Professional paper
Accepted 20.05.2002

KAMIL MALEČEK
DAGMAR SZARKOVÁ

A Method for Creating Ruled Surfaces and its Modifications

Metoda stvaranja pravčastih ploha i njihovih modifikacija

SAŽETAK

U članku je dana netradicionalna metoda za definiranje pravčastih ploha. Ta metoda omogućuje jednostavnu konstrukciju izvodnica pravčaste plohe prvenstveno pomoću računala, a ne samo u klasičnom smislu. Opisana je metoda za definiranje i konstrukciju poznatih ploha, ali i za modeliranje novih. Uvedeni matematički opis omogućuje stvaranje interaktivnog modeliranja ploha pomoću računala i vrlo brzi dizajn plohe te projekcije njezinih odabranih dijelova. Slike prikazuju računalne grafičke izlaze.

Ključne riječi: pravčaste plohe, razvojne plohe, vitopere plohe

A Method for Creating Ruled Surfaces and its Modifications

ABSTRACT

The paper presents a non-traditional method for defining ruled surfaces. This method enables a simple construction of the ruled surfaces generating lines, not only with the classical means, but first of all with a computer. The method for defining and constructing known surfaces and also modelling of new surfaces is described here. The introduced mathematical description enables creation of the interactive modelling of surfaces by using a computer and very quick surface design and projection of its arbitrary segments. The pictures are presenting the graphical output from a computer.

Key words: developable surface, ruled surface, skew surface

MSC 2000: 65D17, 51N05, 51N20

1 Definition of a ruled surface and construction of generating lines

We will work in the Euclidean space $\mathbf{E}_3$ and in the vector space $V(\mathbf{E}_3)$ with the Cartesian coordinates system $\langle O, x_1, x_2, x_3 \rangle$.

Let these vector functions be set:

$$\begin{aligned} \mathbf{y}_1(x_1) &= (x_1, 0, f(x_1)), \quad x_1 \in I_1, \\ \mathbf{y}_2(x_2) &= (0, x_2, g(x_2)), \quad x_2 \in I_2. \end{aligned} \tag{1}$$

Let the real functions f and g in (1) be continuous and differentiable on the intervals I_1 and I_2. These intervals can contain many points for which derivative of the functions f and g are improper. Vector functions (1) describe curves $k_1 \subset x_1x_3$ and $k_2 \subset x_2x_3$. We assume that these curves k_1 and k_2 are not intersected (Fig. 1).

Let the curve m be defined by the vector function (2) in the plane x_1x_2

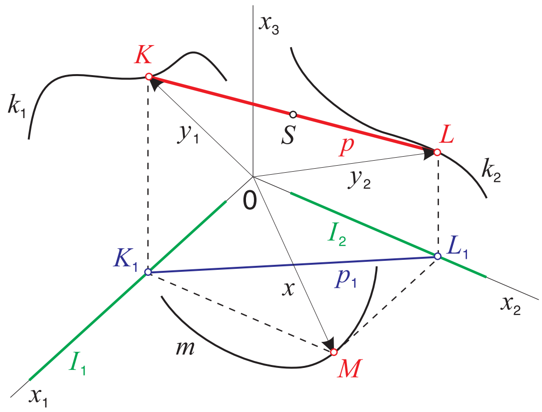

Fig. 1

$$\mathbf{x}(t) = (x(t), y(t), 0), \quad t \in I, \tag{2}$$

where for any $t \in I$ is $x(t) \in I_1$, $y(t) \in I_2$ and $\dfrac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{x}'(t)$ is a non-zero vector.

Now, we will construct the generating line p in this way (Fig. 1):

a) We choose a point M on the curve m and mark its orthogonal projections to the axes x_1 and x_2 as K_1 and L_1.

b) Points K and L are points located on curves k_1 and k_2 respectively, while K_1 and L_1 are their orthogonal projections to the plane x_1x_2.

c) The line p joins the points K and L.

The line p is a generating line of the ruled surface φ and with this method we would construct next generating lines of the surface φ.

