A fixed point theorem for multi-maps satisfying an implicit relation on metrically convex metric spaces

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Abstract. *In this paper, we give a fixed point theorem for multi-valued mapping satisfying an implicit relation on metrically convex metric spaces. This result extends and generalizes some fixed point theorem in the literature.*

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1. Introduction

Let (X, d) be a metric space. Then X is said to be metrically convex if for every pair $x, y \in X, x \neq y$, there is a point $z \in X$ such that $d(x, y) = d(x, z) + d(z, y)$. We need the following lemma in the sequel.

Lemma 1 [[1]]. *Let K be a non-empty and closed subset of a metrically convex metric space X. Then for any $x \in K$ and $y \notin K$, there exists a point $z \in \partial K$ such that $d(x, y) = d(x, z) + d(z, y)$, where ∂K denotes the boundary of K.*

Let $CB(X)$ denote the family of all non-empty closed and bounded subsets of X. Denote for $A, B \in CB(X)$

$$D(A, B) = \inf\{d(a, b) : a \in A, b \in B\}, \tag{1}$$

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\} \tag{2}$$

and

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\}. \tag{3}$$

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Note that $D(A,B) \leq H(A,B) \leq \delta(A,B)$. Function H is a metric on $CB(X)$ and is called a Hausdorff metric. It is well known that if X is a complete metric space, then so is the metric space $(CB(X), H)$.

Itoh [4] proved a fixed point theorem for non-self maps $F : K \to CB(X)$ satisfying certain contraction condition in terms of Hausdorff metric H on $CB(X)$ under the boundary condition $F(\partial K) \subset K$. Rhoades [7] generalized this result to a wider class of non-self multi-maps on K. Recently Dhage [2] has proved a fixed point theorem for non-self multi-maps on K satisfying a slightly stronger contraction condition than that in Rhoades [7] and under a weaker boundary condition. In *Section 2* of this paper we give an implicit relation and some examples for this relation. In *Section 3*, we prove a fixed point theorem for non-self multi-maps on K satisfying an implicit relation.

2. Implicit relation

Implicit relations on metric space have been used in many articles (see [3], [5], [6], [8]).

Let R_+ be the set of all non-negative real numbers and let $\mathcal{T}$ be the set of all continuous functions $T : R_+^5 \to R$ satisfying the following conditions:

T_1 : $T(t_1, ..., t_5)$ is non-decreasing in t_1 and non-increasing in $t_2, ..., t_5$.

T_2 : there exist two constants $a, b \geq 0$, $2a + 3b < 1$ such that the inequality

$$T(u, v, v, w, v + w) \leq 0 \tag{4}$$

implies $u \leq \max\{(a + b)v + bw, (a + b)w + bv\}$.

T_3 : $T(u, 0, 0, u, u) > 0, T(u, 0, u, 0, u) > 0$ and $T(u, u, 0, 0, 2u) > 0$, $\forall u > 0$.

Remark 1. *Note that, if $u = w$ in T_2, then the inequality $T(u, v, v, w, v+w) \leq 0$ implies $u \leq \dfrac{a+b}{1-b}v$.*

Now we give some examples.

Example 1. *Let $T(t_1, ...t_5) = t_1 - \alpha \max\{t_2, t_3, t_4\} - \beta t_5$, where $\alpha, \beta \geq 0$ and $2\alpha + 3\beta < 1$.*

T_1 : Obvious. T_2 : Let $T(u, v, v, w, w + v) = u - \alpha \max\{w, v\} - \beta(w + v) \leq 0$. Thus $u \leq \max\{(\alpha + \beta)v + \beta w, (\alpha + \beta)w + \beta v\}$. T_3 : $T(u, 0, 0, u, u) = T(u, 0, u, 0, u) = u(1-\alpha-\beta) > 0$ and $T(u, u, 0, 0, 2u) = u(1-\alpha-2\beta) > 0, \forall u > 0$. Therefore $T \in \mathcal{T}$.

