Mathematical representation models and applications on seismic tomography

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REVIEW

New mathematical techniques have contributed substantially to the improvement of the geophysical prospecting methods, like traveltime seismic tomography. Thanks to these new techniques, the time to solve an inverse problem has been reduced dramatically making seismic tomography applicable to a great number of problems even in three dimensions. New raytracing and wavefront techniques provide a more flexible parameterization. Advancement from the least squares technique to today's back-projection method's, for example, has improved tomographic methods.

Key words: tomography, raytracing, wave front, grid points, travel time

INTRODUCTION

What does tomography mean as a word? Its origin is found in the Greek word "tomo", which means slice. Therein lies the basic idea: if we take many 2-D slices, according to the central slice theorem[28], we can reconstruct the whole 3-D image of an object. Thanks to the same theorem, we can easily construct 2-D sections from 1-D lines, which can be measured in experiments.

Seismic travel time tomography can be defined as the reconstruction of the Earth's velocity model, using the seismic waves travel time deviations from a reference velocity model, better known as starting or background model. A starting model is an initial guess, an estimate of a velocity model. Seismic tomography as we know it today originated in 1974 as "3-D inversion". At first, seismologists were very skeptical about the new method and its results. The whole attitude changed dramatically in the mid-1980's, when iterative methods were introduced, to facilitate the calculation of large and sparse matrixes that occurred from the seismological data.[6,23] Believability in the method was linked to the first global tomographic results[9,37,11],which correlated satisfactory with the geoid. As credibility of the method grew seismologists renamed "3-D inversion" to what is known today as seismic tomography.

We can classify seismic tomography in two main categories; travel time and amplitude tomography. In this paper we will focus only on travel time tomography. Regarding the nature of the seismic waves, travel time tomography can be divided into refraction, reflection and diffraction tomography. Referring to the source, whether it is a natural earthquake or a shot, we carve up tomography into passive and active tomography, respectively.

Seismic tomography is an imaging technique that uses seismic waves generated by earthquakes and explosions to create computer-generated, three-dimensional images of Earth's interior. This is how seismologists infer the different layers in the Earth. How is this done? The time it takes for a seismic wave to arrive at a seismic station from an earthquake can be used to calculate the speed along the wave's ray path. By using first arrival times of P waves recorded by seismic stations all over the world, scientists are able to define slower or faster regions deep in the Earth

The simplest case of seismic tomography is to estimate P-wave velocity. Several methods have been developed for this purpose, e.g., refraction traveltime tomography, finite-frequency traveltime tomography, reflection traveltime tomography, waveform tomography.[36]

To obtain a higher-resolution image one has to abandon the infinite-frequency approximations of ray theory that are applicable to the time of the wave 'onset' and instead measure travel times (or amplitudes) over a time window of some length using cross-correlation. Finite-frequency tomography takes the effects of wave diffraction into account, which makes the imaging of smaller objects or anomalies possible.[24]

The raypaths are replaced by volumetric sensitivity kernels, often named 'banana-doughnut' kernels in global tomography, because their shape may resemble a banana, whereas their cross-section looks like a doughnut, with, at least for direct P and S waves, zero sensitivity of the travel time on the geometrical ray path. In finite-frequency tomography, travel time and amplitude anomalies are frequency-dependent, which leads to an increase in resolution.

To exploit the information in a seismogram to the fullest, one uses waveform tomography. In this case, the seismograms are the observed data. In seismic exploration, the forward model is usually governed by the acoustic wave equation. This is an approximation to the elastic wave propagation.[36] Elastic waveform tomography is much more difficult than acoustic waveform tomography. The acoustic wave equation is numerically solved by some numerical schemes such as finite-difference and finite-element methods. In global tomography the inverse problem for elastic waves can be handled by adjoint methods.

PARAMETERIZATION

Lets consider two closely spaced points in the medium; the inverse of the local wavespeed associated with these points is the slowness. There are three kinds of slowness models: homogeneous and heterogeneous cells (2-D) or blocks (3-D) of constant slowness values and rectangular grids with slowness values assigned to the grid points with different interpolation schemas to specify the values between the grid points. In general, the use of cells is the most common parameterization but it is facing difficulties, since the sharp boundaries between the cells cannot be resolved. As for the grid parameterization, a fine regular and irregular grid parameterization exist. The former parameterization is the purely tomographic approach while the latter one is closer to forward modelling. Regular grid has the advantage of simplicity but it can cause over-parameterization when high resolution is required. Recent studies are focused on irregular grid using Delaunay triangles or Veronoi polygons to avoid such problems.[4,41] The travel time for a ray is:

$$T = \int s\,1/V(r)\,ds \tag{1}$$

where $V(r)$ is the unknown velocity for the ray-path S. We want to determine $V(r)$ from N travel time measurements. Let To be the travel time for the starting model:

$$T_0 = \int s_0\,1/V_0(r)\,ds \tag{2}$$

Whether we are not sure about the estimate of the starting model we can use a tau-p method from picked arrival times to form a reliable starting velocity model.[2] Using Fermat.s principle, we can ignore the true ray-path and use the ray-path of the starting model instead of it. The delay time is:

$$\delta T = T - T_0 = \int s\,1/V(r)\,ds - \int s_0\,1/V_0(r)\,ds \approx \int s_0\left(1/V - 1/V_0\right)ds \approx$$
$$\approx -\int s_0\left(\delta V(r) - V_0(r)^2\right)ds \tag{3}$$

where:

$$\delta V(r) = V(r) - V_0(r) \tag{3a}$$

Equation (3) comprises a linear system of equations, which can be changed in such a way to become more suitable for computer processing. We parameterize the medium with I interpolation functions h_i, which is the basis of the subspace of the Hilbert space of all possible models $V(r)$:

$$\delta V(r) = \sum_k \gamma_k h_k(r) \tag{4}$$

where k spans the integers from 1 to I, and function γ_k is the weight of the function h_k. Considering a cell-parameterization we have:

$h_i\,(r)=1$, if r in cell I and

$$h_i\,(r)=0,\ \text{anywhere else} \tag{5}$$

Equation (3) can now be defined as:

$$\delta T = \sum_k -\int s_0\left[1/V_0(r)^2 \gamma_k h_k(r)\right]ds = \sum_k A_{ik}\gamma_k \tag{6}$$

where:

$$A_{ik} = -\int s_0\left[1/V_0(r)_2 h_k(r)\right]ds \tag{6a}$$

Equation (6) can be formulated for each shot in a matrix form and in terms of slowness as:

$$Ms = t \tag{7}$$

Where s is the slowness vector, t the time vector and M is the matrix of lij, where lij is the length of the i-th ray-path through j-th cell.

SOLUTION

Foremost, we have to calculate the matrix elements Aik, which implies the finding of the ray-path. Two methods are commonly used in seismic tomography to find the ray path: *ray-tracing* and wavefront methods. The two-point ray-tracing finds ray-paths along which seismic energy is propagating and calculates the travel time. For a layered media, rays are traced by solving the differential equations with a Runge-Kutta predictor-corrector scheme. To define in a better way the ray geometry and the slowness, we mainly have two methods: shooting and bending, inspired both from ray-tracing. The former is based on continuous iterations until the end of a ray to meet a limit condition or by interpolating between close rays using hermite cubic interpolation.[7] The latter uses a parameterization for a ray-path by the support points V_i of a third order B-spline. The location of the ray is a function of the four nearest points:

$$Q_i(u) = b_{-2}\,V_{i-2} + b_{-1}V_{-1} + b_0 V_i + b_1 V_{i+1} \tag{8}$$

Where b_1 depends on u, $0 \le u \le 1$, and is known for the different values of u. A conjugate gradient method[33] is needed to find which of the support points V_1 minimizes the time given by (1). Červený[5] uses the quadratic slowness, $1/V^2$, instead of slowness since it offers the simplest analytical solution in inhomogeneous medium. Latter techniques pored over the drawbacks of ray-tracing including the works of Zelt and Ellis[49] who invented a ray-tracing technique with a trapezoidal parameterization providing a rapid traveltime calculation and of Sethian and Popovici[34] who presented the *fast marching* technique that can model turning rays, but with unsatisfactory accuracy.

The *wavefront* (surfaces of equal travel time) methods are alternative techniques to ray-tracing. This method determines minimum ray-paths and traveltimes by expanding a wavefront in the whole model. Recent advances in the ray-tracing methods have mainly been focused on wavefront methods rather than ray-tracing, for two basic reasons: ray-tracing is valid only for smooth velocity structures and it is significantly slower than any wavefront method. Vidale[43.42] modified the wavefront method and the eikonal equation solver introducing a finite difference procedure, to propagate traveltimes through a uniformly sampled grid. The eikonal solver finds the wavefront that forms concentric shells about the source and conducts the ray-paths from their shape. A defect of Vidalia's method[27] is that it fails when velocity contrasts are of the order of

$$u_2 / u_1 > \sqrt{2} \tag{9}$$

(Hole et al.[14] modified Vidalia's algorithm to a more rapid algorithm using variable grid spacing. SPR, for

Shortest Path Ray-tracing, developed primarily by Saito[32,31] and Moser[20], expands the wavefront in the entire velocity network being in that way more stable to velocity contrasts. Zhang, et al.[53,54] developed a SPR method with a graphic template that allowed only straight rays within a constant cell, calculating rapidly and with good accuracy ray-paths and traveltimes on a large number of grids. In the *wavefront construction*[44,45] the entire wavefront is represented by a triangular mesh providing good accuracy but implying time-consuming calculations. GRT, for grid ray-tracing[55], combines the advantages of both *wavefront construction* and *fast marching*, by tracing rays within a local grid. GRT is about 8 times faster than any wavefront method. The main shortcoming of the wavefront methods is that they used to calculate only first arrivals. Moser[20], Hole and Zelt[13] and Zhang and Toksöz[54] modified the method to calculate also latter arrivals.

