Uniform density u and corresponding I_u - convergence*

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Abstract. *The concept of a uniform density of subsets A of the set N of positive integers was introduced in [1] and [2]. Corresponding I_u - convergence to the notion of uniform density u can be found in [8]. This paper studies I_u - convergence in detail.*

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We recall some known notions. Let $A \subseteq N$. If $m, n \in N$, by $A(m,n)$ we denote the cardinality of the set $A \cap [m,n]$. Numbers

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{A(1,n)}{n}, \qquad \overline{d}(A) = \limsup_{n\to\infty} \frac{A(1,n)}{n}$$

are called the lower and the upper asymptotic density of the set A, respectively. If there exists the limit

$$\limsup_{n\to\infty} \frac{A(1,n)}{n},$$

then $d(A) = \underline{d}(A) = \overline{d}(A)$ is said to be the asymptotic density of A. The uniform density of $A \subseteq N$ was introduced in [1] and [2] as follows: Put

$$a_n = \min_{m\geq 0} A(m+1, m+n), \qquad a^n = \max_{m\geq 0} A(m+1, m+n).$$

It can be shown (see [2]) that the following limits exist

$$\underline{u}(A) = \lim_{n\to\infty} \frac{a_n}{n}, \qquad \overline{u}(A) = \lim_{n\to\infty} \frac{a^n}{n}$$

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and they are called the lower and the upper uniform density of the set A, respectively. If $\underline{u}(A) = \overline{u}(A)$, then $u(A) = \underline{u}(A)$ is called the uniform density of A. It is clear that for each $A \subseteq N$ we have

$$\underline{u}(A) \leq \underline{d}(A) \leq \overline{d}(A) \leq \overline{u}(A). \tag{1}$$

Hence if there exists $u(A)$, then there also exists $d(A)$ and $u(A) = d(A)$. The converse is not true (see *Example 1.*).

The concept of statistical convergence was introduced in [4] (see also [3], [5], [10], [11]) as follows: Let $x = (x_n)_1^\infty$ be a sequence of complex numbers. The sequence x is said to be statistically convergent to a complex number L provided that for every $\epsilon > 0$ we have $d(A_\epsilon) = 0$, where $A_\epsilon = \{n \in N : |x_n - L| \geq \epsilon\}$. If $x = (x_n)_1^\infty$ converges statistically to L, then we write lim - stat $x_n = L$.

A generalized approach to convergence is done in [6] by means of the notion of an ideal I of subsets of N (i.e. I is an additive and hereditary class of sets).

A sequence x is said to be I - convergent to L provided that for every $\epsilon > 0$ the set A_ϵ belongs to I, we write I - lim $x_n = L$. Put $I = I_d = \{A \subset N : d(A) = 0\}$, then I_d - convergence coincides with statistical convergence. Hence lim - stat $x_n = L$ $= I_d$ - lim x_n. In the case $I = I_u = \{A \subset N : u(A) = 0\}$ we obtain I_u - convergence. If $x = (x_n)_1^\infty$ is I_u - convergent to L, we write I_u - lim $x_n = L$.

We can easily verify that if I_u - lim $x_n = L_1$, I_u - lim $y_n = L_2$, then I_u - lim $(x_n + y_n) = L_1 + L_2$ and if a is constant, then I_u - lim $ax_n = aL_1$. By M_1 we denote the set of all I_u - convergent sequences; M_1 is a linear space. Analogously, we have for M_0, the set of all statistically convergent sequences (see [11]). Let c be the set of all convergent sequences. By (1) we have $c \subseteq M_1 \subseteq M_0$.

The following examples show that $c \neq M$ and $M_1 \neq M_0$ even in case of bounded sequences.

Example 1. *Let P be the set of all primes. Define $x_k = 1$ for $k \in P$ and $x_k = 0$ otherwise. Because of $u(P) = 0$ (see [2]), we have that $x = (x_k)_1^\infty$ is I_u - convergent to 0, but not convergent.*

Example 2. *It is easy to see that for the set*

$$A = \bigcup_{k=1}^{\infty} \{10^k + 1, 10^k + 2, \dots, 10^k + k\}$$

we have $d(A) = 0$, $\underline{u}(A) = 0$, $\overline{u}(A) = 1$. Put $x_k = 1$ for $k \in A$ and $x_k = 0$ for $k \notin A$. Then I_d - lim $x_k = 0$, but $x = (x_k)_1^\infty$ is not I_u - convergent.