Example 2. *Let $T(t_1, ..., t_5) = t_1 - m \max\{t_2, t_3, t_4, \frac{1}{2}t_5\}$, where $0 \leq m < \frac{1}{2}$.*

T_1 : Obvious. T_2 : Let $T(u, v, v, w, w + v) = u - m \max\{w, v\} \leq 0$. Thus $u \leq \max\{mw, mv\}$ and so T_2 is satisfying with $a = m$, $b = 0$. T_3 : $T(u, 0, 0, u, u) = T(u, 0, u, 0, u) = T(u, u, 0, 0, 2u) = u(1 - m) > 0, \forall u > 0$. Therefore $T \in \mathcal{T}$.

Example 3. *Let $T(t_1, ..., t_5) = t_1 - (\alpha t_2 + \beta t_3 + \gamma t_4)$, where $\alpha, \beta, \gamma \geq 0, 2\alpha + 2\beta + \gamma < 1$ and $\alpha + \beta - \gamma \geq 0$.*

T_1 : Obvious. T_2 : Let $T(u, v, v, w, w + v) = u - (\alpha v + \beta v + \gamma w) \leq 0$. Thus $u \leq (\alpha+\beta)v+\gamma w \leq \max\{(\alpha+\beta)v+\gamma w, (\alpha+\beta)w+\gamma v\}$ and so T_2 is satisfying with

$a=\alpha+\beta-\gamma, b=\gamma$. $T_3: T(u,0,0,u,u)=u(1-\gamma)>0, T(u,0,u,0,u)=u(1-\beta)>0$ and $T(u,u,0,0,2u)=u(1-\alpha)>0, \forall u>0$. Therefore $T\in\mathcal{T}$.

Example 4. *Let $T(t_1,...t_5)=t_1-\alpha t_2-\beta\max\{t_3,t_4\}-\gamma t_5$, where $\alpha,\beta,\gamma\geq 0$ and $2\alpha+2\beta+3\gamma<1$.*

T_1 : Obvious. T_2 : Let $T(u,v,v,w,w+v)=u-\alpha v-\beta\max\{w,v\}-\gamma(w+v)\leq 0$. Thus $u\leq\max\{(\alpha+\beta+\gamma)v+\beta w,(\alpha+\beta+\gamma)w+\beta v\}$ and so T_2 is satisfying with $a=\alpha+\beta+\gamma, b=\gamma$. $T_3: T(u,0,0,u,u)=T(u,0,u,0,u)=u(1-\beta-\gamma)>0$ and $T(u,u,0,0,2u)=u(1-\alpha-2\gamma)>0, \forall u>0$. Therefore $T\in\mathcal{T}$.

3. Main result

Now we give our main theorem.

Theorem 1. *Let (X,d) be a metrically convex complete metric space and K a non-empty closed subset of X. Let $F: K\to CB(X)$ be a multi-map satisfying*

$$T(\delta(Fx,Fy),d(x,y),D(x,Fx),D(y,Fy),D(x,Fy)+D(y,Fx))\leq 0, \tag{5}$$

for all $x,y\in K$, where $T\in\mathcal{T}$. Further, if $Fx\cap K\neq\phi$ for each $x\in\partial K$, then F has a fixed point $p\in K$ such that $Fp=\{p\}$ and F is continuous at p in the Hausdorff metric on X.

Proof. Let be arbitrary and consider a sequence $\{x_n\}$ in K as follows: Let $x_0=x$ and take a point $x_1\in Fx_0\cap K$ if $Fx_0\cap K\neq\phi$. Otherwise choose a point $x_1\in\partial K$ such that

$$d(x_0,x_1')=d(x_0,x_1)+d(x_1,x_1') \tag{6}$$

for some $x_1'\in Fx_0\subset X\backslash K$. Similarly, pick $x_2\in Fx_1\cap K$ if $Fx_1\cap K\neq\phi$, otherwise choose a point $x_2\in\partial K$ such that

$$d(x_1,x_2')=d(x_1,x_2)+d(x_2,x_2') \tag{7}$$

for some $x_2'\in Fx_1\subset X\backslash K$. Continuing this way we have

$$x_{n+1}\in Fx_1\cap K \text{ if } Fx_1\cap K\neq\phi, \tag{8}$$

or $x_{n+1}\in\partial K$ satisfying

$$d(x_n,x_{n+1})+d(x_{n+1},x_{n+1}')=d(x_n,x_{n+1}') \tag{9}$$

for some $x_{n+1}'\in Fx_n\subset X\backslash K$.