2 Parametric representation of the ruled surface φ

We obtain the coordinates of the points K and L with the substitution of (2) to (1). Then

$$K = [x(t), 0, F(t)] \quad \text{and} \quad L = [0, y(t), G(t)],$$

where $F(t) = f(x(t))$ and $G(t) = g(y(t))$.
Let the generating line p be defined for example by the centre

$$S = \left[\frac{x(t)}{2}, \frac{y(t)}{2}, \frac{F(t)+G(t)}{2}\right]$$

of the line segment KL and by the direction vector

$$\mathbf{p}(t) = \frac{1}{2}\left(x(t), -y(t), F(t)-G(t)\right). \qquad (3)$$

Then the ruled surface φ has the following parametric representation:

$$\begin{aligned} x_1 &= \frac{x(t)}{2}(1+u), \\ x_2 &= \frac{y(t)}{2}(1-u), \\ x_3 &= \frac{F(t)+G(t)}{2} + u\frac{F(t)-G(t)}{2}, \\ & t \in I, \quad u \in R. \end{aligned} \qquad (4)$$

Example 1: The surface of an elliptic movement

The vector functions (1) are

$$\begin{aligned} \mathbf{y}_1(x_1) &= (x_1, 0, q_1), \quad x_1 \in R, \\ \mathbf{y}_2(x_2) &= (0, x_2, q_2), \quad x_2 \in R, \end{aligned} \qquad (5)$$

where q_1 and q_2 are non-zero constants from R, $q_1 \neq q_2$.

Curves k_1 and k_2 are lines, $k_1 \parallel x_1$ and $k_2 \parallel x_2$. Let the curve m become a circle defined by the vector function

$$\mathbf{x}(t) = (a\cos t, a\sin t, 0), \quad t \in \langle 0, 2\pi\rangle. \qquad (6)$$

The surface φ defined in this way has the following parametric representation according to equation (4):

$$\begin{aligned} x_1 &= \frac{a\cos t}{2}(1+u), \\ x_2 &= \frac{a\sin t}{2}(1-u), \\ x_3 &= \frac{q_1+q_2}{2} + u\frac{q_1-q_2}{2}, \\ & t \in \langle 0, 2\pi\rangle, \quad u \in R. \end{aligned} \qquad (7)$$

In Fig. 2a the curves k_1, k_2 and m are shown.
In Fig. 2b a surface segment, which is called the surface of an elliptic movement is shown.

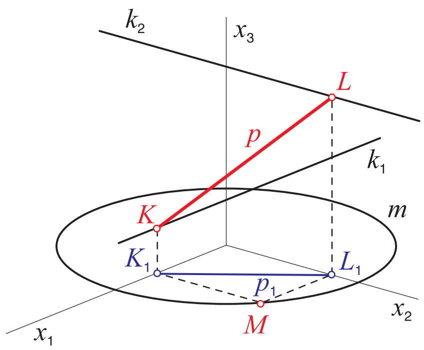

Fig. 2a

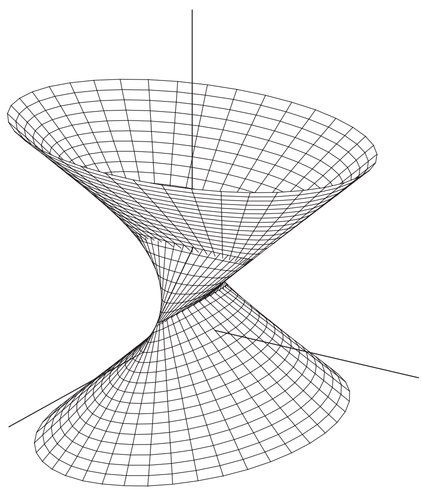

Fig. 2b

3 The section of the surface φ by the plane x_1x_2

If the curve k_3 is the section of the surface φ by the plane x_1x_2, $(x_3 = 0)$, then we get its parametric representation from (4)

$$x_1 = \frac{G(t)x(t)}{G(t)-F(t)},\; x_2 = \frac{-F(t)y(t)}{G(t)-F(t)},\; x_3 = 0,\; t \in I\,. \quad (8)$$

If for any $t \in I$ is

$$G(t) = F(t), \quad (9)$$

then the corresponding generating line $p \parallel x_1x_2$ and its intersection point with the plane x_1x_2 is a point at infinity.

