Semi-aposyndetic continuum X is metrizable if and only if it admits a Whitney map for $C(X)$

Ivan Lončar*

Abstract. *The main purpose of this paper is to prove the metrizability of semi-aposyndetic continuum X which admits a Whitney map for $C(X)$.*

Key words: *metrizability, Whitney map*

AMS subject classifications: 54F15, 54C30

Received February 1, 2005 Accepted April 19, 2006

1. Introduction

All spaces in this paper are compact Hausdorff and all mappings are continuous. The weight of a space X is denoted by $w(X)$.

A *generalized arc* is a Hausdorff continuum with exactly two non-separating points (end points) x, y. Each separable arc is homeomorphic to the closed interval $I = [0, 1]$.

We say that a space X is *arcwise connected* if for every pair x, y of points of X there exists a generalized arc L with end points x, y.

Let X be a space. We define its hyperspaces as the following sets:

$$\begin{array}{l} 2^X = \{F \subseteq X : F \text{ is closed and nonempty}\}, \\ C(X) = \{F \in 2^X : F \text{ is connected}\}, \\ C^2(X) = C(C(X)), \\ X(n) = \{F \in 2^X : F \text{ has at most } n \text{ points}\},\ n \in \mathbb{N}. \end{array} \tag{1}$$

For any finitely many subsets S_1, ..., S_n, let

$$\langle S_1, ..., S_n \rangle = \left\{ F \in 2^X : F \subset \bigcup_{i=1}^{n} S_i, \text{ and } F \cap S_i \neq \emptyset, \text{ for each } i \right\}. \tag{2}$$

The topology on 2^X is the Vietoris topology, i.e., the topology with a base

$$\{< U_1, ..., U_n >: U_i \text{ is an open subset of } X \text{ for each } i \text{ and each } n < \infty \},$$

*Faculty of Organization and Informatics Varaždin, University of Zagreb, Pavlinska 2, HR-42 000 Varaždin, e-mail: `ivan.loncar1@vz.htnet.hr`

and $C(X)$, $X(n)$ are subspaces of 2^X. Moreover, $X(1)$ is homeomorphic to X.

Let X and Y be the spaces and let $f : X \to Y$ be a mapping. Define $2^f : 2^X \to 2^Y$ by $2^f(F) = f(F)$ for $F \in 2^X$. By [9, p. 170, Theorem 5.10] 2^f is continuous and $2^f(C(X)) \subset C(Y)$, $2^f(X(n)) \subset Y(n))$. The restriction $2^f|C(X)$ is denoted by $C(f)$.

Let Λ be a subspace of 2^X. By a *Whitney map* for Λ [10, p. 24, (0.50)] we will mean any mapping $g : \Lambda \to [0, +\infty)$ satisfying

a) if $A, B \in \Lambda$ such that $A \subset B$ and $A \neq B$, then $g(A) < g(B)$ and

b) $g(\{x\}) = 0$ for each $x \in X$ such that $\{x\} \in \Lambda$.

If X is a metric continuum, then there exists a Whitney map for 2^X and $C(X)$ ([10, pp. 24-26], [3, p. 106]). On the other hand, if X is non-metrizable, then it admits no Whitney map for 2^X [1]. It is known that there exist non-metrizable continua which admit and ones which do not admit a Whitney map for $C(X)$ [1].

The notion of an irreducible mapping was introduced by Whyburn [12, p. 162]. If X is a continuum, a surjection $f : X \to Y$ is *irreducible* provided no proper subcontinuum of X maps onto all of Y under f. Some theorems for the case when X is semi-locally-connected are given in [12, p. 163].

A mapping $f : X \to Y$ is said to be *hereditarily irreducible* [10, p. 204, (1.212.3)] provided that for any given subcontinuum Z of X, no proper subcontinuum of Z maps onto $f(Z)$.

A mapping $f : X \to Y$ is *light* (*zero-dimensional*) if all fibers $f^{-1}(y)$ are hereditarily disconnected (zero-dimensional or empty) [2, p. 450], i.e., if $f^{-1}(y)$ does not contain any connected subsets of cardinality larger that one ($\dim f^{-1}(y) \leq 0$). Every zero-dimensional mapping is light, and in the realm of mappings with compact fibers the two classes of mappings coincide. Every hereditarily irreducible mapping is light. If $f : X \to Y$ is monotone and hereditarily irreducible, then f is one-to-one.

We shall use the notion of inverse system as in [2, pp. 135-142]. An inverse system is denoted by $\mathbf{X} = \{X_a, p_{ab}, A\}$.

An element $\{x_a\}$ of the Cartesian product $\prod\{X_a : a \in A\}$ is called a *thread* of $\mathbf{X}$ if $p_{ab}(x_b) = x_a$ for any $a, b \in A$ satisfying $a \leq b$. The subspace of $\prod\{X_a : a \in A\}$ consisting of all threads of $\mathbf{X}$ is called the limit of the inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ and is denoted by $\lim \mathbf{X}$ or by $\lim\{X_a, p_{ab}, A\}$ [2, p. 135].