By the construction of $\{x_n\}$ we can write

$$\{x_n\}=P\cup Q\subset K, \tag{10}$$

where

$$P=\{x_n\in\{x_n\}: x_n\in Fx_{n-1}\} \tag{11}$$

and

$$Q=\{x_n\in\{x_n\}: x_n\in\partial K, x_n\notin Fx_{n-1}\}. \tag{12}$$

Then for any two consecutive terms x_n, x_{n+1} of the sequence $\{x_n\}$, we observe that there are only the following three possibilities:

(i) $x_n, x_{n+1} \in P$,

(ii) $x_n \in P, x_{n+1} \in Q$, and

(iii) $x_n \in Q$, $x_{n+1} \in P$.

First we show that $\{x_n\}$ is a Cauchy sequence in K. Now for any $x_n, x_{n+1} \in \{x_n\}$, we have the following estimates:

Case 1: Suppose that $x_n, x_{n+1} \in P$. Now since $x_{n-1}, x_n \in K$, we can use the inequality (5), then we have

$$\begin{aligned} T\,(\delta(Fx_{n-1}, Fx_n), d(x_{n-1}, x_n), D(x_{n-1}, Fx_{n-1}),& \\ D(x_n, Fx_n), D(x_{n-1}, Fx_n) + D(x_n, Fx_{n-1})) \leq 0& \end{aligned} \tag{13}$$

and so

$$T(d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_n)+d(x_n, x_{n+1})) \leq 0. \tag{14}$$

From *Remark 1* there exist two constants $a, b \geq 0$, $2a+3b < 1$ such that $d(x_n, x_{n+1}) \leq \dfrac{a+b}{1-b} d(x_{n-1}, x_n)$, where $\dfrac{a+b}{1-b} < \frac{1}{2}$ since $2a + 3b < 1$.

Case 2: Let $x_n \in P$ and $x_{n+1} \in Q$. Then $d(x_n, x_{n+1}) + d(x_{n+1}, x'_{n+1}) = d(x_n, x'_{n+1})$ for some $x'_{n+1} \in Fx_n$. Clearly,

$$\begin{cases} d(x_n, x_{n+1}) \leq d(x_n, x'_{n+1}) \\ d(x_n, x'_{n+1}) \leq \delta(Fx_{n-1}, Fx_n). \end{cases} \tag{15}$$

Now following arguments similar to those in Case 1, we obtain

$$d(x_n, x'_{n+1}) \leq \frac{a+b}{1-b} d(x_{n-1}, x_n). \tag{16}$$

From (15) and (16) it follows that

$$d(x_n, x_{n+1}) \leq \frac{a+b}{1-b} d(x_{n-1}, x_n). \tag{17}$$

Case 3: Suppose that $x_n \in Q$ and $x_{n+1} \in P$. Note that then $x_{n-1} \in P$ and there is a point $x'_n \in Fx_{n-1}$ such that

$$d(x_{n-1}, x_n) + d(x_n, x'_n) = d(x_{n-1}, x'_n). \tag{18}$$

Now,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, x'_n) + d(x'_n, x_{n+1}) \\ &\leq d(x_{n-1}, x'_n) + \delta(Fx_{n-1}, Fx_n). \end{aligned} \tag{19}$$

On the other hand, since $x_{n-1}, x_n \in K$, we can use inequality (5), then we have

$$\begin{aligned} T\,(\delta(Fx_{n-1}, Fx_n), d(x_{n-1}, x_n), D(x_{n-1}, Fx_{n-1}), D(x_n, Fx_n),& \\ D(x_{n-1}, Fx_n) + D(x_n, Fx_{n-1})) \leq 0.& \end{aligned} \tag{20}$$

Thus we have

$$T\,(d(x'_n,x_{n+1}),d(x_{n-1},x_n),d(x_{n-1},x'_n),d(x_n,x_{n+1}),\\ d(x_{n-1},x_{n+1})+d(x_n,x'_n))\le 0. \tag{21}$$