The section of the surface φ of elliptic movement by the plane x_1x_2 (from the example 1 according to (8)) has the following parametric representation

$$x_1 = \frac{aq_2}{q_2-q_1}\cos t, \quad x_2 = \frac{-aq_1}{q_2-q_1}\sin t,$$
$$x_3 = 0, \qquad t \in \langle 0, 2\pi \rangle. \quad (10)$$

This section is the ellipse k_3 with the centre in the origin of the coordinate system, values of the semiaxes are

$$\left|\frac{aq_2}{q_2-q_1}\right| \quad \text{and} \quad \left|\frac{aq_1}{q_2-q_1}\right|.$$

In the case when the equation (9) expresses identity for the interval I all generating lines of the surface φ are parallel to the plane x_1x_2 and the section of the surface by the plane x_1x_2 cannot be described by equations (8).

If we choose the vector functions (1) as

$$\begin{aligned} \mathbf{y}_1(x_1) &= (x_1, 0, f(x_1)), \quad x_1 \in I_1, \\ \mathbf{y}_2(x_2) &= (0, x_2, f(x_2)), \quad x_2 \in I_1, \quad I_1 = I_2 \quad (11) \end{aligned}$$

and the curve m is a line parametrized by the vector function

$$\mathbf{x}(t) = (t, t, 0), \quad t \in R, \quad (12)$$

then $F(t) = G(t) = f(t)$.

The ruled surface φ has the following parametric representation according to (4):

$$\begin{aligned} x_1 &= \frac{t}{2}(1+u), \\ x_2 &= \frac{t}{2}(1-u), \\ x_3 &= f(t), \quad t \in I_1, \quad u \in R \quad (13) \end{aligned}$$

and it is a cylindrical surface. Its generating lines are parallel to the plane x_1x_2. The curves k_1 and k_2 are congruent. Revolving the curve k_1 about the axis x_3 by the angle 90^o we would get the curve k_2.

The section k_3 of the cylindrical surface (13) by the plane x_1x_2 is composed from the surface generating lines. Their number is equal to the number of common points of the curve k_1 and the axis x_1.

Example 2: Circular cylindrical surface

Curves k_1 and k_2 are semicircles with centres $S_1 \in x_1$, $S_2 \in x_2$, with the same radius r and $|OS_1| = |OS_2| = p$. The semicircles are parametrized by the vector functions

$$\begin{aligned} \mathbf{y}_1(x_1) &= \left(x_1, 0, \sqrt{r^2-(x_1-p)^2}\right), \\ & \qquad x_1 \in \langle p-r, p+r \rangle, \\ \mathbf{y}_2(x_2) &= \left(0, x_2, \sqrt{r^2-(x_1-p)^2}\right), \\ & \qquad x_2 \in \langle p-r, p+r \rangle, \quad 0 < p-r\,. \end{aligned}$$

The surface φ is a half of the circular cylindrical surface which has the following parametric representation according to (13)

$$\begin{aligned} x_1 &= \frac{t}{2}(1+u), \\ x_2 &= \frac{t}{2}(1-u), \\ x_3 &= \sqrt{r^2-(t-p)^2}, \\ & \qquad t \in \langle p-r, p+r \rangle, \quad u \in R\,. \quad (14) \end{aligned}$$

Fig. 3a illustrates curves k_1, k_2, m, and the section of the surface by the plane x_1x_2 which is created by lines l_1 and l_2.

In Fig. 3b the segment of the circular cylindrical surface is shown.

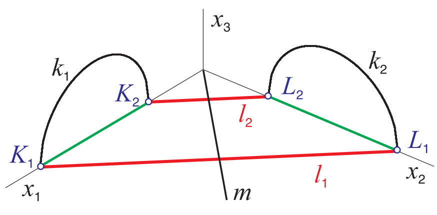

Fig. 3a

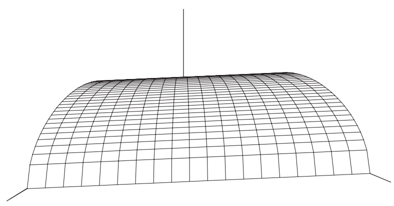

Fig. 3b

4 Developable and skew surfaces φ

The generating line of the surface φ is defined by choice of the parameter $t \in I$. When we substitute the parametric representation (2) of the curve m to the vector functions (1) and differentiate the vector functions (1) according to argument t, then we get:

$$\mathbf{y}'_1(t) = x'(t)\left(1, 0, \left[\frac{\mathrm{d}f(x_1)}{\mathrm{d}x_1}\right]_{x_1=x(t)}\right) \quad \text{and}$$

$$\mathbf{y}'_2(t) = y'(t)\left(0, 1, \left[\frac{\mathrm{d}g(x_2)}{\mathrm{d}x_2}\right]_{x_2=y(t)}\right).$$

These vector functions for chosen $t \in I$ are defining the direction vectors of the tangent lines of curves k_1 and k_2. To make the generating line of the surface φ torsal, vectors $\mathbf{y}'_1(t)$, $\mathbf{y}'_2(t)$ and (3) must be linearly dependent. From this condition we get equation:

$$x'(t)y'(t)\left(F(t) - x(t)\left[\frac{\mathrm{d}f(x_1)}{\mathrm{d}x_1}\right]_{x_1=x(t)} - \left(G(t) - y(t)\left[\frac{\mathrm{g}(x_2)}{\mathrm{d}x_2}\right]_{x_2=y(t)}\right)\right) = 0. \quad (15)$$

If the equation (15) is identity on the interval I, the surface φ is created by torsal lines only and it is a developable surface. If the equation (15) is not identity, the surface φ is a skew surface on which torsal generating lines can exist.

The equation (15) of the surface of an elliptic movement has the form:

$$a^2(q_1 - q_2)\sin t \cos t = 0, \quad t \in \langle 0, 2\pi\rangle.$$

Then the lines for parameters $t = 0,\ \pi/2,\ \pi,\ 3\pi/2$ are torsal lines located in the planes x_1x_3 and x_2x_3.

In the case when the cylindrical surface has the parametric presentation (13) we can simply verify that the equation (15) is an identity and the cylindrical surface will be a developable surface. The intersection points K_1 and L_1 of the line l_1 with the semicircles k_1 and k_2 from the example 2 (Fig. 3a) are examples of points in which derivative of the functions f and g is improper. The tangent lines of the curves k_1 and k_2 in the points K_1 and L_1 are parallel with the axis x_3. Analogously for the line l_2.

5 Continuity between the surfaces φ and skew surfaces

Continuity between the mentioned ruled surfaces φ and skew surfaces, which are defined by three basic curves, is clearly seen on the surface of an elliptic movement. If the section of the surface φ by the plane x_1x_2 is the curve k_3, then it is possible to define the surface φ by basic curves k_1, k_2 and k_3. The generating lines of the surface φ are lines intersecting the basic curves.

6 Envelope of orthographic views of the ruled surface generating lines in the plane x_1x_2

Generating lines of the ruled surface are orthogonally projected to the plane x_1x_2 and parametric representation of these orthographic views can be given by the first two equations in (4) without the parameter u:

$$y(t)x_1 + x(t)x_2 - x(t)y(t) = 0, \quad t \in I. \quad (16)$$

The equation (16) is the equation of a one-parametric line system and its envelope can be found by differentiating of the equation (16) according to parameter t:

$$y'(t)x_1 + x'(t)x_2 - x'(t)y(t) - x(t)y'(t) = 0. \quad (17)$$

From the equations (16) and (17) we get:

$$x_1 = \frac{x^2(t)y'(t)}{x(t)y'(t) - x'(t)y(t)},$$

$$x_2 = \frac{-y^2(t)x'(t)}{x(t)y'(t) - x'(t)y(t)},$$

$$x_3 = 0, \qquad t \in I. \quad (18)$$

If an envelope exists and it is a curve marked as m', then the equations (18) are its parametric representation. The points for which $x(t)y'(t) - x'(t)y(t) = 0$ do not have to be necessarily troublesome points. This problem will be not investigated here.

The envelope m' depends only on the curve m, what is evident from the equations (18) and the geometric view, too.