We recall the notion of strong p - Cesàro convergence and almost convergence. A sequence $x = (x_k)_1^\infty$ is said to be strong p - Cesàro convergent $(0 < p < \infty)$ to a number L if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - L|^p = 0$$

(see [3]). By w_p denote the set of all strong p - Cesàro convergent sequences. A bounded sequence $x = (x_k)_1^\infty$ is almost convergent to a number L if every Banach limit of x is equal to L, which is equivalent to the condition

$$\lim_{p\to\infty} \frac{1}{p} \sum_{i=1}^{p} x_{n+i} = L$$

uniformly in n(see [9], [10], p 59-62). By F we denote the set of all almost convergent sequences.

It is shown in [9] that almost convergence and statistical convergence are not compatible even in the case of bounded sequences.

The following *Theorem 1* shows that in the case of bounded sequences I_u- convergence and almost convergence can be compared.

Theorem 1. *Suppose $x = (x_k)_1^\infty$ is a bounded sequence. If x is I_u - convergent to L, then x is almost convergent to L.*

Proof. Let $p, n \in N$ be arbitrary. We estimate

$$S(n,p) = \Big|\frac{x_{n+1} + x_{n+2} + \ldots + x_{n+p}}{p} - L\Big|.$$

We have

$$S(n,p) \le S^{(1)}(n,p) + S^{(2)}(n,p), \tag{2}$$

where

$$S^{(1)}(n,p) = \frac{1}{p} \sum_{\substack{1 \le j \le p,\\ n+j \in A_\epsilon}} |x_{n+j} - L|,$$

$$S^{(2)}(n,p) = \frac{1}{p} \sum_{\substack{1 \le j \le p,\\ n+j \notin A_\epsilon}} |x_{n+j} - L|.$$

By using the definition of $A_\epsilon = \{n \in N : |x_n - L| \ge \epsilon\}$ we have

$$S^{(2)}(n,p) < \epsilon \qquad \text{for every} \quad n = 1, 2, \ldots . \tag{3}$$

The boundedness of $x = (x_k)_1^\infty$ implies that there exists $M > 0$ such that

$$|x_k - L| \le M \qquad (k = 1, 2, \ldots). \tag{4}$$

Then (4) implies

$$S^{(1)}(n,p) \le M\frac{A_\epsilon(n+1, n+p)}{p} \le M\frac{\max\limits_m A_\epsilon(m+1, n+p)}{p} = M\frac{a^p}{p}.$$

Using the last estimation which holds for every $n = 1, 2, \ldots$ and (2), (3) we obtain the assertion of *Theorem 1.* $\square$

Remark 1. *The converse of the previous theorem does not hold. For instance, let $y = (y_k)_1^\infty$ be the sequence defined by $y_k = 1$ if n is even and $y_k = 0$ if n is odd. The sequence y is almost convergent to 1/2 but it is not I_u - convergent.*

In [3] a connection between strong p - Cesàro convergence and statistical convergence is articulated. In the case of bounded sequences both of these kinds of convergence are equivalent. A similar result can be obtained for I_u - convergence. First of all, we define a new kind of convergence, so-called uniformly strong p - Cesàro convergence, which is a generalization of the notion of strong almost convergence (see [8]).

Definition 1. *The sequence $x = (x_k)_1^\infty$ is said to be uniformly strong p-Cesàro convergent $(0 < p < \infty)$ to a number L if*

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=n+1}^{n+k} |x_i - L|^p = 0$$

uniformly in n.

By uw_p denote the set of all uniformly strong p - Cesàro convergent sequences. It is immediate that $uw_p \subset w_p$ $(0 < p < \infty)$. *Example 2* shows that the inclusion is strict.

