun dials.

The book *Cosmographicus Liber*, which he published in 1524, was very appreciated in astronomy and navigation. This is supported by the fact it was published at least 30 times in 14 languages and remained popular until the end of the 16th century. Subsequently, Reigner Gemma Frisius published his work on trigonometric network in it, which made the book even more valuable.

He published sine tables and astronomic tables in 1534. He also published *Astronomicum Caesareum*, which was dedicated to emperor Charles V, in 1540.

2 Origins of Trigonometric Networks

Three greats of European science contributed to the establishment of trigonometric networks. They are: Reigner Gemma Frisius, Tycho Brahe and Willebrord Snel van Royen (Snellius).

2.1 Reigner Gemma Frisius (1508-1555)

The first proposal of using a trigonometric network (triangulation) as an accurate method of determining the position of points on the Earth was made by Flemish

Fig. 3. Flemish mathematician, cartographer, physician and astronomer Reigner Gemma Frisius (1508-1555), woodcut by E. de Boulonois, about 1682 (URL 6)

Slika 3. Flamanski matematičar, kartograf, liječnik i astronom Reigner Gemma Frisius (1508-1555), drvorez E. de Boulonoisa oko 1682. godine (URL 6)

sl.1, URL 1), napisao je 1464. godine *Pet knjiga o trokutima svih vrsta* (*De triangulis omnimodis libri quinque*), koje su tiskane 1533. godine. U prvoj knjizi objavio je sva dotadanja znanja iz trigonometrije koja su utjecala na razvoj te vrste znanosti u cijeloj Europi. Na taj način to je djelo dalo doprinos razvoju tehničkih znanja. Od tada je trigonometrija postala neovisna o astronomiji. On je također od 1468. do 1471. sastavljao astronomske tablice, koje je nazvao efemeridama. *Prve tiskane tablice* objavljene su u Nürnbergu 1474. godine pod naslovom *Efemerides* (Šentija 1981, sv. 7, str. 42).

Regiomontan je mnogo truda uložio u računanje trigonometrijskih tablica, koje su u tiskanom obliku objavljene 1490. godine, tj. nakon njegove smrti. Tablice sinusa sastavio je s korakom od 1' (kutne minute), ali i tangensa s nešto većim korakom.

Regiomontan je po arapskom uzoru izradio više mjernih instrumenata među kojima planisferni astrolab i torket (torquetum ili turquetom). Osim toga, izradio je i sunčani sat u koji je ugradio magnetsku iglu i tako proširio uporabljivost sunčane ure na svim geografskim širinama, pa se ona mogla koristiti i na putovanjima.

Bio je na putovanju u Konstantinopolu (današnjem Istanbulu), gdje je pronašao velik broj starih grčkih rukopisa, te ih ponio sa sobom u Italiju i na taj način spasio od zaborava.

Može se reći da je Regiomontan dao velik doprinos razvoju znanosti u inače mračnom srednjem vijeku.

Nijemac *Petrus Apianus* poznat kao Peter Apian, Peter Bennewitz i Peter Bienewitz (1495–1552) (sl. 2, URL 2 i URL 3), bio je poznati stručnjak za astronomiju, matematiku, geografiju, kartografiju, mjerništvo i sunčane satove.

Knjiga *Cosmographicus Liber*, koju je objavio 1524. godine, bila je vrlo cijenjena u astronomiji i navigaciji. To se vidi i po tome što je bila ponovno tiskana najmanje 30 puta, i to na 14 jezika, i ostala je popularna sve do kraja 16. stoljeća. Reigner Gemma Frisius je poslije u njoj tiskao kao dodatak svoj rad o trigonometrijskoj mreži, što je toj knjizi dalo dodatnu vrijednost.

Tablice sinusa, kao i astronomske tablice objavio je 1534. godine. Osim toga objavio je 1540. godine rad *Astronomicum Caesareum*.

2. Začeci trigonometrijskih mreža

Tri velikana europske znanosti dali su svoje doprinose prvim uspostavljanjima trigonometrijskih mreža. To su Reigner Gemma Frisius, Tycho Brahe i Willebrord Snel van Royen (Snellius).

2.1. Reigner Gemma Frisius (1508-1555)

Prvi prijedlog uporabe trigonometrijske mreže (triangulacije) kao točne metode određivanja položaja točaka na Zemlji opisao je flamanski matematičar, kartograf, liječnik i astronom Reigner Gemma Frisius (1508–1555, sl.3, URL 4 i URL 5). To je učinio u svom radu *Libellus de locarum describendorum ratione*, koji je objavio kao dodatak u Apianovoj *Cosmographia Liber* 1533. godine, a bio je tiskan u još 28 pretisaka te knjige.

Frisius je u 19. poglavlju prvi put opisao kako se može izračunati geografska dužina mjesta uporabom sata, određujući razliku lokalnog mjesnog vremena i apsolutnog vremena. Osim toga napisao je da se mjerenjem astrolabom i satom može odrediti geografska širina i dužina, ali da za to treba imati satove čija točnost ne ovisi o stanju atmosfere. (Mehanički sat izumljen je 1509. godine (vidi Regiomontan). Prije toga bili su uglavnom pješčani satovi.)

Njegova knjiga *Lagana metoda praktične aritmetike* (*Arithmeticae practicae methodus facilis*) iz 1540. godine bila je najpopularniji sveučilišni udžbenik matematike sa šezdesetak izdanja sve do 1600. godine (Šentija 1977, sv. 3, str. 125).

Zanimljivo je napomenuti da je slavni nizozemski kartograf Gerardus Mercator (1512–1594) bio njegov student 1534. godine, te da su poslije surađivali.

U opisu trigonometrijske mreže najprije je dao definiciju kuta koji se danas naziva *magnetski azimut*, a poslije će se vidjeti da to nije definirao na takav način na koji to danas činimo. Rekao je da je za mjerenje čitavog područja sa svima gradovima i mjestima potreban instrument mathematician, cartographer, physician and astronomer Reigner Gemma Frisius (1508 – 1555) (Fig.3, URL 4 and URL 5). This was done in *Libellus de locarum describendorum ratione*, which was published as a supplement to Apian's *Cosmographia Liber* in 1533, and was printed in another 28 editions of the book.

In the 19th chapter, Frisius was also the first to describe how to calculate the longitude of a place by using a clock, determining the difference between the local time and the absolute time. In addition, he wrote that by measuring with an astrolabe and a clock, one can determine the latitude and the longitude, but the clocks' accuracy has to be independent of the atmospheric state (the mechanical clock was invented in 1509 (see Regiomontanus), prior to which hourglasses were used mostly).

His book *Easy Method of Practical Arithmetics* (*Arithmeticae practicae methodus facilis*) from 1540 was the most popular university mathematics textbook, with about 60 editions until 1600 (Šentija 1977, vol. 3, p. 125).

Interestingly, the great Dutch cartographer Gerardus Mercator (1512–1594) was his student in 1534, and they also collaborated.

Describing a trigonometric network, he first defined the angle we refer to as the magnetic azimuth, and it will be seen it was not done as in the present practice. He stated that in order to measure an entire area with all cities and places, one requires an instrument which contains a circle divided into four quadrants, each divided into 90°. He noted that one end of the sight direction has to go through the centre of the circle and that the other end of the sighting equipment can be moved on the margin of the circle. According to Frisius, the circle needs to be horizontal and the line connecting the centre of the circle and the zero on the circle division needs to be directed toward the northern magnetic pole, which can be achieved with a "maritime compass".

His method is described in Fig. 4, where he considered that:

magnetic azimuths were surveyed at a tower in ANTWERP:

- to Gent 80° west of the north direction,
- to Lier 30° south of the east direction,
- to Merchelen 8° west of the south direction,
- to Leuven 4° east of the south direction,
- to Brussels 25° west of the south direction,
- to Middelburg 30° north of the west direction
- to Bergen op Zoom 20° west of the north direction.

In Brussels, he considered the surveyed magnetic azimuths:

- to Leuven almost 14° south of the east direction,
- to Mechelen and Lier on one straight line 47° north of the east direction,
- to Gent 29° west of the north direction,
- to Middelburg 33° west of the north direction
- to Bergen op Zoom 9° east of the north direction.

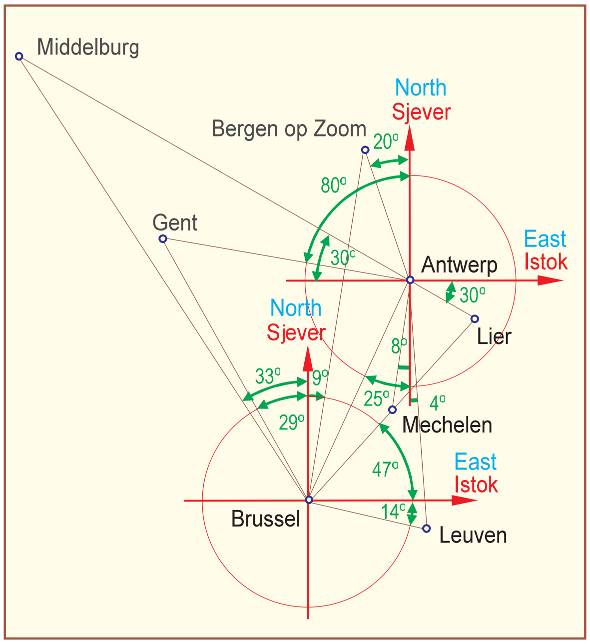


Fig. 4. Representation of an idea of a trigonometric network according to Reigner Gemma Frisius from 1533 (according to URL 6)

Slika 4. Prikaz ideje trigonometrijske mreže prema Reigneru Gemmi Frisiusu iz 1533. godine (prema URL 6)

He also noted the illustration was the only reason he imagined azimuths from Brussels toward Bergen op Zoom and Middelburg, because in reality those two places can not be seen from Brussels.

Finally, Frisius stated the spherical Earth can not be represented on a map plane without distortions. However, the error will not be large for 50×50 or 100×100 mile area, but the spherical shape of the Earth would have to taken into account for measuring Europe.

There are questions about Gemma applying his theory about trigonometric network in practice. Some state his health was a hindrance. Nevertheless, some believe Gemma measured angles in a trigonometric network with a cross-staff and worked on a map of Lorraine which was unfortunately lost.

During the middle ages, angles were measured with a *cross-staff* or a *Jacob's staff* (Latin *baculus Jacob*, French *Crosse de Saint Jacques*, German *Jacobsstab*), and Gemma Frisius and Tycho Brahe referred to it as *radius geometricus.* It was written about Persian mathematician Avicenna in the 11th century, and this measuring concept arrived in Spain in 1342, when Levi ben Gerson (1288-1344) worked in Spain (south France according to other sources). The transversal staff of a cross-staff can slide on the sliding staff (Fig. 5), so the size of the α angle depends on the distance from the end of the sliding staff. In order to measure elevation angles of the

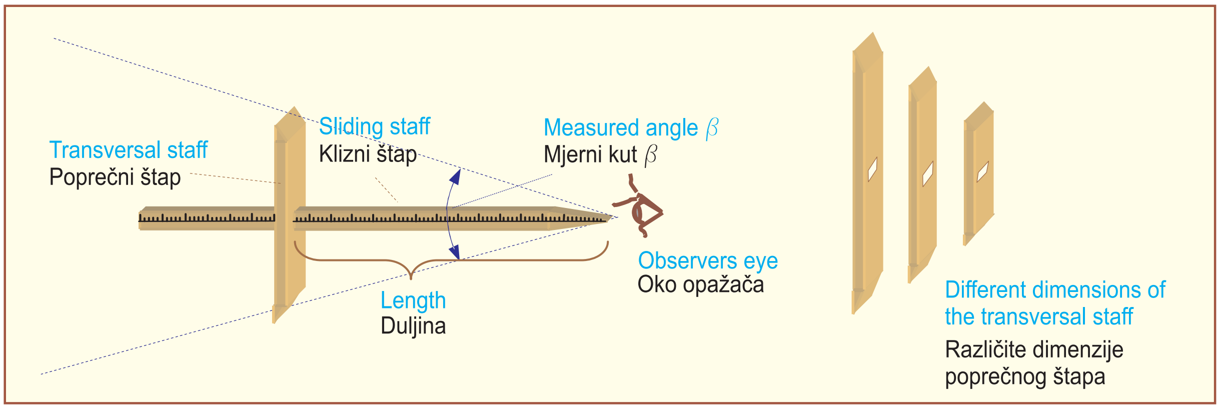


Fig. 5. (up) Representation of cross-staff (Jacob's staff) use

Slika 5. (gore) Prikaz upotrebe križnoga (Jakobova) štapa

Fig. 6. (right) Use of cross-staff (Jacob's staff) for measuring angles between celestial bodies, vertical, as well as horizontal angles on the Earth (Korošec 1978, from Apian's book about instruments)

Slika 6. (desno) Uporaba križnoga (Jakobova) štapa za mjerenje kutova između nebeskih tijela, vertikalnih ili horizontalnih kutova na Zemljinoj površini (Korošec 1978, iz Apianove knjige o instrumentima)

koji sadrži krug podijeljen na četiri kvadranta i da svaki kvadrant mora biti podijeljen na 90° (kutnih stupnjeva). Napomenuo je da središtem kruga mora prolaziti jedan kraj pravca dogledanja i da se drugi kraj tog pribora za viziranje može pomicati po obodu kruga. Po njemu taj krug mora biti horizontalan, a linija koja spaja središte kruga i nulu na podjeli kruga mora biti usmjerena prema sjevernom magnetskom polu, što se može postići ''pomorskim kompasom''.