If the curve m is a circle parametrized by the function (6), then according to (18) the envelope m' has the following parametric representation:

$$x_1 = a\cos^3 t,\ x_2 = a\sin^3 t,\ x_3 = 0,\ t \in \langle 0, 2\pi\rangle, \quad (19)$$

the curve m' is an asteroid (Fig. 4a).

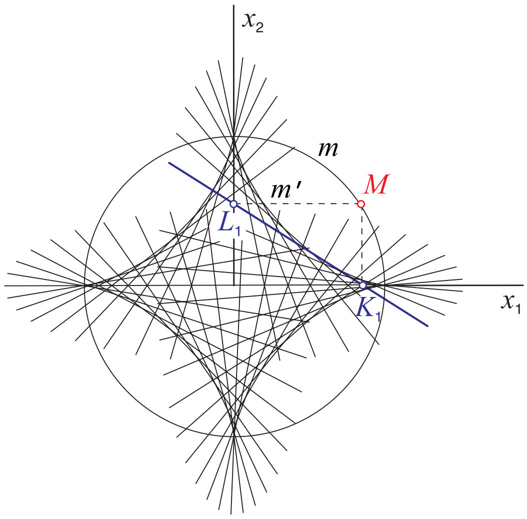

Fig. 4a

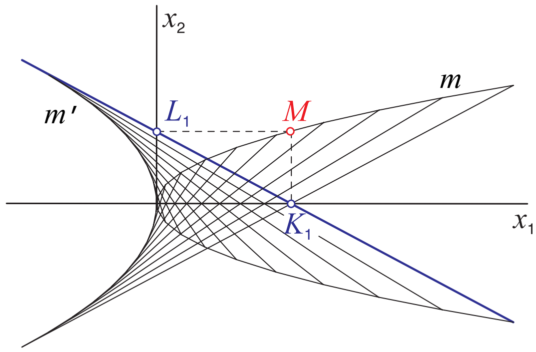

Fig. 4b

Orthographic views of the cylindrical surface generating lines (example 2) in the plane x_1x_2 are examples for the one-parametric system of lines which has not any envelope.

7 Modification of the method for the creation of ruled surfaces

The idea described above allows us to define and create ruled surfaces by a method which we could call dual for defining and creating the surfaces φ. The surface is defined by the curves k_1 and k_2 which are parametrized by the functions (1). Let the curve m' be without singular points in the plane x_1x_2. Tangent lines of the curve m' create a one-parametric system of lines. Let K_1 and L_1 be the intersections of one tangent line (which is the intersecting line with the axes x_1 and x_2 too) of the one-parametric system with the axis x_1 and x_2. We can construct the surface generating line p by means of points K_1 and L_1 with the same method as in the first part (see Fig. 1).

Example 3:

Let the surface φ be defined by curves k_1 and k_2, which are parametrized by the vector functions (5) and the curve $m' \subset x_1x_2$ is a parabola expressed by parametric representation

$$x_1 = -\frac{1}{2p}t^2, \quad x_2 = t, \quad x_3 = 0, \quad t \in R\,.$$

The vector function

$$\mathbf{y}(v) = \left(-\frac{1}{2p}t^2 - \frac{1}{p}tv, t+v, 0\right), \quad v \in R \tag{20}$$

of the parameter v describes the system of tangent lines to the parabola m' for any value of parameter $t \in R$ (Fig. 4b).

The intersections of the tangent lines with the axes x_1 and x_2 are the points K_1 and L_1 with the following coordinates

$$K_1 = \left[\frac{1}{2p}t^2, 0, 0\right] \quad \text{and} \quad L_1 = \left[0, \frac{t}{2}, 0\right]\,. \tag{21}$$

The coordinates of the points K and L are

$$K = \left[\frac{1}{2p}t^2, 0, q_1\right] \quad \text{and} \quad L = \left[0, \frac{t}{2}, q_2\right]\,.$$

The surface parametric representation according to (4) has the following form:

$$\begin{aligned} x_1 &= \frac{t^2}{4p}(1+u), \\ x_2 &= \frac{t}{4}(1-u), \\ x_3 &= \frac{q_1+q_2}{2} + u\frac{q_1-q_2}{2}, \\ & t \in R, \quad u \in R\,. \end{aligned} \tag{22}$$

The set of points M which are projected orthogonally to the points K_1 and L_1 on the axes x_1 and x_2 can be parameterized according to (21) by the function

$$\mathbf{x}(t) = \left(\frac{t^2}{2p}, \frac{t}{2}, 0\right), \quad t \in R\,.$$

The points M are therefore located on the parabola m which is the generatrix of the ruled surface φ constructed by the method described in the first part.