The solution of the equation (7) is challenging. With real data there always exists possibility that no ray crosses a cell (an ill-posed system). The other problem is the amount of data that we use in iterative solution techniques or inversion. Early tomographic methods used exact least squares techniques to solve the resulting system of equations, but model limitations (due to the restrained computer capabilities) on the order of 1 000 unknowns were coming up. This number is not that large or as much of a concern today, e.g. the tomographic analysis of Zelt and Barton[47] involves more than 50 000 travel times. To surpass these obstacles, iterative matrix solvers were introduced. McMehan[18] and Neumann-Denzau and Behrens[22], improved an older iterative method of Kaczmarc and renamed it to ART, for Algebraic Reconstruction Technique. This method was badly conditioned and extremely slow for seismic tomography. Gilbert[10] invented a more efficient method, named SIRT, for Simultaneous Iterative Reconstruction Technique. Both ART and SIRT are mainly applicable when pixels or voxels (the three dimensional analog of pixels) are used as the basis function (5). Both of these techniques use backprojection in an iterative manner to solve the system of equations. Backprojection is an iterative process to estimate the average slowness. Instead of backprojecting travel time residuals along ray-paths, other formulations backproject phase residuals along wavepaths[35] which take into account the finite frequency effects in travel time data.

In order to reconstruct velocities and interfaces we solve the regularized inverse problem. The trial-and-error forward modelling is a time-consuming and laborious process which fails to provide an estimate of parameter uncertainty and resolution, as inversion does. In general, the non-linear system of equations (7) is being linearized, considering that the velocity structure is divided into a reference or starting model, that is assumed to be known, and an unknown perturbation, which is considered as very small. When the model is non-linearized, the solution will be independent of the model parameterization, so we apply the Tikhonov method[40] in order to reconstruct the model with a Laplacian operator. Broadly speaking, for a regularised problem we can invert various types of data, e.g. reflection, refraction, crosswell, both refraction and reflection data, known as joint inversion, for better performance and various types of model parameters (e.g. slowness, reflector geometries). The aim of the regularized inversion is to minimize a tradeoff parameter concerning the data misfit. In early tomographic problems fitting data was the major care, but recent studies prove that we can fit data to any small misfit magnitude according to the constraints of the model parameters, although the solution may not be physically consistent.[38] Given that a crucial demerit of inversion is its nonuniqueness[30], which leads to multiple solutions of the problem, the main concern is how to obtain a stable and unique solution that doesn't provide unneeded structures. Fitting travel times with a least squares criterion can't always provide the best solution, although the vast majority of the tomographic methods implement some variant of the principal least-square method, by selecting a model that minimizes a certain measure of travel time error, e.g. the damped least-squares technique[17], the conjugate gradient methods, the biconjugate gradient algorithm[25,26] etc. Another approach[54] is to regularize average slowness (traveltime divided by ray-length) and apparent slowness (traveltime derivative with respect to surface distance) than traveltimes or to apply smoothness constraints or derivative operators to find the simplest structure that fits the data under a given tolerance.

A generally accepted opinion is that 2-D seismic inversion can give an incorrect picture of the subsurface[21] so 3-D inversions are needed in order to put up a better image. In the last two decades, three dimensional seismic methods have been developed but early studies, (Thurber[39], Kanasewich and Chiu[15],) faced many problems due to limitations in computer resources and data coverage. Zelt[47] (1994) inverted simultaneously refraction and reflection travel time data in order to provide a three dimensional starting model. The recent developments in 3D refraction methods[48,3,8,12] have partially solved many problems of the past, but there is still a long way to go, e.g. difficulties still occur when large velocities contrasts exist, wavespeeds cannot be resolved reliably in the main refractor.[16]

CONCLUSIONS

With the help of new mathematical techniques, seismic tomography has been evolved to a widely used technique covering a broad range of applications from global tomography to near surface geophysics. The basis of a tomographic problem is the inversion of a matrix. The requested precision as well as today's technical needs demand us to solve systems of the order of more than, e.g. 10 000 unknowns. The inversion of such matrices cannot be reliably handled with conventional techniques, so new inversion techniques like, e.g. SIRT, have been introduced. Today, to deal with similar problems, modern backprojection techniques are used. Lately, many efforts have been focused on more advanced methods, like the inversion using genetic algorithms.

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