Opis njegove metode prikazan je na sl. 4, gdje su prema Frisiusu:

na tornju u Antwerpenu izmjereni magnetski azimuti:

- na Gent 80° zapadno od pravca sjevera
- na Lier 30° južno od pravca prema istoku
- na Mechelen 8° zapadno od pravca prema jugu
- na Leuven 4° istočno od pravca prema jugu
- na Bruxelles 25° zapadno od pravca prema jugu
- na Middelburg 30° sjeverno od pravca zapada i
- na Bergen op Zoom 20° zapadno od pravca prema sjeveru.

U Bruxellesu je uzeo da ima izmjerene magnetske azimute:

- na Leuven gotovo 14° južno od pravca prema istoku
- na Mechelen i Lier na jednom pravcu 47° sjeverno od pravca prema istoku,
- na Gent 29° zapadno od pravca prema sjeveru,
- na Middelburg 33° zapadno od pravca prema sjeveru i
- na Bergen op Zoom 9° istočno od pravca prema sjeveru.

Pritom je napomenuo da je samo zbog ilustracije zamislio azimute iz Bruxellesa prema Bergen op Zoomu i Middelburgu, jer se ta dva mjesta u stvarnosti ne mogu vidjeti iz Bruxellesa.

Na kraju rada naveo je da se sferna Zemlja ne može prikazati u ravnini karte bez deformacija. Međutim, za područje od 50 milja × 50 milja ili 100 milja ×100 milja ta pogreška neće biti velika, ali kad bi se mjerila Europa, morao bi se uzeti u obzir Zemljin sferni oblik.

Fig. 7. Great Danish astronomer Tycho Brahe (1508-1555, URL 7)

Slika 7. Slavni danski astronom Tycho Brahe (1508-1555, URL 7)

Fig. 8. King of Denmark and Norway Frederick II, sponsor of Tycho Brahe (URL 12)

Slika 8. Kralj Danske i Norveške Frederick II., sponzor Tyche Brahea (URL 12)

North Star, one had to move the transversal staff until the horizon could be seen under its lower part and the North Star could be seen immediately above the upper part of the transversal staff. Since 1650, cross-staves have had the possibility to set 4 values of transversal staves, i.e. for 90°, 60°, 30° and 10°. The cross-staff was accepted by navigators for astronomic measurements, as well as surveyors for determining tower heights (Fig. 6), hence vertical angles, but also horizontal, i.e. generally angles between various objects in nature.

It can be pointed out that Tycho Brahe was in contact with Gemma's family. Namely, Brahe had the first instruments for measuring angles, *radius astronomicus* (cross-staff) and *annulus astronomicus* (astronomic ring), produced by Walter Arscenius, Gemma Frisius's grandson. It is thus assumed Brahe knew about Gemma Frisius's work.

2.2 Tycho Brahe (1546-1601)

Tycho Brahe (original Danish name Tyge Ottesen Brahe de Knutstorp, 1546–1601, Fig. 7) was a great Danish astronomer, but also a cartographer, and arguably the originator of trigonometric measurements. He came from an old Danish noble family (URL 7). Brahe had a great financial support from the king of Denmark and Norway, Frederick II (Fig. 8) and built the Uraniborg castle (15 m x 15 m) on the small Island of Ven. He named it according to Greek mythology after Urania, the goddess of astronomy, one of nine daughters of the Greek god Zeus and Mnemozin, who were muses and protectors of science and arts. Next to Uraniborg, Brahe constructed the *Stjerneborg* (star castle) observatory (Fig. 9) in the earth to reduce the impact of wind and other vibrations on measurements. This small Island of Ven (area 7,2 km^2) is located in Öresund, between the Danish island of Sjællanda and Sweden, and which belonged to Denmark at the time, and nowadays belongs to Sweden. Brahe determined the positions of 777 fixed stars and discovered a star supernova in the Cassiopeia constellation on November 11, 1572 and thus became a famous astronomer. In astronomic measurements, he employed numerous astronomic instruments, most of which he constructed himself (URL 11). Brahe published descriptions of his instruments and results of his measurements in *De Stela Nova* in 1573. Unfortunately, his instruments were not preserved, neither was the observatory. Willem Janszoom Blaeu (1571–1638), a famous Dutch cartographer and instrument constructor, lived at Brahe's on the Island of Ven from 1594 to 1596.

Brahe introduced innovations in astronomic measurements in three directions:

- He improved the measurement precision by introducing finer sights, enlarging instruments, and he also searched for better materials for his constructions. Limbs of Brahe's instruments had marked divisions with interval of 10' (angular minutes), and he employed the so-called *transversals* in order to obtain a more accurate reading. He used plumbs to level instruments. Namely, the level was invented by French Melchisedec Thévenot in 1660 (Macarol 1977).

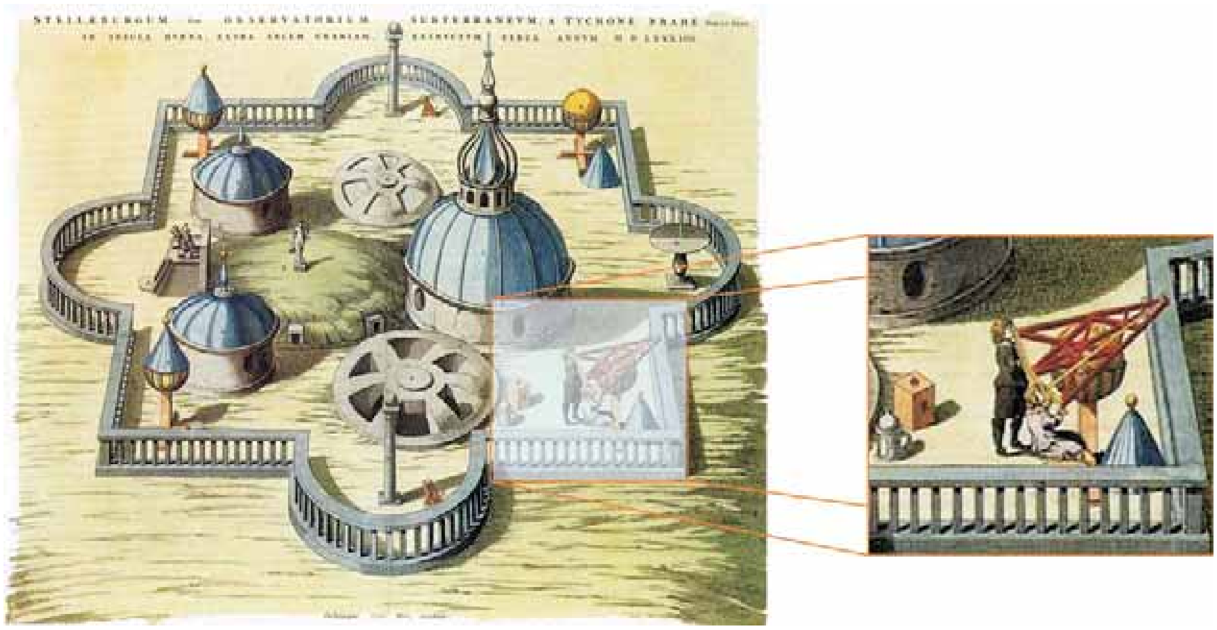


Fig. 9. Brahe's observatory Stjerneborg according to a representation in the Atlas Major by Johan Blaeu, published in Amsterdam in 1662. The enlarged part of the figure shows two observers next to an astronomic sextant (URL 9)

Slika 9. Braheova zvjezdarnica Stjerneborg prema prikazu u Atlasu Major Johana Blaeua, koji je objavljen u Amsterdamu 1662. godine. Na povećanom dijelu slike vide se dva opažača uz astronomski sekstant (URL 9)

Sumnja se da je Frisius svoju teoriju o trigonometrijskoj mreži primijenio u praksi. Neki tvrde da je bio spriječen zbog slaba zdravlja. Međutim, ima i onih koji tvrde da je Frisius mjerio kutove u trigonometrijskoj mreži s pomoću križnoga štapa i da je radio na karti Lorraine, koja je nažalost izgubljena.

Za mjerenje kutova u srednjem vijeku upotrebljavao se *križni štap* ili, kako se još zvao, *Jakobov štap* (engleski *cross-staff*, latinski *baculus Jacob*, francuski *Crosse de Saint Jacques*, njemački *Jacobsstab*), a Gemma Frisius i Tycho Brahe nazivali su ga *radius geometricus.* O njemu je pisao perzijski matematičar Avicenna u 11. stoljeću, a koncept te vrste mjerenja stigao je u Španjolsku 1342. godine, kada je Levi ben Gerson (1288-1344) ondje radio (a prema nekim drugim izvorima u južnoj Francuskoj). Poprečni štap križnoga štapa može kliziti po kliznom štapu (sl. 5), pa tako prema udaljenosti od početka kliznog štapa ovisi veličina kuta α. Pri mjerenju elevacijskoga kuta zvijezde Polarnice križnim štapom trebalo je pomicati poprečni štap tako dugo da se horizont vidi ispod njegova donjeg dijela i da se zvijezda Polarnica vidi neposredno iznad gornjega dijela poprečnoga štapa. Nakon 1650. godine križni štapovi imali su mogućnost postavljanja 4 veličine poprečnih štapova, tj. za 90°, 60°, 30° i 10°. Križni štap prihvatili su pomorci za astronomska mjerenja, ali i mjernici za određivanje visina tornjeva (sl. 6), dakle vertikalnih kutova, ali i horizontalnih, tj. općenito kutova između raznih objekata u prirodi.

Spomenimo da je Tycho Brahe bio povezan s članovima Frisiusove obitelji. Naime, Brahe je imao prve instrumente za mjerenje kutova *radius astronomicus* (križni štap) i *annulus astronomicus* (astronomski ring), koje je izradio Walter Arscenius, unuk Gemme Frisiusa. Pretpostavlja se da je Brahe morao poznavati rad Gemme Frisiusa.