The both modifications of the presented method are illustrated in Fig. 4b showing the construction of the surface φ projected orthogonally to the plane x_1x_2. A similar construction can be seen in Fig. 4a, where the curve m is a circle and the curve m' is an asteroid.

Description of the surface φ construction is shown in Fig. 5a. The segment of the surface is shown in Fig. 5b.

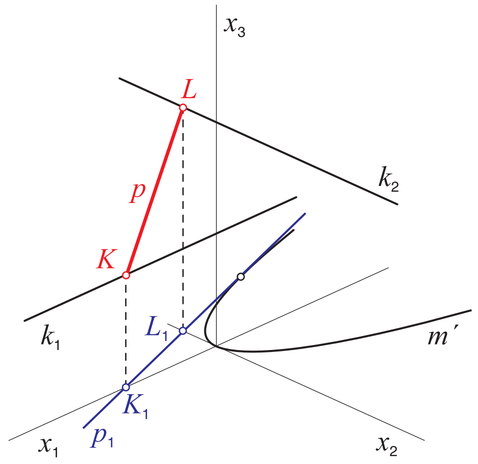

Fig. 5a

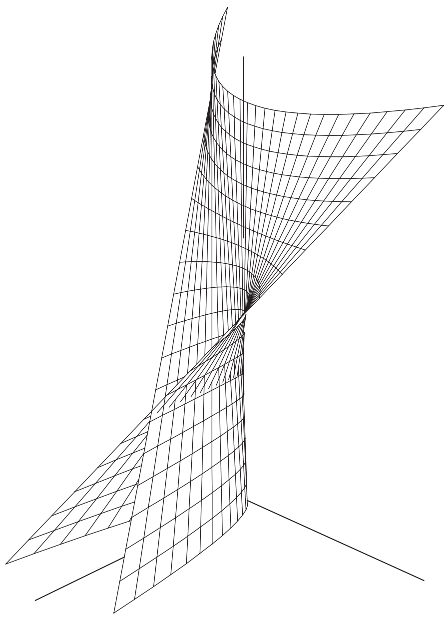

Fig. 5b

The section of the surface φ by the plane x_1x_2 has the following parametric representation according to (8):

$$x_1 = \frac{q_2}{2p(q_2-q_1)}t^2, \quad x_2 = \frac{-q_1}{2(q_2-q_1)}t, \quad x_3 = 0, \quad t \in R,$$

the curve k_3 is a parabola.

The equation (15) has the form

$$\frac{1}{2p}t(q_2-q_1) = 0$$

and therefore the surface has only one torsal basic line correspondent to the parameter $t = 0$.

Now we will show some examples of the surfaces φ.

Example 4: Conical surface

The vector functions (1) are

$$\begin{aligned}\mathbf{y}_1(x_1) &= (x_1, 0, q_1), \quad x_1 \in R,\\ \mathbf{y}_2(x_2) &= \left(0, x_2, \frac{1}{2p}x_2^2 + q_2\right), \quad x_2 \in R\,.\end{aligned}$$

The curve k_1 is a line, the curve k_2 is a parabola. Let the curve m be a line parallel to the axis x_2 defined by the vector function

$$\mathbf{x}(t) = (k, t, 0), \quad t \in R, \tag{23}$$

where k is a non-zero constant from R (Fig. 6a).