Using (18) we have

$$T\,(d(x'_n,x_{n+1}),d(x_{n-1},x'_n),d(x_{n-1},x'_n),d(x_n,x_{n+1}),\\ d(x_{n-1},x_n)+d(x_n,x_{n+1})+d(x_n,x'_n))\le 0 \tag{22}$$

and so

$$T\,(d(x'_n,x_{n+1}),d(x_{n-1},x'_n),d(x_{n-1},x'_n),d(x_n,x_{n+1}),\\ d(x_{n-1},x'_n)+d(x_n,x_{n+1}))\le 0. \tag{23}$$

From T_2 there exist two constants $a,b\ge 0$, $2a+3b<1$ such that

$$d(x'_n,x_{n+1})\le \max\left\{\begin{array}{c}(a+b)d(x_{n-1},x'_n)+bd(x_n,x_{n+1}),\\ (a+b)d(x_n,x_{n+1})+bd(x_{n-1},x'_n)\end{array}\right\}. \tag{24}$$

Therefore using (19) we have

$$d(x_n,x_{n+1})\le d(x_{n-1},x'_n)+\max\left\{\begin{array}{c}(a+b)d(x_{n-1},x'_n)+bd(x_n,x_{n+1}),\\ (a+b)d(x_n,x_{n+1})+bd(x_{n-1},x'_n)\end{array}\right\} \tag{25}$$

Now from (16) in Case 2 applied to $n-1$, we have

$$d(x_{n-1},x'_n)\le \frac{a+b}{1-b}d(x_{n-2},x_{n-1}) \tag{26}$$

and hence from (25)

$$\begin{aligned} d(x_n,x_{n+1}) &\le \frac{a+b}{1-b}d(x_{n-2},x_{n-1})\\ &\quad+\max\left\{\begin{array}{c}\dfrac{(a+b)^2}{1-b}d(x_{n-2},x_{n-1})+bd(x_n,x_{n+1}),\\ (a+b)d(x_n,x_{n+1})+\dfrac{b(a+b)}{1-b}d(x_{n-2},x_{n-1})\end{array}\right\}\\ &=\max\left\{\begin{array}{c}\dfrac{(a+b)(1+a+b)}{1-b}d(x_{n-2},x_{n-1})+bd(x_n,x_{n+1}),\\ (a+b)d(x_n,x_{n+1})+\dfrac{(1+b)(a+b)}{1-b}d(x_{n-2},x_{n-1})\end{array}\right\}. \end{aligned}$$

This implies

$$d(x_n,x_{n+1})\le \max\{\frac{(a+b)(1+a+b)}{(1-b)^2},\frac{(1+b)(a+b)}{(1-b)(1-a-b)}\}d(x_{n-2},x_{n-1}). \tag{27}$$

Note that $q=\max\{\frac{(a+b)(1+a+b)}{(1-b)^2},\frac{(1+b)(a+b)}{(1-b)(1-a-b)}\}<1$. To see this, $2a+3b<1$ yields

$$\begin{aligned} &a+b<1-2b-a\\ &\Rightarrow a+b+ab+b^2<1-2b-a+ab+b^2\\ &\Rightarrow \frac{(a+b+ab+b^2)}{1-2b-a+ab+b^2}<1\\ &\Rightarrow \frac{(a+b)(1+b)}{(1-b)(1-a-b)}<1. \end{aligned} \tag{28}$$

Similarly, again from $2a + 3b < 1$ we have

$$\begin{aligned}
&1 > 3b \\
&\Rightarrow \frac{3}{2} > \frac{1}{1-b} \\
&\Rightarrow 1 > \frac{1}{2(1-b)} + \frac{1}{4} \\
&\Rightarrow 1 > (\frac{1}{1-b} + \frac{1}{2})\frac{1}{2} \\
&\Rightarrow 1 > (\frac{1}{1-b} + \frac{a+b}{1-b})\frac{a+b}{1-b} \\
&\Rightarrow 1 > \frac{(1+a+b)(a+b)}{(1-b)^2}.
\end{aligned} \tag{29}$$