2.2 Tycho Brahe (1546-1601)

Tycho Brahe (pravo mu je ime Tyge Ottesen Brahe de Knutstorp, 1546–1601, sl. 7) bio je slavni danski astronom i kartograf, a može se reći i začetnik geodetskih trigonometrijskih mjerenja. Potjecao je iz stare danske plemićke obitelji (URL 7). Brahe je uz veliku financijsku potporu kralja Danske i Norveške Fredericka II. (sl. 8) na malom otoku Hven izgradio dvorac Uraniborg (dimenzija 15 m x 15 m). Nazvao ga je prema grčkoj božici astronomije Uraniji, jednoj od devet kćeri vrhovnoga grčkog boga Zeusa i Mnemozine, koje su bile muze i zaštitnice umjetnosti i znanosti. Pokraj Uraniborga Brahe je izgradio zvjezdarnicu Stjerneborg ("zvjezdani dvorac", sl. 9), ukopanu u zemlju da bi smanjio utjecaj vjetra i ostalih vibracija na mjerenje. Otok Hven (površine 7,2 km^2) smješten je u Öresundu, između danskog otoka Sjællanda i Švedske, koji je u Braheovo doba pripadao Danskoj, a danas pripada Švedskoj. Brahe je odredio položaje 777 zvijezda stajačica i otkrio supernovu zvijezdu u zviježđu Kasiopeje 11. studenoga 1572. te je tako postao slavan. Pri astronomskim mjerenjima koristio se većim brojem astronomskih instrumenata koje je većinom sam izradio (URL 11). Opise svojih instrumenata i rezultate mjerenja objavio je u publikaciji *De Stela Nova* 1573. godine. Nažalost, ni

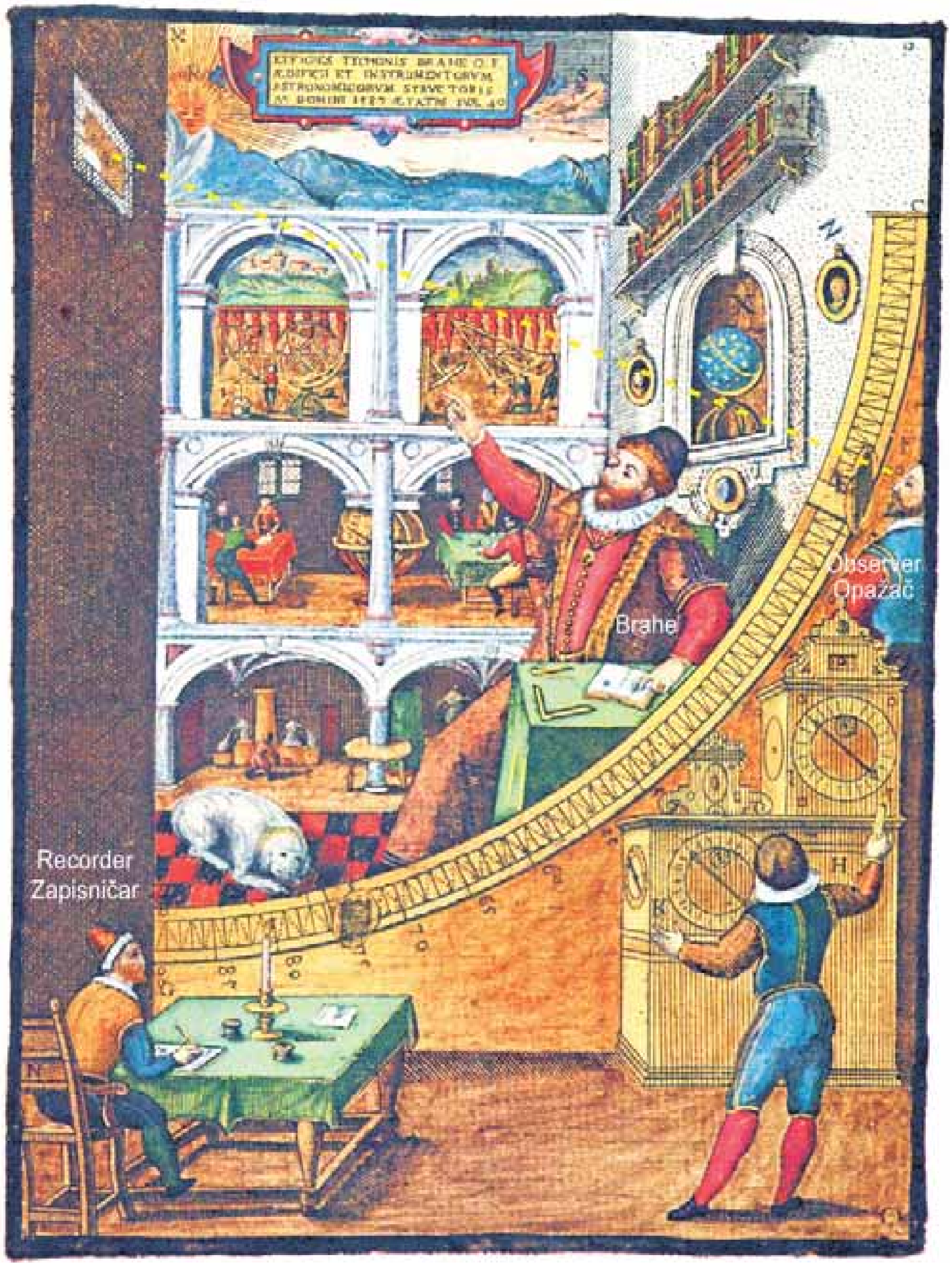


Fig. 10. Large wall quadrant by Tycho Brahe according to a picture from Blaue's Atlas Major from 1633 (URL 10)

Slika 10. Veliki zidni kvadrant Tyche Brahea prema slici iz Blaeuova Atlasa Major iz 1633. godine (URL 10)

- He used the average of several measurements to replace the existing less precise way of choosing the value which looked the best to the astronomer. Brahe had several instruments, observers and helpers. It was therefore possible for him to spot possible systematic errors for a given instrument or surveyor.
- Continuous measurement by following the position of a celestial body in its movement was a significant innovations without which the transition from a circle to a small eccentricity ellipse. Namely, the old Greek were assured planets and the Sun revolved around the Earth in circular orbits. Thus Johannes Kepler had enough observations of Mars's position on his disposal for deriving the famous Kepler's laws.

Comparing Brahe's measurements with more contemporary ones, it can be seen his measuring uncertainties were less than 4' (angular minutes). However, Brahe was assured his measuring uncertainty in determining the direction toward celestial bodies was about 5'' to 10'' (angular seconds) (URL 9). It needs to be said that such a high measurement precision could not have been obtained by any astronomer at the time, or earlier. It used to be said he drew maximum accuracy from astronomic instruments.

The main instrument in Brahe's observatory was a large wall quadrant with a 5 cubit (1,94 m) radius (Fig. 10), the limb of which had marked divisions of 10' (angular minutes). His observatory contained numerous

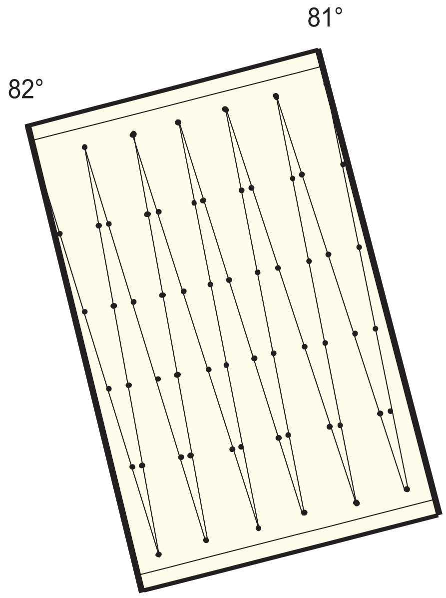


Fig. 11. Part of the detailed limb division between marked degrees on some Brahe's instruments (according to Haasbroek, 1968)

Slika 11. Dio detaljne podjele limba (transverzal) između označenih stupnjeva na nekim Braheovim instrumentima (prema Haasbroeku 1968)

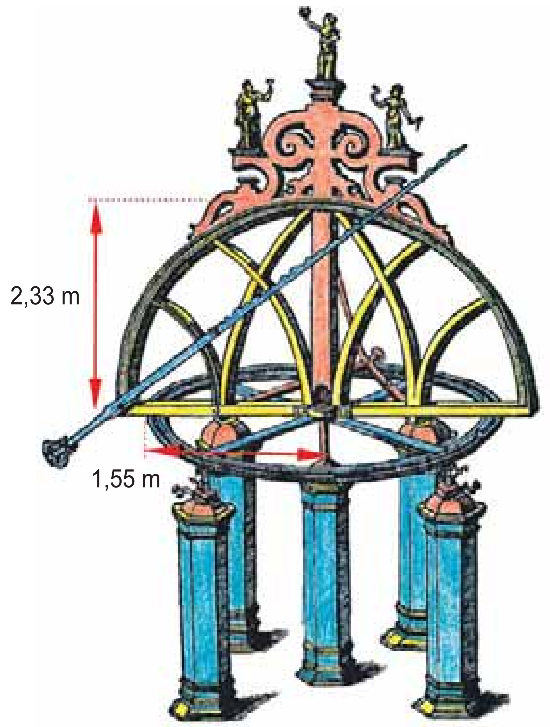


Fig. 12. Large astronomic azimuthal semicircular instrument by Tycho Brahe (according to Haasbroek, 1968). At the top, there is a statue of Urania (Greek goddess of astronomy), left and right of geometry and arithmetic. Dimensions are marked in the figure (according to Haasbroek, 1968)

Slika 12. Veliki astronomski azimutalni polukružni instrument Tyche Brahea (prema Haasbroeku 1968). Na vrhu se nalazi statua Uranije (grčke božice astronomije), lijevo i desno geometrije i aritmetike. Dimenzije su označene na slici (prema Haasbroeku 1968)

jedan njegov instrument nije sačuvan, kao ni zvjezdarnica. Kod Brahea na otoku Hvenu boravio je od 1594. do 1596. godine Willem Janszoom Blaeu (1571–1638), poznati nizozemski kartograf i izrađivač instrumenata.

Brahe je uvodio inovacije u astronomska mjerenja u tri smjera:

- Poboljšao je preciznost mjerenja uvođenjem finijih *nišana*, povećanjem dimenzija instrumenata, a tražio je i bolje materijale za svoje konstrukcije. Limbovi Braheovih instrumenata imali su označenu podjelu s intervalom od 10' (kutnih minuta), a u svrhu dobivanja točnijeg očitanja upotrebljavao je takozvane *transverzale*. Za horizontiranje instrumenata koristio se viskovima. Naime, libelu za horizontiranje pronašao je Francuz Melchisedec Thévenot 1660. godine (Macarol 1977).
- Srednjom vrijednošću iz više mjerenja zamijenio je dotadašnji manje precizan način da se od nekoliko mjerenja iste veličine odabere ona vrijednost koja je prethodnim astronomima izgledala najbolja. Brahe je imao više instrumenata i više opažača i pomagača. Tako je mogao uočiti eventualne sustavne greške kod pojedinih instrumenata ili mjerača.
- Kontinuirano mjerenje praćenjem položaja nebeskog tijela u njegovu gibanju bila je bitna inovacija bez koje bi se teško prešlo s kruga na elipsu malog ekscentriciteta. Naime, stari Grci bili su uvjereni da se planeti i Sunce kreću oko Zemlje po kružnim orbitama. Tako je Johannes Kepler za izvođenje svojih čuvenih zakona imao na raspolaganju dovoljan broj opažanih položaja Marsa.