Theorem 2.

a) If $0 < p < \infty$ and a sequence $x = (x_k)_1^\infty$ is uniformly strong p - Cesàro convergent to L, then it is I_u - convergent to L.

b) If $x = (x_k)_1^\infty$ is bounded and I_u - convergent to L, then it is uniformly strong p - Cesàro convergent to L for every p, $0 < p < \infty$.

Proof.

a) Let x be uniformly strong p - Cesàro convergent to L, $0 < p < \infty$. Suppose $\epsilon > 0$. Then, for every $n \in N$ we have

$$\sum_{j=1}^{k} |x_{n+j} - L|^p \geq \sum_{\substack{1 \leq j \leq k, \\ |x_{n+j} - L| \geq \epsilon}} |x_{n+j} - L|^p \geq \epsilon^p A_\epsilon(n+1, n+k),$$

and further,

$$\frac{1}{k}\sum_{j=1}^{k} |x_{n+j} - L|^p \geq \epsilon^p \frac{\max_{m\geq 0} A_\epsilon(m+1, m+k)}{k} = \epsilon^p \frac{a^k}{k}$$

for every $n = 1, 2, 3 \ldots$. This implies $\lim_{k\to\infty} \frac{a^k}{k} = 0$, and $u(A_\epsilon) = 0$, so that I_u-lim x_n $= L$.

b) Now, suppose that x is a bounded sequence and I_u - lim $x_n = L$. Let $0 < p < \infty$ and $\epsilon > 0$. According to the assumption, we have

$$u(A_\epsilon) = 0. \tag{5}$$

The boundedness of $x = (x_k)_1^\infty$ implies that there exists $M > 0$ such that $|x_k - L| \leq M$ $(k = 1, 2, \ldots)$. Observe that for every $n \in N$, we have that

$$\begin{aligned}\frac{1}{k}\sum_{j=1}^{k} |x_{n+j} - L|^p &= \frac{1}{k} \sum_{\substack{1 \leq j \leq k, \\ n+j \in A_\epsilon}} |x_{n+j} - L|^p + \frac{1}{k} \sum_{\substack{1 \leq j \leq k, \\ n+j \notin A_\epsilon}} |x_{n+j} - L|^p \\ &\leq M \frac{\max_{m\geq 0} A_\epsilon(m+1, m+k)}{k} + \epsilon^p \leq \epsilon^p + M\frac{a^k}{k}\end{aligned} \tag{6}$$

Using (5) and (6) we obtain $\lim_{k\to\infty} \frac{1}{k}\sum_{j=1}^{k} |x_{n+j} - L|^p = 0$, uniformly in n. □

Corollary 1. *If $x = (x_k)_1^\infty$ is a bounded sequence, then x is I_u - convergent to L if and only if x is uniformly strong p - Cesàro convergent to L for every p, $0 < p < \infty$.*

In [3], [5] and [11] it is shown that statistical convergence can be characterized by the convergence in the usual sense along a great set of indexes, great in the sense of asymptotic density. The following theorem shows that the I_u - convergence can be characterized by the convergence along a great set of indexes, great now being in the sense of uniform density. In [6] it is shown that a similar statement is not true for the I - convergence where I is an arbitrary ideal.

Theorem 3. *A sequence $x = (x_k)_1^\infty$ is I_u - convergent to L if and only if there exists a set*

$$K = \{k_1 < k_2 < \ldots < k_n < \ldots\} \subseteq N$$

such that $u(K) = 1$ and $\lim_{n\to\infty} x_{k_n} = L$.

Proof. If there exists a set with the mentioned properties and ϵ is an arbitrary positive number, we can choose a number $m \in N$ such that for each $n > m$ we have

$$|x_{k_n} - L| < \epsilon. \tag{7}$$

Let $A_\epsilon = \{n \in N : |x_{k_n} - L| \geq \epsilon\}$. Then, on the basis of (7), we have

$$A_\epsilon \subseteq N - \{k_{m+1}, k_{m+2}, \ldots\}$$

where on the right-hand side there is a set with the uniform density 0. Therefore, $u(A_\epsilon) = 0$; hence I_u - $\lim x_k = L$.