Now for any $n \in N$, we have

$$d(x_{2n}, x_{2n+1}) \leq qd(x_{2n-2}, x_{2n}) \leq q^n d(x_0, x_1). \tag{30}$$

Since n is arbitrary, one has

$$d(x_n, x_{n+1}) \leq q^n d(x_0, x_1). \tag{31}$$

Then from Cases 1-3, it easily follows that $\{x_n\}$ is a Cauchy sequence in K. As K is closed, it is complete and hence $\lim_n x_n = p$ exists. We show that p is a fixed point of F. Without loss of generality, we may assume that $x_{n+1} \in Fx_n$ for some $n \in N$. Then using (5) we have

$$T(\delta(Fx_n, Fp), d(x_n, p), D(x_n, Fx_n), D(p, Fp), D(x_n, Fp) + D(p, Fx_n)) \leq 0, \tag{32}$$

and letting $n \to \infty$ we have

$$T(D(p, Fp), 0, 0, D(p, Fp), D(p, Fp)) \leq 0. \tag{33}$$

From T_3 we have $D(p, Fp) = 0$ and so $p \in Fp$.

Further, we have

$$T(\delta(Fp, Fp), d(p, p), D(p, Fp), D(p, Fp), D(p, Fp) + D(p, Fp)) \leq 0, \tag{34}$$

and so

$$T(\delta(Fp, Fp), 0, 0, 0, 0) \leq 0. \tag{35}$$

Again from T_1 and T_3 we have $\delta(Fp, Fp) = 0$ and so $Fp = \{p\}$.

To show the uniqueness of p, let $q(\neq p)$ be another fixed point of F. Then

$$T(\delta(Fp, Fq), d(p, q), D(p, Fp), D(q, Fq), D(p, Fq) + D(q, Fp)) \leq 0, \tag{36}$$

and so

$$T(d(p, q), d(p, q), 0, 0, 2d(p, q)) \leq 0. \tag{37}$$

Again from T_3 we have $p = q$.

Finally, we prove the continuity of F at p. Let $\{z_n\} \subset X$ be any sequence such that $z_n \to p$ as $n \to \infty$. Now

$$T(\delta(Fz_n, Fp), d(z_n, p), D(z_n, Fz_n), D(p, Fp), D(z_n, Fp) + D(p, Fz_n)) \leq 0 \quad (38)$$

and letting $n \to \infty$ we have

$$T(\lim_n H(Fz_n, Fp), 0, \lim_n H(Fp, Fz_n), 0, \lim_n H(Fp, Fz_n)) \leq 0. \quad (39)$$

From T_3 we have $\lim_n H(Fz_n, Fp) = 0$, showing that F is continuous at p. This completes the proof. □

Remark 2. *Theorem 1 of [2] follows from* Example 1 *and* Theorem 1*.*

Remark 3. *We can have some new results from other examples and* Theorem 1.

References

[1] L. M. Blumenthal, *Theory and Applications of Distance Geometry*, Clarendon Press, Oxford, 1943.

[2] B. C. Dhage, *A fixed point theorem for non-self multi-maps in metric spaces*, Comment. Math. Univ. Carolinae **40**(1999), 251-258.

[3] M. Imdad, S. Kumar, M. S. Khan, *Remarks on some fixed point theorems satisfying implicit relations*, Rad. Math. **11**(2002), 135-143.

[4] S. Itoh, *Multi-valued generalized contraction and fixed point theorems*, Comment. Math. Univ. Carolinae **18**(1977), 247-248.

[5] V. Popa, *Some fixed point theorems for compatible mappings satisfying an implicit relation*, Demonstratio Math. **32**(1999), 157-163.

[6] V. Popa, *A general coincidence theorem for compatible multivalued mappings satisfying an implicit relation*, Demonstratio Math. **33**(2000), 159-164.

[7] B. E. Rhoades, *A fixed point theorem for a multi-valued non-self mappings*, Comment. Math. Univ. Carolinae **37**(1996), 401-404.

[8] S. Sharma, B. Desphande, *On compatible mappings satisfying an implicit relation in common fixed point consideration*, Tamkang J. Math. **33**(2002), 245-252.