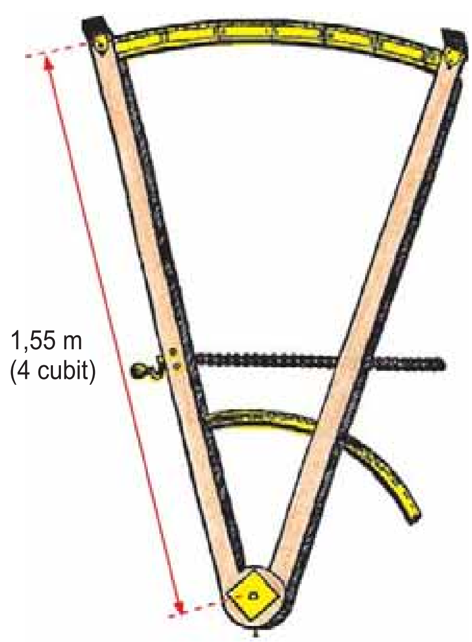


Fig. 13. Goniometer (according to Haasbroek, 1968)

Slika 13. Goniometar (prema Haasbroeku 1968)

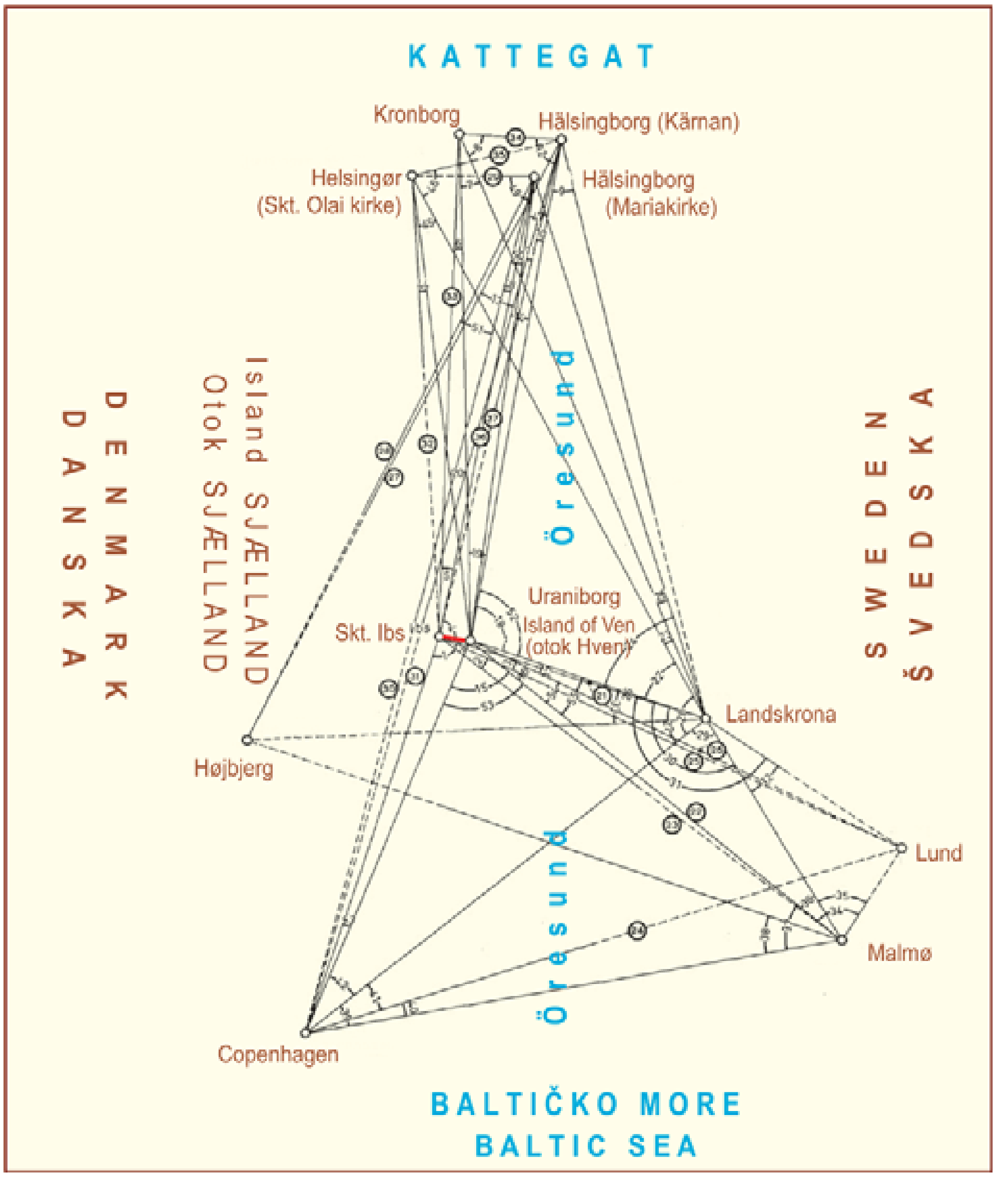


Fig. 14. Representation of Brahe's trigonometric network composed of 11 trigonometric points which is quite deformed (according to Haasbroek, 1968)

Slika 14. Prikaz Braheove trigonometrijske mreže sastavljene od 11 trigonometrijskih točaka, koji je dosta deformiran (prema Haasbroeku 1968)

instruments, for example a large astronomic azimuthal semicircle (Fig. 12), astronomic sextant (in the enlarged part of Fig. 9), etc.

When Brahe was supposed to measure angles in places where he could not employ his Observatory, he used a portable goniometer (Fig. 13), and a cross-staff (Fig. 5), especially in the beginning of his scientific work.

Brahe left the Island of Ven after arguing with the new Danish king Christian IV in 1597, and in 1599 he was invited by emperor of the Holy Roman Empire of the German Nation Rudolf II and came to Prague to work as his astrologer and astronomer. In the castle Benátky nad Jizerou, about 40 km from Prague, i.e. located northeast of Prague between Stará Boleslav and Mladá Boleslav, he built a new astronomic observatory, where he worked for a year. The emperor transferred him back to Prague, where he stayed until he passed away in 1601.

Johannes Kepler came from Graz to Prague to be Brahe's assistant. After a brief period, dying Brahe left him his astronomic measurements of planet positions, which he brought from Denmark wishing to prove the Earth is in the centre of the world. However, Kepler was a supporter of Copernicus and his heliocentric system of the world, i.e. with the Sun in the centre of the world (universe). Kepler published his *First Law of Elliptic Movement of Planets Around the Sun* and *Second Law of Equal Areal Velocities* in his work *Astronomia nova* in Prague in 1609. Kepler's third law linked the orbit time of a planet around the Sun and their distance from the Sun, and was published in *Harmonicess mundi* in 1619.

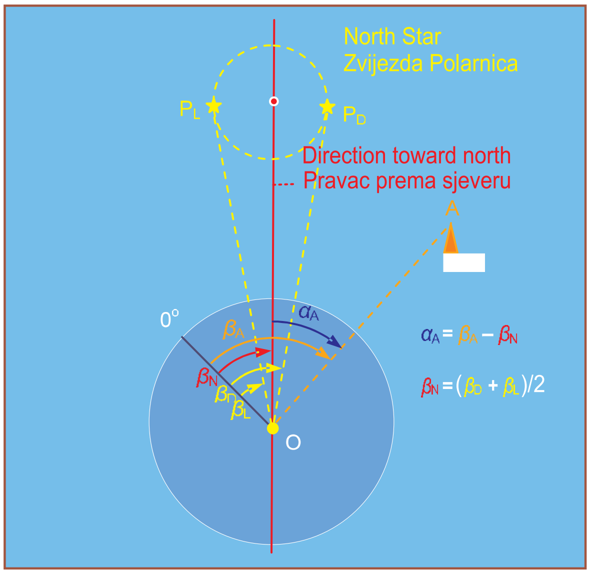


Fig. 15. Determination of the direction toward the north by measuring horizontal angles toward extreme elongations, i.e. positions in the east and west of the North Star and the azimuth toward an object A

Slika 15. Određivanje pravca prema sjeveru mjerenjem horizontalnih kutova prema ekstremnim elongacijama, tj. položajima na istoku i na zapadu zvijezde Polarnice i azimuta prema nekom objektu A

Uspoređivanjem Braheovih mjerenja sa suvremenijima može se vidjeti da su njegove mjerne nesigurnosti bile manje od 4' (kutne minute). Međutim, Brahe je bio uvjeren da njegova mjerna nesigurnost u određivanju pravca prema nebeskim tijelima iznosi oko 5'' do 10'' (kutnih sekundi) (URL 9). Treba naglasiti da tako visoku preciznost mjerenja nije mogao ostvariti ni jedan astronom iz njegova doba, a niti prije. Govorilo se da on izvlači iz astronomskih instrumenata maksimalnu točnost.

Glavni instrument u Braheovu opservatoriju bio je veliki zidni kvadrant s radijusom 5 cubitsa (1,94 m, sl. 10) na čijem su limbu bile označene podjele od 10' (kutnih minuta). Na opservatoriju je imao veći broj instrumenata, na primjer veliki astronomski azimutski polukrug (sl. 12), astronomski sekstant (na povećanom dijelu sl. 9), i dr.

Kad je Brahe trebao mjeriti kutove izvan svojeg opservatorija, koristio je transportabilni goniometar (sl.13), a i križni štap (sl. 5), posebice na početku svojega znanstvenog rada.

Brahe je napustio otok Hven 1597. godine zbog neslaganja s novim danskim kraljem Christianom IV., a 1599. se na poziv cara Svetoga Rimskog Carstva Njemačke narodnosti Rudolfa II. nastanio u Pragu kao njegov astrolog i astronom. U dvorcu Benátky nad Jizerou, smještenom oko 40 km sjeveroistočno od Praga između mjesta Stará i Mladá Boleslav, izgradio je novi astronomski opservatorij, gdje je radio jednu godinu. Zatim ga je car premjestio natrag u Prag, gdje je ostao sve do svoje smrti 1601. godine.

Johannes Kepler došao je za Braheova asistenta iz Graza u Prag. Brahe mu je na samrti ostavio sva svoja astronomska mjerenja položaja planeta koja je donio iz Danske uz želju da s pomoću njih dokaže da je Zemlja u središtu svijeta. Međutim, Kepler je bio pristaša Kopernika i njegova heliocentričkog sustava svijeta, tj. sa Suncem u središtu svijeta (svemira). Kepler je svoj *Prvi zakon eliptičkog gibanja planeta oko Sunca* i *Drugi zakon jednakih plošnih brzina* objavio u Pragu 1609. u radu *Astronomia nova*. Trećim zakonom Kepler je povezao vrijeme ophoda planeta oko Sunca i njihove udaljenosti od Sunca, što je objavio u radu *Harmonicess mundi* 1619. godine.

Malo je poznato da je slavni danski astronom Tycho Brahe od 1578. do 1579. uspostavio trigonometrijsku mrežu (sl. 14). Njegova je namjera bila da to postane baza za kartu cijeloga Danskoga kraljevstva (Haasbroek 1968, str. 7). O tom radu Tyche Brahea malo je poznato jer su u svim njegovim biografijama autori pisali najviše o astronomskim mjerenjima, a ne i o geodetskim mjerenjima za potrebe izrade karte Danske.

Dvorac Uraniborg bio je približno u središtu Braheove trigonometrijske mreže, koja mu je trebala biti baza za kartu Danske. Pritom neke kutove nije mjerio izravno, već ih je izveo iz svojih mjerenja.

Kutove u Uraniborgu nije mjerio, već ih je izračunao iz azimuta prema:

- Kopenhagenu (udaljenom 26 km),
- Malmøu (38,6 km),
- Lundu (38,5 km),
- Landskronau (9,3 km),
- Kronborgu (15,3 km),
- Helsingøru Skt. Olai kirke (15,1 km) i
- Skt. Ibs gamle kirke (1,3 km).

Brahe je mjerio duljinu polazne stranice svoje trigonometrijske mreže izraženu u tadašnjim jedinicama za duljine *passus geometricus*. Dužina passus geometricus nije bila poznata u metrima sve do 1943., kada je N. E. Norlund odredio da je 1 *passus* = 1,551 m. Brahe je izravnim mjerenjem na otoku Hvenu odredio da duljina stranice Uraniborg – Skt. Ibs iznosi 830 passus geometricus (1287,90 m).

Brahe je posvetio veliku pozornost astronomskom dijelu svoje trigonometrijske mreže, njezinoj orijentaciji i određivanju geografskih koordinata točaka.