We say that an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is *σ-directed* if for each sequence $a_1, a_2, ..., a_k, ...$ of the members of A there is an $a \in A$ such that $a \geq a_k$ for each $k \in \mathbb{N}$.

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces with the natural projections $p_a : \lim \mathbf{X} \to X_a$, for $a \in A$. Then $2^{\mathbf{X}} = \{2^{X_a}, 2^{p_{ab}}, A\}$, $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ and $\mathbf{X}(n) = \{X_a(n), 2^{p_{ab}}| X_b(n), A\}$ form inverse systems.

Lemma 1. *[5, Lemma 2]. Let* $X = \lim \mathbf{X}$. *Then* $2^X = \lim 2^{\mathbf{X}}$, $C(X) = \lim C(\mathbf{X})$ *and* $X(n) = \lim \mathbf{X}(n)$.

The following theorem is an external characterization of non-metric continua which admit a Whitney map for $C(X)$ [7, p. 398, Theorem 2.3].

Theorem 1. *Let X be a non-metric continuum. Then X admits a Whitney map for $C(X)$ if and only if for each σ-directed inverse system* $\mathbf{X} = \{X_a, p_{ab}, A\}$

of continua which admit Whitney maps for $C(X_a)$ and $X = \lim \mathbf{X}$ there exists a cofinal subset $B \subset A$ such that for every $b \in B$ the projection $p_b : \lim \mathbf{X} \to X_b$ is hereditarily irreducible.

In the sequel we shall use the following result [11, p. 226, Exercise 11.52].

Lemma 2. *If X is a continuum and if A and B are mutually disjoint subcontinua of X, then there is a component K of $X \setminus (A \cup B)$ such that $\mathrm{Cl}K \cap A \neq \emptyset$ and $\mathrm{Cl}K \cap B \neq \emptyset$.*

2. Preliminary results and definitions

The theorems stated in this section will be used in proving the main theorems in the section below.

We shall use the notion of a network of a topological space.

A family $\mathcal{N} = \{M_s : s \in S\}$ of subsets of a topological space X is a *network* for X if for every point $x \in X$ and any neighbourhood U of x there exists an $s \in S$ such that $x \in M_s \subset U$ [2, p. 170]. The *network weight* of a space X is defined as the smallest cardinal number of the form card($\mathcal{N}$), where $\mathcal{N}$ is a network for X; this cardinal number is denoted by $nw(X)$.

Theorem 2. *[2, p. 171, Theorem 3.1.19]. For every compact space X we have $nw(X) = w(X)$.*

The following theorem is the main theorem of this section.

Theorem 3. *Let X be a continuum. Then $w(C(X) \setminus X(1)) = \aleph_0$ if and only if $w(X) = \aleph_0$.*

Proof. If $w(X) = \aleph_0$, then $w(C(X)) = \aleph_0$. Hence, $w(C(X) \setminus X(1)) = \aleph_0$. Conversely, if $w(C(X) \setminus X(1)) = \aleph_0$, then there exists a countable base $\mathcal{B} = \{B_i : i \in \mathbb{N}\}$ of $C(X) \setminus X(1)$. For each B_i let $C_i = \cup\{x \in X : x \in B, B \in B_i\}$, i.e. the union of all continua B contained in B_i.

Claim 1. *The family $\{C_i : i \in N\}$ is a network of X*. Let X be a point of X and let U be an open subsets of X such that $x \in U$. There exists an open set V such that $x \in V \subset \mathrm{Cl}V \subset U$. Let K be a component of $\mathrm{Cl}V$ containing x. By Boundary Bumping Theorem [11, p. 73, Theorem 5.4] K is non-degenerate and, consequently, $K \in C(X) \setminus X(1)$. Now, $\langle U \rangle \cap (C(X) \setminus X(1))$ is a neighbourhood of K in $C(X) \setminus X(1)$. It follows that there exists a $B_i \in \mathcal{B}$ such that $K \in B_i \subset \langle U \rangle \cap (C(X) \setminus X(1))$. It is clear that $C_i \subset U$ and $x \in C_i$ since $x \in K \subset U$. Hence, the family $\{C_i : i \in N\}$ is a network of X.

Claim 2. $nw(X) = \aleph_0$. Apply Claim 1 and the fact that $\mathcal{B}$ is countable.

Claim 3. $w(X) = \aleph_0$. By Claim 1 we have $nw(X) = \aleph_0$. Moreover, by *Theorem 2* $w(X) = \aleph_0$. □

Corollary 1. *If X is a continuum, then $w(C^2(X) \setminus C(X)(1)) = \aleph_0$ if and only if $w(X) = \aleph_0$.*

Proof. By *Theorem 3* $w(C(X)) = \aleph_0$. This means that $w(X) = \aleph_0$ since X is homeomorphic to $X(1) \subset C(X)$. □

3. Main theorem

The concept of aposyndesis was introduced by Jones in [4].

A continuum is said to be *semi-aposyndetic* [3, p. 238, Definition 29.1], if for every $p \neq q$ in X, there exists a subcontinuum M of X such that $\mathrm{Int}_X(M)$ contains one of the points p, q and $X \setminus M$ contains the other one. Each locally connected continuum is semi-aposyndetic.