The surface φ has the following parametric representation according to (4):

$$\begin{aligned}x_1 &= \frac{k}{2}(1+u),\\ x_2 &= \frac{t}{2}(1-u),\\ x_3 &= \frac{2p(q_1+q_2)+t^2}{4p} + u\frac{2p(q_1-q_2)-t^2}{4p},\\ & t \in R, \quad u \in R\,.\end{aligned} \tag{24}$$

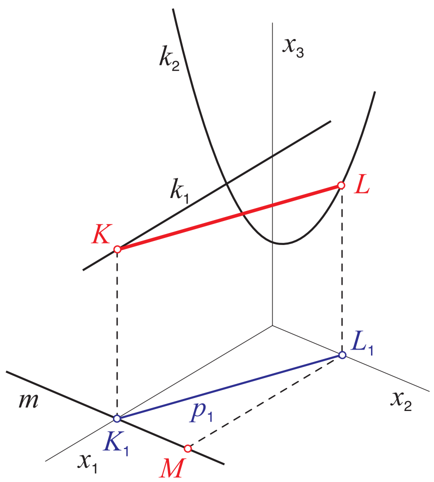

Fig. 6a

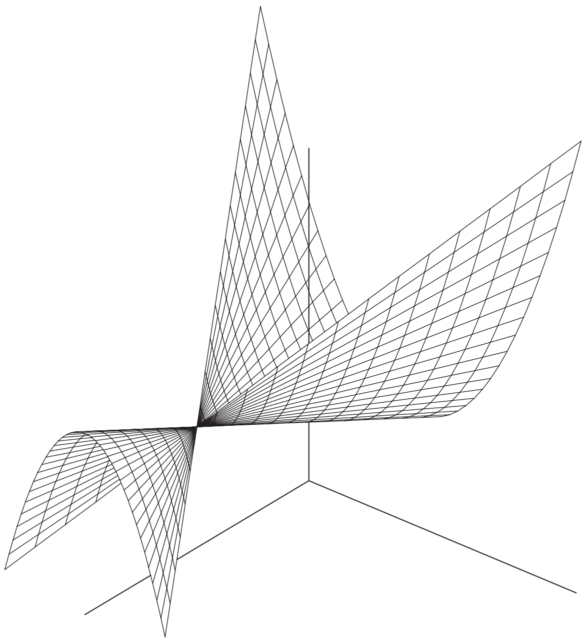

Fig. 6b

It is evident that the surface φ is a conical surface with a vertex in the point K, so this surface is developable. In this case the equation (15) is an identity, because in the vector function (23) is $x'(t) = 0$ for all $t \in R$. The segment of the surface is illustrated in Fig. 6b.

We could construct generating lines by the means of a one-parametric system of lines in the plane x_1x_2, too. In this case, the system of lines would be a pencil of lines with the centre in the point K_1 (without the line m). The line p_1 is one line from the pencil of lines (see Fig. 6a). This is nothing new for us, it is a classical construction of conical surface generating lines.

Example 5: Frezier's cylindroid

The vector functions (1) are

$$\mathbf{y}_1(x_1) = \left(x_1, 0, \sqrt{r^2-(x_1-p)^2}\right), \quad x_1 \in \langle p-r, p+r\rangle, \quad 0 < p-r,$$

$$\mathbf{y}_2(x_2) = \left(0, x_2, \sqrt{r^2-(x_2-p)^2}+q\right), \quad x_2 \in \langle p-r, p+r\rangle,$$

where q is a non-zero constant from R.

The curves k_1 and k_2 are semicircles as in the example 2 for the cylindrical surface, but the circle k_2 is translated by the translation vector $(0,0,q)$. The curve m is a line defined by the vector function (12), Fig. 7a.

This surface is so called Frezier's cylindroid and its segment is shown in Fig. 7b. The surface is a skew surface which has two torsal generating lines. The equation (15) has the form

$$\sqrt{r^2-(t-p)^2} + \frac{t(t-p)}{\sqrt{r^2-(t-p)^2}} = \sqrt{r^2-(t-p)^2} + q + \frac{t(t-p)}{\sqrt{r^2-(t-p)^2}}$$

and this is fulfilled only for the points $t = p \pm r$, in which derivative of the functions f and g is improper. Orthographic views of torsal lines in the plane x_1x_2 are the lines l_1 and l_2 (Fig. 7a).

It is possible to construct cylindroid generating lines analogously using the one-parametric system of lines as at a conical surface. It this case the system of lines is parallel to the line l_1.