U Uraniborgu Brahe je mogao izmjeriti azimut i vertikalni kut s pomoću fiksno postavljenog astronomskog azimutalnog instrumenta (sl. 12). Pravac prema sjeveru određivao je mjerenjem horizontalnih kutova β_D i β_L u položajima kada se zvijezda Polarnica nalazila u ekstremnim elongacijama, tj. ekstremnim položajima na istoku i na zapadu od pravca sjevera (sl. 15). Tako je mjerenjem utvrdio da se pravac prema sjeveru nalazi točno u sredini tih očitanih horizontalnih kutova, tj. po formuli:

$$\beta_N = (\beta_D + \beta_L)/2.$$

However, it is little known that the renowned Danish astronomer Tycho Brahe established the trigonometric network from 1578 to 1579 (Fig. 14). He intended for it to become the base for a map of the entire Danish Empire (Haasbroek 1968, p. 7). This work by great Tycho Brahe was little known in literature because in all his biographies, authors wrote about astronomic measurements, but not geodetic measurements needed to produce a map of Denmark.

The Uraniborg castle was approximately in the centre of Brahe's trigonometric network, which was supposed to become a base for a map of Denmark. He did not measure some angles directly, but derived them from his measurements.

Thus he did not directly measure angles in Uraniborg, but calculated them from the azimuth toward:

- Copenhagen (26 km away),
- Malmø (38.6 km),
- Lund (38.5 km),
- Landskrona (9.3 km),
- Kronborg (15.3 km),
- Helsingør Skt. Olai kirke (15.1 km) and
- Skt. Ibs gamle kirke (1.3 km).

Brahe measured the length of the starting side of his trigonometric network expressed in existing measure of length, *passus geometricus.* The length of passus geometricus in meters was not known until 1943, when N. E. Nørlund determined that 1 *passus* = 1,551 m. Brahe's direct measurements on the Island of Ven determined that the length of the Uraniborg – Skt. Ibs side equals 830 passus geometricus (1287.90 m).

As a great astronomer, Brahe paid much attention to the astronomic part of his trigonometric network, its orientation and determination of geographic coordinates.

In Uraniborg, Brahe was able to measure the azimuth and vertical angle with the fixed astronomic azimuthal instrument (Fig. 12). He determined the direction toward the north by measuring horizontal angles β_D and β_L in positions when the North Star was in extreme elongation, i.e. extreme positions in the east and west from the north direction (Fig. 15). Thus Brahe used measurements to determine the direction toward the north is exactly in the middle of the read horizontal angles, i.e. according to the formula:

$$\beta_N = (\beta_D + \beta_L)/2.$$

In this way, Brahe did not have to measure time which was a great problem at the time. He was able to finally determine the azimuth by measuring the angular distance to an object on the Earth α_A on a horizontal circle and subtracting the calculated angle where the direction α_N toward the north is, i.e. according to the formula:

$$\beta_N = (\beta_D + \beta_L)/2.$$

By inspecting this numerical analysis of Brahe's azimuth determination, it can be seen Brahe determined azimuths with the measuring uncertainty of 4' (angular minutes).

Brahe measured latitudes in several places, and according to analysis by Nørlund, Brahe determined them with the measuring uncertainty of only 2' (angular minutes).

The method of calculating differences between *latitudes* from measured azimuths on standpoint (A) and the measured distance from the standpoint to the point (B) which was sighted was already given by Gemma Frisius in *Libellus*. Thus Brahe was able to determine the difference of latitudes between two places according to the formula:

$$\lambda_B - \lambda_A = \Delta\lambda_B \approx \frac{d_{A-B} \sin \alpha_{A-B}}{R \cos \varphi_B} \rho ,$$

where:

λ_A – standpoint (A) longitude,

λ_B – place B longitude,

d_{A-B} – distance between points B and A in meters,

α_{A-B} – azimuth from standpoint A toward point B,

R – Earth radius in meters,

φ_B – latitude of place B, the azimuth toward which was measured and

ρ = 3438' (angular minutes), if $\Delta\lambda$ is expressed in minutes.

In the same way, one can determine the difference between latitudes:

$$\varphi_B - \varphi_A = \Delta\varphi_B \approx \frac{d_{A-B} \cos \alpha_{A-B}}{R} \rho ,$$

where: φ_A – latitude of the standpoint A and φ_B – latitude of the place B.

Unfortunately, Brahe never completely calculated his trigonometric network. In the paper by Haasbroek (1968), Brahe's trigonometric network was calculated and identical points in the network of the Danish Geodetic Institute were found. Thus it was possible to derive a point transformation of Brahe's trigonometric network and transform them to the coordinate system of the Danish Geodetic Institute, i.e. establish a connection between the two coordinate systems. Comparison of transformed coordinates of points from Brahe's trigonometric network with coordinates of the same points in the coordinate system of the Geodetic Institute yielded coordinate differences. These coordinate differences represent point positional errors of Brahe's trigonometric network. They were determined in relation to positions determined by the Danish Geodetic Institute and can be said to be determined practically absolutely accurate with high quality instruments and in a more contemporary way. These errors can be seen in Fig. 16. These results represent the accuracy of Brahe's trigonometric network.

The largest positional error vector equals -100 m at the Lund point, which is not as large if it is considered the

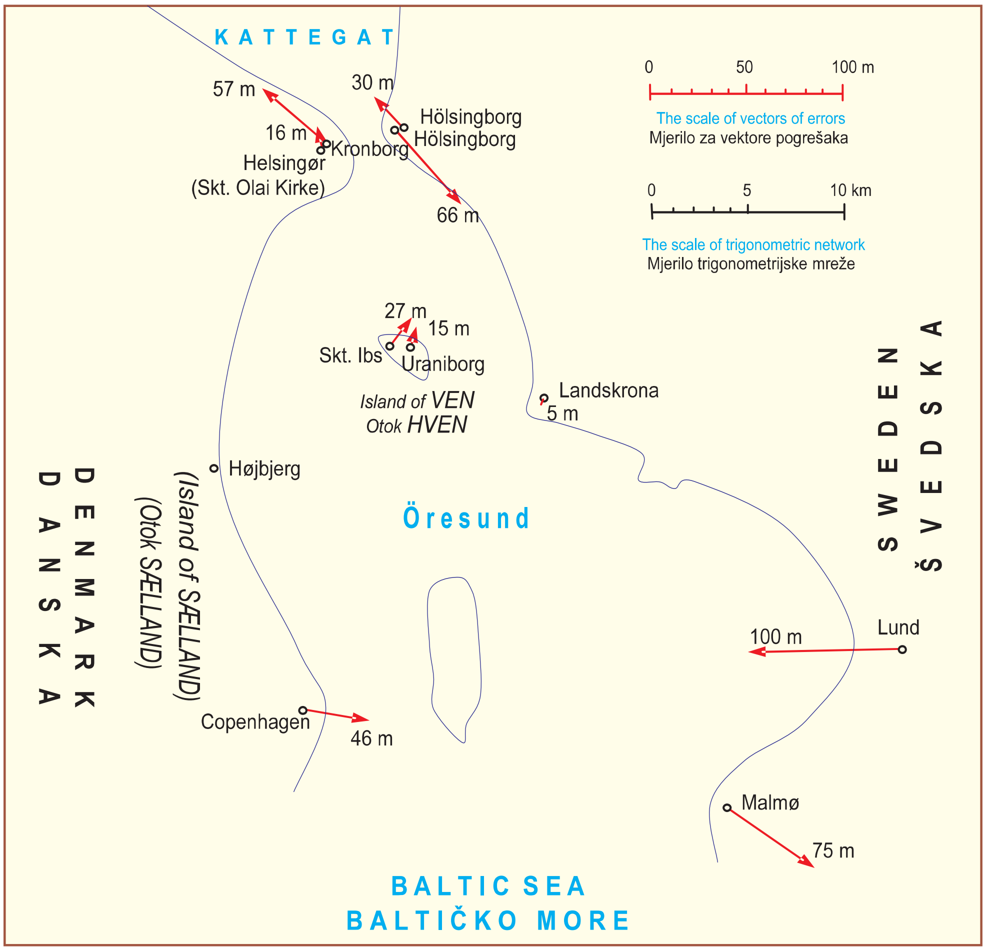


Fig. 16. Trigonometric point positional error vectors in Brahe's trigonometric network determined according to measurements by the Geodetic Institute (according to Haasbroek, 1968)

Slika 16. Vektori položajnih pogrešaka trigonometrijskih točaka u Braheovoj trigonometrijskoj mreži određenih prema mjerenjima Geodetskog instituta (prema Haasbroeku 1968)

Na taj način Brahe nije trebao mjeriti vrijeme, što je u njegovo doba bio veliki problem. Zatim je azimut mogao konačno odrediti mjerenjem kuta prema objektu na Zemlji β_A na horizontalnom krugu i oduzimanjem izračunatoga kuta gdje se nalazi pravac prema sjeveru β_N, tj. po formuli

$$\beta_N = (\beta_D + \beta_L)/2.$$

Iz provedene numeričke analize Braheovih određivanja azimuta vidljivo je da je azimute određivao s mjernom nesigurnošću od 4' (kutne minute).

Geografske širine Brahe je mjerio na više mjesta, a prema analizi Nørlunda, odredio ih je s mjernom nesigurnosti od samo 2' (kutne minute).

Metodu za računanje razlika *geografskih* dužina iz izmjerenih azimuta na stajalištu (A) i izmjerene udaljenosti od stajališta do neke točke (B) na koju je bilo vizirano, dao je već prije Gemma Frisius u svojemu radu *Libellus*. Tako je Brahe mogao odrediti razliku geografskih dužina između dvaju mjesta po formuli:

$$\lambda_B - \lambda_A = \Delta\lambda_B \approx \frac{d_{A-B} \sin\alpha_{A-B}}{R\cos\varphi_B}\rho ,$$

gdje je:

λ_A – geografska dužina mjesta stajališta (A),

λ_B – geografska dužina mjesta B,

$d_{A\text{-}B}$ – udaljenost između točaka B i A izražena u metrima,

$\alpha_{A\text{-}B}$ – azimut od stajališta A prema točki B,

R – radijus Zemlje izražen u metrima,

φ_B – geografska širina mjesta B prema kojem je mjeren azimut i

ρ = 3438' (kutnih minuta), ako je Δλ izražen u minutama.

Na isti način može se odrediti i razlika geografskih širina računskim putem između dvaju mjesta:

$$\varphi_B - \varphi_A = \Delta\varphi_B \approx \frac{d_{A-B}\cos\alpha_{A-B}}{R}\rho ,$$

gdje su: φ_A – geografska širina stajališta A i φ_B – geografska širina mjesta B.

point was determined with only five outer directions. Generally, the network contains lots of small angles with adverse effects on determination of positional coordinates, and the starting measured side of Brahe's trigonometric network is very short. After a transformation, the Uraniborg – Skt. Ibs side equalled 1277.7 m, i.e. 10.2 m shorter.

In his *Opera Omnia II*, Tycho Brahe assumed the perimeter of the Earth is 5400 German miles. Knowing that 1 German mile = 7420,6714 m, Earth's radius according to Brahe's estimate is R = 6382 km, which is very close to contemporary estimates R=6370,283 km, where Earth is approximated with a ball.

Unfortunately, Tycho Brahe never completely calculated his trigonometric network, maybe because work on the map of Denmark never was started in practice, besides the work he had done. The other assumption is that Brahe perceived such a network's accuracy would not be satisfactory, especially the four points in the north with very small angles.

Tycho Brahe's work on the trigonometric network was not perfect, but it is primarily an introduction to Snellius's trigonometric network and all trigonometric networks thereafter. He also introduced the astronomic way of calculating azimuths. This was much more accurate than Frisius's proposal of measuring magnetic azimuths.

2.3 Willebrord Snel van Royen (Snellius) (1580–1626)

Willebrord Snellius (real last name Snel van Royen, URL 13) was a great Dutch astronomer, mathematician, physicist, and arguably geodesist (Fig. 17). He became famous due to his law on refraction, nowadays known as the *Snell's Law*, which was formulated in 1621. However, Snellius was also the first to apply the trigonometric network and calculated it for determination of distance between two remote points. After measuring latitudes on both ends of an arc, he used the length to determine the meridian arc length, i.e. the Earth's radius.