Now suppose that a sequence $x = (x_k)_1^\infty$ is I_u - convergent to L. Let K_j be the complement of the set $A_{1/j}$ for $j = 1, 2, \ldots$,

$$K_j = N - \left\{n \in N : |x_{k_n} - L| \geq \frac{1}{j}\right\}$$

Then, by the definition of I_u - convergence, we have

$$u(K_j) = 1 \qquad \text{for} \quad j = 1, 2, \ldots .$$

By the definition of K_j we have

$$K_1 \supseteq K_2 \supseteq \ldots \supseteq K_j \supseteq K_{j+1} \supseteq \ldots . \tag{8}$$

Let us choose an arbitrary number $s_1 \in K_1$. By the definition of K_j there exists a number $s_2 > s_1$, $s_2 \in K_2$ such that for each $n \geq s_2$ we have

$$\frac{\min_{m\geq 0} K_2(m+1, m+n)}{n} > \frac{1}{2}.$$

Again on the basis of the definition of K_j there exists a number $s_3 > s_2$, $s_3 \in K_3$, such that for each $n \geq s_3$ we have

$$\frac{\min_{m\geq 0} K_3(m+1, m+n)}{n} > \frac{2}{3}.$$

In this manner we can construct an increasing sequence of positive integers

$$s_1 < s_2 < \ldots < s_j < \ldots$$

such that $s_j \in K_j$ and that for each $n \geq s_j$ we have

$$\frac{\min_{m\geq 0} K_j(m+1, m+n)}{n} > 1 - \frac{1}{j} \qquad \text{for} \quad j = 1, 2, \ldots . \tag{9}$$

Define K as follows:
if $1 \leq k \leq s_1$, then $k \in K$; suppose that $j \geq 1$ and that $s_j < k \leq s_{j+1}$, then $k \in K$ if and only if $k \in K_j$. Let $K = \{k_1 < k_2 < \ldots < k_n < \ldots\}$. According to (8) and (9), for each n, $s_j \leq n < s_{j+1}$ we have

$$\frac{\min_{m\geq 0} K(m+1, m+n)}{n} \geq \frac{\min_{m\geq 0} K_j(m+1, m+n)}{n} > 1 - \frac{1}{j}.$$

From this it is obvious that $u(K) = 1$.

Let $\epsilon > 0$ be given and select j such that $1/j < \epsilon$. Let $n \geq s_j$, $n \in K$. Then there exists a number $r \geq j$ such that $s_r \leq n < s_{r+1}$. According to the definition of K, $n \in K_r$, we have

$$|x_n - L| < \frac{1}{r} \leq \frac{1}{j} < \epsilon.$$

Thus $|x_n - L| < \epsilon$ for each $n \geq s_j$, $n \in K$. Hence $\lim_{n\to\infty} x_{k_n} = L$. □

Corollary 2. *If a sequence $x = (x_k)_1^\infty$ is uniformly strong p - Cesàro convergent $(0 < p < \infty)$ or I_u - convergent to L, then there exists a sequence $y = (y_k)_1^\infty$ convergent to L and a sequence $z = (z_k)_1^\infty$ I_u - convergent to 0, such that $x = y + z$ and $u(B) = 0$, where $B = \{n \in N : z_k \neq 0\}$.*

Proof. First observe that if x is uniformly strong p - Cesàro convergent to L $(0 < p < \infty)$, then x is I_u - convergent to L. From the previous theorem there exists a set

$$K = \{k_1 < k_2 < \ldots < k_n < \ldots\} \subseteq N$$

such that $u(K) = 1$ and $\lim_{n\to\infty} x_{k_n} = L$. We define y and z as follows: If $k \in K$, put $y_k = z_k$ and $z_k = 0$, and if $k \notin K$, we put $y_k = L$ and $z_k = x_k - L$. □

Remark 2. *If a sequence $x = (x_k)_1^\infty$ is uniformly strong p - Cesàro convergent $(0 < p < \infty)$ or I_u - convergent to L, then x has a subsequence which converges to L.*

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