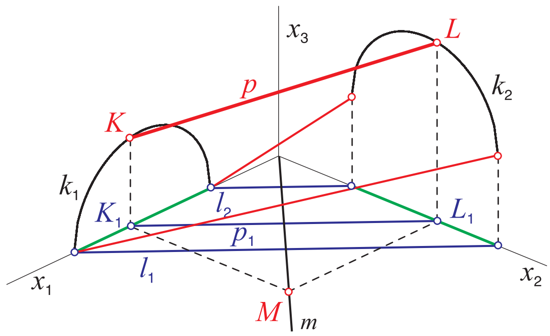

Fig. 7a

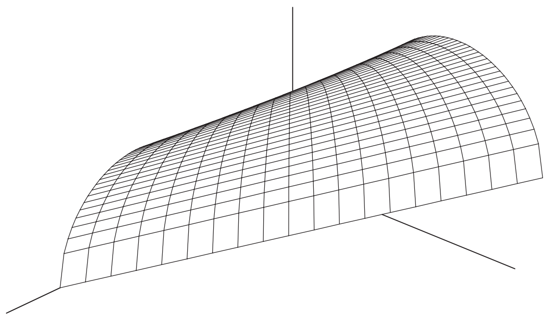

Fig. 7b

At the end of this paper there are illustrated two complicated surfaces (see Figs 8 and 9). The segment of the surface demonstrated in Fig. 8 is defined by curves k_1, k_2 and m, where the curve k_1 is the Witch of Agnési, k_2 is a parabola and the curve m is an epicycloid. In Fig. 9 is a segment of the surface for which the curve k_1 is a parabola, the curve k_2 is Witch of Agnési and the curve m is a circle with its centre in the origin.

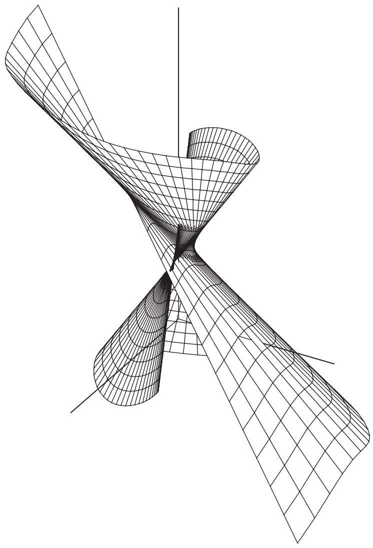

Fig. 8

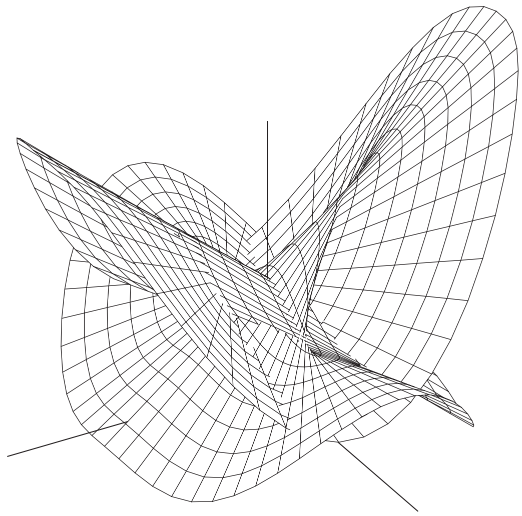

Fig. 9

References

[1] MEDEK, V., ZÁMOŽÍK, J.: *Konštruktívna geometria pre technikov*, Alfa-Bratislava, SNTL-Praha, 1978

[2] BUDINSKÝ B., KEPR. B: *Základy diferenciální geometrie s technickými aplikacemi, SNTL*, Praha 1970

Kamil Maleček
Department of Mathematics
Faculty of Civil Engineering
Czech Technical University
Thákurova 7
166 29 Praha 6, Czech Republic
tel: +420 2 24354384
e-mail: kamil@mat.fsv.cvut.cz

Dagmar Szarková
Department of Mathematics
Faculty of Mechanical Engineering
Slovak University of Technology
Námestie Slobody 17
812 31 Bratislava, Slovakia
tel: +421 2 57296 394
e-mail: szarkova@sjf.stuba.sk