His *Eratostehenes Batavus, de terrae ambitus vera quantitate* (*Dutch Eratosthenes*) (Fig. 18) was published in 1617, and it also contained descriptions and results of his measurements between cities of Alkmaar and Bergen op Zoom (URL 13). This work is considered *the beginning of scientific geodesy, the beginning of production of accurate maps and cadastral plans.*

His *Doktrina triangulorum* about trigonometry was printed in 1627, i.e. a year after he passed away.

Snellius set up the trigonometric network with 14 trigonometric points (Fig. 19) between Bergen op Zoom and Alkmaar, 130 km from each other.

The work was divided into following parts:

a) Base measurement and its extension to the starting side Leiden – Deen Haag in Snellius's trigonometric network.

b) Trigonometric network measurement.

Fig. 17. Great Dutch astronomer, mathematician, physicist, and arguably the first geodesist in the contemporary meaning of the word – Willebrord Snellius (URL 14)

Slika 17. Slavni nizozemski astronom, matematičar, fizičar, a može se reći i prvi geodet u današnjem smislu riječi – Willebrord Snellius (URL 14)

c) Trigonometric network calculation and Alkmaar - Bergen op Zoom side calculation.

d) Astronomic measurements: determination of latitude in Alkmaar, Bergen op Zoom and his house in Leiden, and determination of astronomic azimuth from his house in Leiden to Jakobstoren in Den Haag.

e) Transfer of this azimuth to the points of the Leiden Stadhuis – Den Haag Jacobstoren trigonometric network.

f) Calculation of distances between Alkmaar and Bergen op Zoom measured by the meridian passing through Alkmaar and passing from Alkmaar to the intersection with the parallel passing through Bergen op Zoom.

Snellius used the basic length unit Rijnlandse roede (Rhineland rood), which was 3,766 m long. It was divided into 12 feets and each foot into 12 inches. However, he worked with tenths and hundredths of rood.

For astronomic determination of latitude with the help of the North Star, he measured elevation angles of the North Star with a large iron quadrant of $5\frac{1}{2}$ feet (about 1.75 m), and on towers with a 1.10 m semicircle.

Snellius determined the length of the starting side of his trigonometric network Leiden – Den Haag (abbreviated L – Hg) from measured lengths of the base and

Nažalost, Brahe nije nikada u cijelosti izračunao svoju trigonometrijsku mrežu. U radu (Haasbroek 1968) izračunana je Braheova trigonometrijska mreža i nađene su identične točke u mreži Danskoga geodetskog instituta. Tako je bilo moguće izvesti transformaciju točaka Braheove trigonometrijske mreže i svesti ih u koordinatni sustav Danskoga geodetskog instituta, tj. uspostaviti vezu između jednog i drugoga koordinatnog sustava. Uspoređujući transformirane koordinate točaka iz Braheove trigonometrijske mreže s koordinatama tih istih točaka u koordinatnom sustavu Geodetskog instituta dobivene su koordinatne razlike. Te koordinatne razlike pokazuju položajne pogreške točaka Braheove trigonometrijske mreže. One su određene u odnosu na položaje kako ih je odredio Danski geodetski institut, za koje se može reći da su određene gotovo apsolutno točno s kvalitetnim instrumentarijem i na suvremeniji način. Te pogreške zorno su prikazane na sl. 16. Iz tih se rezultata može dobiti predodžba o točnosti Braheove trigonometrijske mreže.

Najveći vektor položajne pogreške iznosi –100 m na točki Lund, što nije tako velika pogreška ako se uzme u obzir da je ta točka određena samo s pet vanjskih vizura. Općenito, u mreži ima puno malih kutova koji nepovoljno djeluju na točnost određivanja položajnih koordinata, a i polazna izmjerena stranica Braheove trigonometrijske mreže je vrlo kratka. Za tu stranicu Uraniborg – Skt. Ibs poslije provedene transformacije dobilo se da ona iznosi 1277,7 m, tj. da je kraća za 10,2 m.

Tycho Brahe je u radu *Opera Omnia II* pretpostavio da je opseg Zemlje 5400 njemačkih milja. Budući da je 1 njemačka milja = 7420,6714 m, dobije se da je radijus Zemlje prema Braheovoj procjeni R = 6382 km, što je vrlo blisko suvremenim procjenama R = 6370,283 km kada se Zemlja aproksimira kuglom.

Nažalost, Tycho Brahe nije nikada u potpunosti izračunao svoju trigonometrijsku mrežu, što može biti motivirano činjenicom da rad na karti Danske nije praktički započet, osim onoga što je on učinio. Druga je pretpostavka da je Brahe vidio kako ga takva mreža neće zadovoljiti po točnosti, i to posebice za one četiri točke na sjeveru, gdje su bili vrlo mali kutovi.

Za Braheov rad na trigonometrijskoj mreži može se reći da nije bio završen, ali je on uz sve to uvod u Snelliusovu trigonometrijsku mrežu i sve trigonometrijske mreže poslije njega. On je također uveo mjerenje azimuta astronomskim načinom. Bilo je to znatno točnije nego što je predložio Frisius s mjerenjem magnetskih azimuta.

2.3 Willebrord Snel van Royen (Snellius) (1580–1626)

Willebrord Snellius (pravo prezime Snel van Royen, URL 13) slavni je nizozemski astronom, matematičar, fizičar, a može se reći i geodet (sl. 17). Postao je slavan po zakonu loma svjetlosti (refrakcije), danas poznat kao *Snelliusov zakon*, što ga je matematički formulirao 1621. godine. Međutim, postao je poznat i po tome što je prvi primijenio trigonometrijsku mrežu za određivanje duljine između dviju udaljenih točaka. Pomoću te dužine poslije mjerenja geografskih širina na početku i na kraju luka odredio je dužinu luka meridijana, a na taj način i radijus Zemlje.

Njegov rad *Eratostehenes Batavus, de terrae ambitus vera quantitate* (*Nizozemski Eratosten,* sl. 18) publiciran je 1617. godine, a u njemu je opisao i dao rezultate svojih mjerenja između gradova Alkmaara i Bergen op Zooma (URL 13). Uzima se da *taj rad predstavlja početak znanstvene geodezije, početak izrade točnih geografskih karata i katastarskih planova*.

Njegov rad o trigonometriji *Doktrina triangulorum* tiskan je 1627., godinu dana nakon njegove smrti.

Snellius je postavio trigonometrijsku mrežu s 14 trigonometrijskih točaka (sl. 19) između mjesta Bergen op Zooma i Alkmaara, udaljenih oko 130 km.

Rad je bio podijeljen na sljedeće dijelove:

a) Mjerenje baze i njezino produženje na polaznu stranicu Leiden – Deen Haag u Snelliusovoj trigonometrijskoj mreži.

b) Mjerenje trigonometrijske mreže.

c) Računanje trigonometrijske mreže i računanje stranice Alkmaar – Bergen op Zoom.

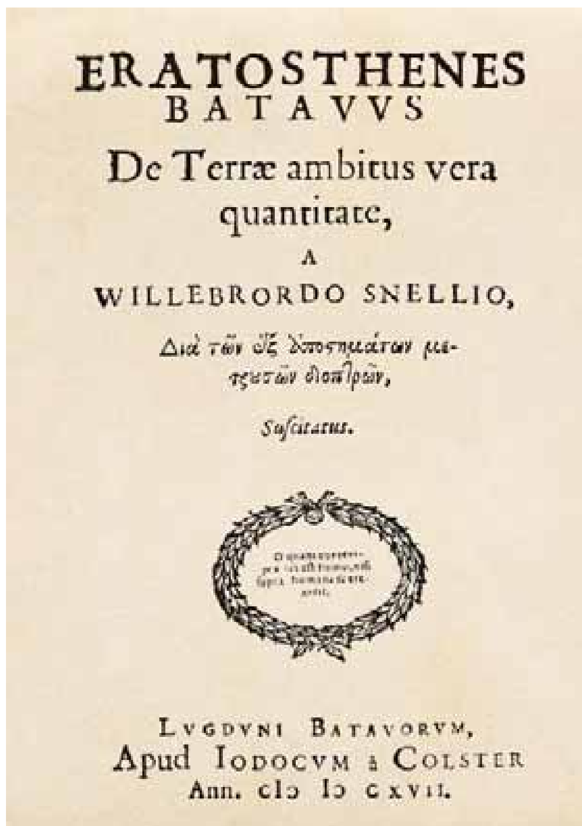


Fig. 18. Covers of Eratosthenes Batavus (Dutch Eratosthenes, Haasbroek 1968)

Slika 18. Korice rada Eratosthenes Batavus (Nizozemski Eratosten, Haasbroek 1968)

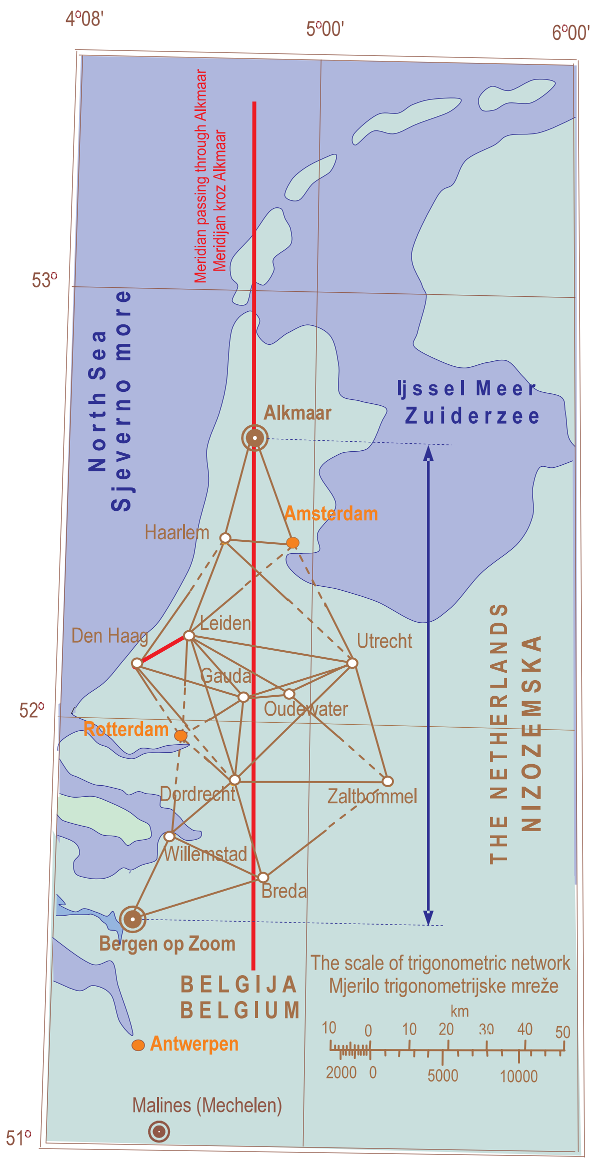


Fig. 19. Snellius's trigonometric network from Bergen op Zoom to Alkmaar was used to determine the Earth's perimeter, i.e. the Earth's radius (according to Haasbroek 1968)

Slika 19. Snelliusova trigonometrijska mreža od Bergen op Zooma do Alkmaara poslužila je za određivanje opsega Zemlje, odnosno njezina radijusa (prema Haasbroeku 1968)

angles, and on the basis of that side length and only from measured angles on trigonometric points he was able to calculate distances between particular trigonometric points. He measured several bases. The *a-e* base (Fig. 20) was measured in 1615 and 1616 with a measuring chain, angles with a quadrant (0.69 m radius) or a semicircle (0.55 m radius) produced by Willem Janszoon Blaeu (renowned cartographer, globe and instrument constructor). These instruments had no telescope. Namely, numerous historians believe telescope construction was first patented by the Dutch Hans Lippershy (1570–1619) in 1608. However, the invention needed to be adjusted for usage in geodesy. Thus in 1611, Kepler constructed telescope which could be fitted with a cross-hairs to point, but some time needed to pass before the invention could be applied in practice.

By measuring directly with a measuring chain, Snellius obtained that the length of the side t – c equals 87.05 roods (327.8 m), and he calculated the length of the side a - e from the length t – c and measured angles and obtained that a - e = 326.45 roods (1229 m) (Fig. 21). Snellius also directly measured the length of the side a – e with a measuring chain. The two results did not match, and he gave priority to the value calculated from the measured side t – c. He then calculated the length of the side L – Z_o = 1093.55 roods (4118 m) from calculated length of the side a – e and measured angles. After that, using the calculated length of the side L – Z_o and angles in points L and Z_o of the triangle L, Hg, Z_o, he calculated the length of the starting side of the trigonometric network L – Hg = 4107.87 roods (15470 m).

Snellius was not focused enough when calculating, so there were some errors. Namely, calculations were not easy to do at his time because there were no computers or even logarithmic tables. Logarithmic tables were invented by John Napier in 1614 (*Mirifici logarithmorum canonis descriptio 1614*), but they were not used right away, so Snellius could not use them either. It can be noted that Snellius did not introduce reductions for eccentric observations because they are lesser than the accuracy of his angle measurements. All his calculations were derived according to plane trigonometry rules for the same reason. Namely, the spherical excess of the largest triangle in his network Leiden – Dordrecht – Utrecht with sides of about 44 km equals only 4'' (angular seconds). His corrections were only restricted to the condition that the sum of angles in each triangle needs to be 180° (Haasbroek 1968, p. 89).

After publishing *Eratosthenes Batavus*, Snellius found some errors in his original observations when he conducted measurements with students in Leiden. He even extended his trigonometric network with some triangles to the south to Mallines (Mechelin) in Belgium. However, he never published it.

About 100 years after Snellius had passed away, his notes were obtained by Petrus van Musschenbroek (1692–1761). He checked Snellius's notes and made numerous changes to them on the basis of his personal measurements. He then recalculated the network and published it in *De magnitudine terrae,* forming a part in *Physicae experimentalis et geometricae de magnete*, which was printed in Leiden in 1729.

By measuring latitudes in Alkmaar and Bergen op Zoom, Snellius obtained the difference between their latitudes, 1° 11' 30''. By measuring with the trigonometric network, he obtained the distance between the two places on the meridian, 127.780 km. Thus he obtained that one degree on the meridian of the Earth corresponds to the distance of 107 388 m. Thus, the Earth's radius

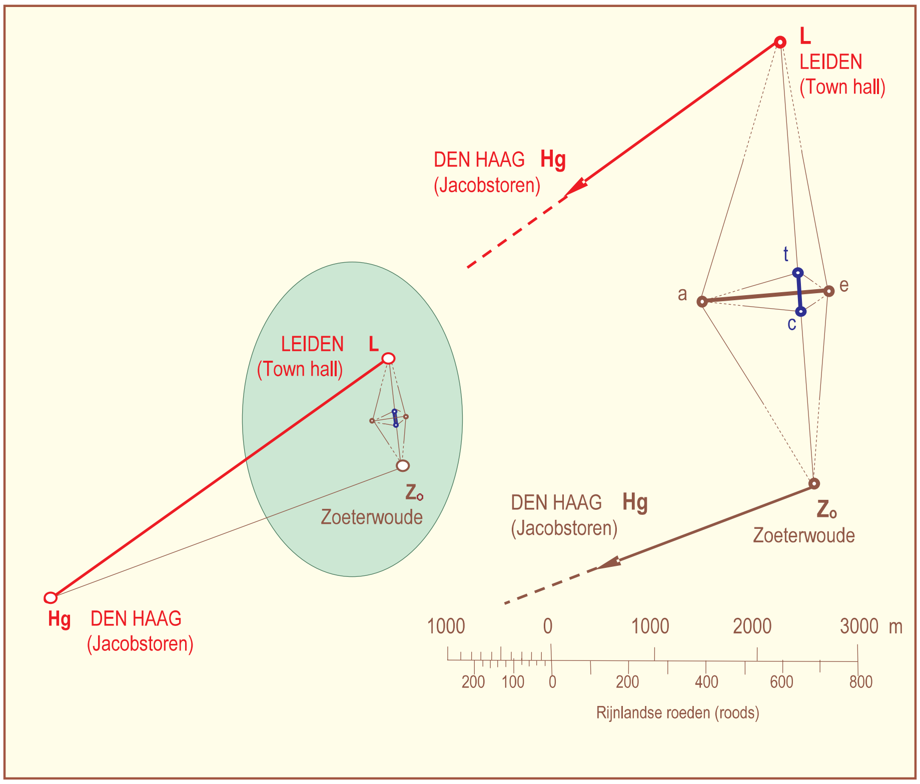


Fig. 20. Determination of the length of the exit (starting) side of Snellius's trigonometric network on the basis of measured base t - c and a - e

Slika 20. Određivanje duljine izlazne (početne) stranice Snelliusove trigonometrijske mreže na osnovi izmjerenih baza t - c i a - e

d) Astronomska mjerenja: određivanje geografske širine u Alkmaaru, Bergen op Zoomu i njegovoj kući u Leidenu, te određivanje astronomskog azimuta iz njegove kuće u Leidenu do Jakobstorena u Den Haagu.

e) Prijenos tog azimuta na točke trigonometrijske mreže Leiden Stadhuis – Den Haag Jacobstoren.

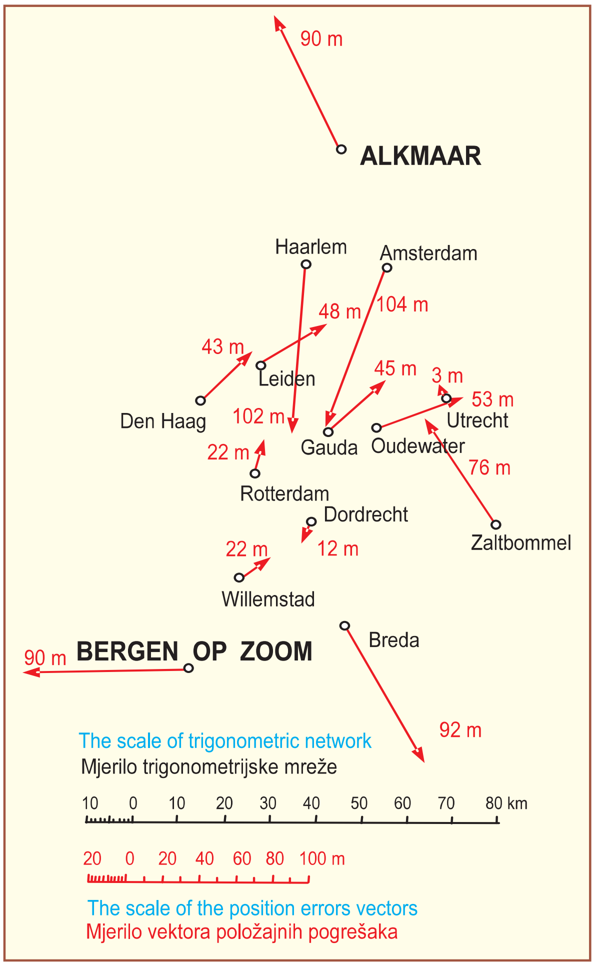
f) Računanje udaljenosti između Alkmaara i Bergen op Zooma mjerene po meridijanu koji prolazi kroz Alkmaar i ide od Alkmaara do presjeka s paralelom koja prolazi kroz Bergen op Zoom.

Snellius je koristio osnovnu jedinicu za duljinu Rijnlandse roede (Rhineland rood), koja je bila duga 3,766 m. Ona je bila podijeljena na 12 stopa, a svaka stopa na 12 palaca. Međutim, on je ipak radio s desetim i stotim dijelovima rooda.

Za astronomska određivanja geografske širine s pomoću zvijezde Polarnice mjerio je elevacijske kutove Polarnice velikim željeznim kvadrantom $5\frac{1}{2}$ stopa (oko 1,75 m), a na tornjevima polukrugom veličine 1,10 m.

Odredio je duljinu polazne stranice svoje trigonometrijske mreže Leiden – Den Haag (skraćeno L – Hg) iz izmjerene dužine baze i kutova, a na osnovi te duljine strane i samo iz mjerenih kutova na trigonometrijskim točkama mogao je izračunati udaljenosti između pojedinih trigonometrijskih točaka. Izmjerio je više baza. Bazu *a-e* (sl. 21) izmjerio je 1615. i 1616. godine mjerničkim lancem, kutove kvadrantom (radijusa 0,69 m) ili polukrugom (radijusa 0,55 m) koje je izradio Willem Janszoon Blaeu (poznati kartograf, izrađivač globusa i instrumenata). Ti instrumenti bili su bez dalekozora. Naime, mnogi povjesničari vjeruju da je konstrukciju dalekozora prvi patentirao Nizozemac Hans Lippershy (1570–1619) 1608. godine. Međutim, taj je pronalazak trebalo još prilagoditi uporabi za mjerenje u geodeziji. Tako je Kepler 1611. godine konstruirao dalekozor u koji se mogao ugraditi nitni križ za viziranje, ali i za primjenu toga pronalaska u praksi trebalo je još vremena.

Izravnim mjerenjem s pomoću mjerničkog lanca Snellius je dobio da duljina stranice t – c iznosi 87,05 roods (327,8 m), a duljinu stranice a – e izračunao je iz duljine t – c i izmjerenih kutova i dobio da je a – e = 326,45 roods (1229 m) (sl. 20). Također je izravno izmjerio duljinu stranice a – e mjernim lancem. Ta dva rezultata nisu se slagala, pa je prednost dao računski dobivenoj veličini iz izmjerene duljine t – c. Zatim je iz izračunane duljine stranice a – e i izmjerenih kutova izračunao duljinu stranice $L – Z_o$ = 1093,55 roods (4118 m). Nakon toga je iz izračunane duljine stranice $L – Z_o$ i kutova u točkama L i Z_o trokuta $LHgZ_o$ izračunao duljinu početne stranice trigonometrijske mreže L – Hg = 4107,87 roods (15470 m).



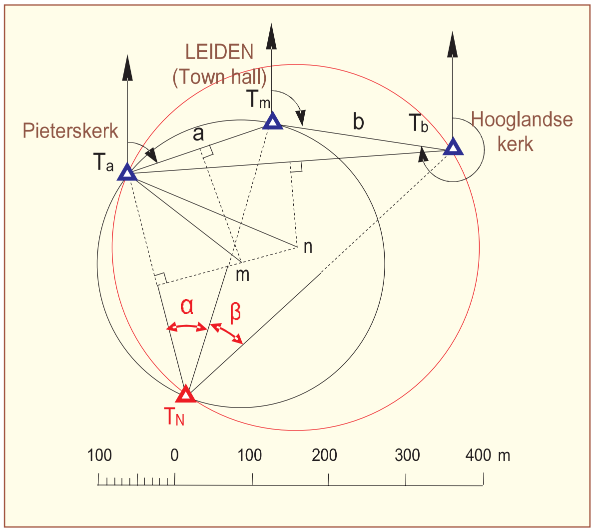


Fig. 21. (left) Vectors of positional errors in Snellius trigonometric network (according to Haasbroek 1968)

Figure 22. (up) Resecting with the Snellius's or Pothenot's method

Slika 21. (lijevo) Vektori položajnih pogrešaka Snelliusovih trigonometrijskih točaka (prema Haasbroeku 1968)

Slika 22. (gore) Presjek natrag na Snelliusov ili Pothenotov način

according to Snellius equals 6150.015 km, i.e. 220 km less than the value known nowadays, i.e. he obtained that the Earth is 3,5% smaller (Haasbroek 1968). According to Vykutil (1982), the Earth's radius according to Snellius is R = 6369 km.

Snellius's trigonometric network was analyzed by Haasbroek (1968). It was followed by an identification check of identical trigonometric points in the field, and coordinates of those points in the Dutch coordinate system *R(ijks) D(riehoeksmeting)*, abbreviated R.D. In order to evaluate the quality of Snellius's trigonometric network, a transformation was done to establish a connection between Snellius's and the Dutch coordinate system. Thus coordinates differences for the same points were obtained which in fact represent errors of Snellius's determination of trigonometric point positional coordinates, which can be seen in Fig. 21.

Fig. 21 shows that largest errors were on points Amsterdam 104 m and Haarlem 102 m and that they are larger than errors in Tycho Brahe's network. This was perhaps unexpected because Brahe's network was based on a very small base and calculations were done with very small angles. However, Snellius's network is especially significant because it was the first trigonometric network completely derived for the needs of determining the perimeter and radius of the Earth.

Snellius's network was followed by a lot of trigonometric networks for determining the Earth's shape and size, but also for production of accurate maps in Europe and the world.

For geodesists, Snellius is also significant because he was the first to determine the coordinates of a point in Leiden by *resecting* (Fig. 22), when angles α and β were measured on the determined point between three points (T_a, T_m, T_b), the position coordinates of which were determined previously.

This is also witnessed by the memorial (Fig. 23) which says:

"Willebrord Snel van Royen (Snellius 1580-1626) lived here. At this place, about 1615, he determined the first point in the history of geodesy by resecting. The memorial was erected by the "Snellius" Geodetic Society. Delft, December 2, 1960."

Interestingly, Snellius visited Tycho Brahe and Johannes Kepler in Prague in 1600 or 1601. Therefore, this was probably how Brahe found out about his work on the trigonometric network in Denmark.

Fig. 23. Memorial dedicated to Snellius, where he lived and where the position of a point was determined for the first time by his method of resection (Haasbroek 1968)

Slika 23. Spomen-ploča posvećena Snelliusu na mjestu gdje je živio i gdje je prvi put određen položaj točke s pomoću presjeka natrag njegovim načinom (Haasbroek 1968)

Snellius pri računanju nije bio dovoljno koncentriran, te su mu se događale pogreške. Naime, u njegovo doba točno računanje bio je težak posao, jer nije bilo računskih strojeva, pa čak ni logaritamskih tablica. Logaritamske tablice izmislio je John Napier 1614. godine (*Mirifici logarithmorum canonis descriptio 1614*), ali one nisu odmah ušle u uporabu, pa ih tako Snellius nije mogao koristiti. Može se također napomenuti da on nije uveo redukcije za ekscentrična opažanja jer su one manje od točnosti mjerenja njegovih kutova. Zbog toga su sva njegova računanja izvedena po pravilima trigonometrije u ravnini. Naime, sferni eksces najvećeg trokuta u njegovoj mreži Leiden – Dordrecht – Utrecht sa stranicama od oko 44 km iznosi samo 4'' (kutne sekunde). Njegove korekcije bile su samo ograničene na uvjet da zbroj kutova u svakom trokutu mora biti jednak 180° (Haasbroek 1968, str. 89).

Snellius je nakon objavljivanja *Eratosthenes Batavusa* pronašao neke pogreške u svojim izvornim opažanjima kad je izvodio mjerenja sa studentima u okolici Leidena. Čak je proširio svoju trigonometrijsku mrežu nekim trokutima na jug do Mallinesa (Mechelin) u Belgiji. Međutim, to nikad nije objavio.

Oko 100 godina nakon Snelliusove smrti njegove su zabilješke došle u ruke Petrusa van Musschenbroeka (1692–1761). On ih je provjerio i učinio velik broj promjena u njima, i to na temelju svojih osobnih mjerenja. Zatim je ponovno izračunao mrežu i to objavio u radu *De magnitudine terrae,* formirajući dio u *Physicae experimentalis et geometricae de magnete*, koji je tiskan u Leidenu 1729. godine.

Iz mjerenja geografskih širina u Alkamaaru i Bergen op Zoomu Snellius je dobio da je razlika njihovih širina 1°11'30''. Mjerenjem s pomoću trigonometrijske mreže dobio je da udaljenost između ta dva mjesta po meridijanu iznosi 127,780 km. Tako je dobio da jednom stupnju po meridijanu na Zemlji odgovara udaljenost od 107 388 m. Iz toga slijedi da je radijus Zemlje po Snelliusu jednak 6150,015 km, tj. 220 km manje od onoga što je danas poznato, odnosno dobio je da je Zemlja manja 3,5% (Haasbroek 1968). Prema drugom izvoru (Vykutil 1982), po Snelliusu radijus Zemlje iznosi R = 6369 km.

U radu (Haasbroek 1968) izvedena je analiza Snelliusove trigonometrijske mreže. Zatim je provjerena identifikacija identičnih trigonometrijskih točaka na terenu, a određene su i koordinate tih točaka u Nizozemskom koordinatnom sustavu *R(ijks) D(riehoeksmeting)* skraćeno R.D. Kako bi se mogla ocijeniti kvaliteta Snelliusove trigonometrijske mreže izvedena je transformacija kojom je uspostavljena veza između Snelliusova i Nizozemskoga koordinatnog sustava. Tako su dobivene koordinatne razlike za iste točke koje zapravo pokazuju pogreške Snelliusovih određivanja položajnih koordinata trigonometrijskih točaka, što je zorno pokazano na sl. 21.

Iz sl. 21 vidi se da su najveće pogreške bile na točkama Amsterdam 104 m i Haarlem 102 m i da su veće

3 Conclusion

Looking at the chronological overview of the emergence of the idea of producing trigonometric network, one can see the idea did not emerge on a single occasion, but that most renowned European scientists of the 17th and 18th century elaborated it: Gemma Frisius, Tycho Brahe and Snellius. All of them greatly contributed to making trigonometric networks a basis for determining the size and shape of the Earth and producing accurate maps and cadastral plans in Europe and the whole world.

Trigonometric networks remained current until the end of the 20th century, i.e. until Earth's artificial satellites were applied to determine positions of points on the Earth. Therefore, trigonometric networks had played an important role for almost four centuries.

Acknowledgements

We would like to thank the reviewers for their useful remarks and contributing to the quality of this research of geodesy's past, i.e. the historical overview of emergence of trigonometric networks. Classic trigonometric networks were very important until the present day when they are being replaced with measurements by using Earth's artificial satellites.

We would also like to thank the Ministry of Science, Education and Sport of the Republic of Croatia for partially financing this paper, which was produced within the project *Development of Scientific Surveying Laboratory for Geodetic Instruments*, No.: 007–0000000–3539.

References / Literatura

Gusić, I. (1995): Matematički rječnik, Element, Zagreb.

Haasbroek, N. D. (1968): Gemma Frisius, Tycho Brahe and Snellius and their Triangulations, Nederlandse Commissie voor Geodesie 14, Delft, Nizozemska.

Korošec, B. (1978): Naš prostor v času in projekciji, Geodetski zavod SRS, Ljubljana.

Macarol, S. (1977): Praktična geodezija, Tehnička knjiga, III. izdanje, Zagreb.

Šentija, J. (ur., 1977): Gemma-Frisius, Opća enciklopedija Jugoslavenskog leksikografskog zavoda, Zagreb, sv. 3, str. 125.

Šentija, J. (ur., 1981): Regiomontanus, Opća enciklopedija Jugoslavenskog leksikografskog zavoda, Zagreb, sv. 7, str. 42.

Vykutil, J. (1982): Vyšší geodézie, Kartografie, Praha.

URL 1: Regiomontan, http://sl.wikipedia.org/wiki/regiomontan (1. 8. 2005.)

URL 2: Peter Apian, http://de.wikipedia.org/wiki/Peter_Apian (24. 10. 2008.)

URL 3: Petrus Apianus, http://en.wikipedia.org/wiki/Petrus_Apianus (25. 10. 2008.)

URL 4: Reigner Gemma Frisius, http://sl.wikipedia.org/wiki/Regnier_Gemma_Frisius (31. 7. 2005.)

URL 5: Reigner Gemma Frisius, http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Gemma_Frisius.html (15. 7. 2005.)

URL 6: Gemma Frisius, http://en.wikipedia.org/wiki/Gemma_Frisius (24. 10. 2008.)

URL 7: Tycho Brahe, http://en.wikipedia.org/wiki/Tycho_Brahe (11. 2. 2008.)

URL 8: Tycho Brahe, http://de.wikipedia.org/wiki/Tycho_Brache (27. 11. 2005.)

URL 9: Tycho Brahe, http://hr.wikipedia.org/wiki/Tycho_Brahe (19. 10. 2008.)

URL 10: Galileo project - Tycho Brahe, http://gelileo.rice.edu/sci/brahe.html (14. 8. 2005.)

URL 11: The astronomical instruments of Tycho Brahe, http://www2.kb.dk/elib/lit/dan/brahe/index-en.htm (27. 10. 2008.)

URL 12: Frederick II, http://en.wikipedia.org/wiki/Image:Frederik_2.jpg (30. 10. 2008.)

URL 13: Willebrord Snellius, http://en.wikipedia.org/wiki/Willebrord_Snellius (11. 2. 2008.)

URL 14: Willebrord Snellius, http://research.leidenuniv.nl/index.php3?c=52 (12. 2. 2008.)

od pogrešaka u mreži Tyche Brahea. To se možda i nije očekivalo jer je Braheova mreža bila oslonjena na vrlo malu bazu i računalo se s vrlo malim kutovima. Međutim, značaj Snelliusove mreže je posebno velik, jer je to prva trigonometrijska mreža kompletno izvedena i izračunata za potrebe određivanja opsega i radijusa Zemlje.

Nakon Snelliusove, slijedio je velik broj trigonometrijskih mreža za određivanje oblika i dimenzija Zemlje, ali i za izradu točnih karata u Europi i svijetu.

Za geodete Snellius je značajan i po tome što je prvi odredio koordinate položaja jedne točke u Leidenu *presjekom natrag* (*presijecanje unutarnjim vizurama*) (sl. 22), kad su na određivanoj točki izmjereni kutovi α i β između tri točke (T_a, T_m, T_b) kojima su koordinate položaja prethodno određene.

O tome svjedoči i spomen-ploča (sl. 23) s natpisom, koji u nešto slobodnijem prijevodu glasi:

“Ovdje je živio Willebrord Snel van Royen (Snellius 1580-1626). Na ovome mjestu on je odredio oko 1615. prvu točku u povijesti geodezije pomoću *presjeka natrag (presjeka unutarnjih vizura)*. Postavilo Geodetsko društvo “Snellius”. Delft, 2. prosinca 1960.”

Zanimljivo je spomenuti da je Snellius posjetio Tycha Brahea i Johannesa Keplera u Pragu 1600. ili 1601. godine. Dakle, vjerojatno je tako saznao za Braheov rad na trigonometrijskoj mreži u Danskoj.

3. Zaključak

Iz kronološkog prikaza nastanka ideje o izradi trigonometrijskih mreža vidi se da ta ideja nije nastala odjednom, već da su je razrađivali najznamenitiji europski znanstvenici 17. i 18. stoljeća: Gemma Frisius, Tycho Brahe i Snellius. Svi su oni dali veliki doprinos tomu da trigonometrijske mreže postanu osnova za određivanje dimenzija i oblika Zemlje i za izradu točnih karata i katastarskih planova u Europi i čitavom svijetu.

Trigonometrijske mreže ostale su aktualne sve do kraja 20. stoljeća, tj. do primjene umjetnih Zemljinih satelita za određivanje položaja točaka na Zemlji. Dakle, trigonometrijske mreže odigrale su vrlo važnu ulogu u gotovo puna četiri stoljeća.

Zahvala

Najljepše zahvaljujemo recenzentima na korisnim primjedbama, kojima su pridonijeli boljoj kvaliteti ovog istraživanja geodetske prošlosti, tj. povijesnog pregleda nastanka trigonometrijskih mreža. Klasične trigonometrijske mreže imale su veliki značaj sve do današnjih dana, kada ih zamjenjuju mjerenja s pomoću umjetnih Zemljinih satelita.

Zahvaljujemo također Ministarstvu znanosti, obrazovanja i športa RH, što je djelomično financiralo ovaj rad, koji je izrađen u okviru projekta “*Razvoj znanstvenog mjeriteljskog laboratorija za geodetske instrumente*” br.: 007–0000000–3539.