Now we shall prove the main theorem of this paper.

Theorem 4. *Semi-aposyndetic continuum X is metrizable if and only if it admits a Whitney map for $C(X)$.*

Proof. If X is metrizable, then it admits a Whitney map for $C(X)$ ([10, pp. 24-26], [3, p. 106]). Conversely, let X admit a Whitney map $\mu : C(X) \to [0, +\infty)$. Suppose that X is non-metrizable. The remaining part of the proof is broken into several steps.

Step 1. *There exists a σ-directed directed inverse system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ of metric compact spaces X_a such that X is homeomorphic to* $\lim \mathbf{X}$ [7, p. 397, Theorem 1.8].

Step 2. *There exists a cofinal subset $B \subset A$ such that for every $b \in B$ the projection $p_b : \lim \mathbf{X} \to X_b$ is hereditarily irreducible.* This follows from *Theorem 1.*

Step 3. *If* $\lim \mathbf{X}$ *is semi-aposyndetic, then for every pair C, D of disjoint non-degenerate subcontinua of* $\lim X$ *there exists a non-degenerate subcontinuum $E \subset \lim X$ such that $C \cap E \neq \emptyset \neq D \cap E$ and $(C \cup D) \setminus E \neq \emptyset$.* We shall consider two cases.

a) If either $\mathrm{Int}_X(C) \neq \emptyset$ or $\mathrm{Int}_X(D) \neq \emptyset$, then it suffices to apply *Lemma 2* to the union $C \cup D$ and obtain a component K of $X \setminus (C \cup D)$ such that $\mathrm{Cl}K \cap C \neq \emptyset$ and $\mathrm{Cl}K \cap D \neq \emptyset$. Then $E = \mathrm{Cl}K$ is a continuum with properties $C \cap E \neq \emptyset \neq D \cap E$ and $(C \cup D) \setminus E \neq \emptyset$ since $\mathrm{Int}_X(C) \cap E = \emptyset$ or $\mathrm{Int}_X(D) \cap E = \emptyset$.

b) Assume that $\mathrm{Int}_X(C) = \emptyset$ and $\mathrm{Int}_X(D) = \emptyset$. There exist $x, y \in C$ such that $x \neq y$. Moreover, there exists a subcontinuum M of $\lim \mathbf{X}$ such that $\mathrm{Int}_{\lim \mathbf{X}}(M)$ contains one of the points x, y and $X \setminus M$ contains the other one since X is semi-aposyndetic. Suppose that $x \in \mathrm{Int}_X(M)$ and $y \in X \setminus M$. If $M \cap D \neq \emptyset$, then we set $E = M$ and we have the continuum E such that $C \cap E \neq \emptyset \neq D \cap E$ and $(C \cup D) \setminus E \neq \emptyset$ since $y \in X \setminus M$. Suppose that $M \cap D = \emptyset$. Applying *Lemma 2* to the union $C \cup D \cup M$ we obtain a component K of $X \setminus (C \cup D \cup M)$ such that $\mathrm{Cl}K \cap (C \cup M) \neq \emptyset$ and $\mathrm{Cl}K \cap D \neq \emptyset$. It is clear that $x \notin \mathrm{Cl}K$. If $\mathrm{Cl}K \cap C \neq \emptyset$, then we set $E = \mathrm{Cl}K$ and obtain a continuum E such that $C \cap E \neq \emptyset \neq D \cap E$ and $(C \cup D) \setminus E \neq \emptyset$ since $x \notin \mathrm{Cl}K$. If $\mathrm{Cl}K \cap C = \emptyset$, then $\mathrm{Cl}K \cap M \neq \emptyset$ and we set $E = \mathrm{Cl}K \cup M$. Now $y \notin E$, $C \cap E \neq \emptyset \neq D \cap E$ and $(C \cup D) \setminus E \neq \emptyset$.

Step 4. *Every $C(p_b) : C(\lim \mathbf{X}) \to C(p_b)(C(\lim \mathbf{X})) \subset C(X_a)$ is one-to-one.* Consider the inverse system $C(\mathbf{X}) = \{C(X_a),\ C(p_{ab}),\ A\}$ whose limit is $C(\lim X)$ (*Lemma 1*). From *Theorem 1* it follows that there exists a subset B cofinal in A such that the projections p_b are hereditarily irreducible and $C(p_b)$ are light for every $b \in B$, see [10, p. 204, (1.212.3)]. Since $\lim \mathbf{X}$ is homeomorphic to $\lim\{X_b, p_{bc}, B\}$, we may assume that $B = A$. Let $Y_a = C(p_a)(C(X))$. Furthermore, $C(p_a)^{-1}(X_a(1)) = X(1)$ since from the hereditary irreducibility of p_a it follows that no non-degenerate subcontinuum of X maps under p_a onto a point. We infer that $C(p_a)^{-1}[Y_a \setminus X_a(1)] = C(X) \setminus X(1)$. Let us prove that the restriction $C(p_a)|[C(X) \setminus X(1)]$ is one-to-one. Suppose that $C(p_a)|[C(X) \setminus X\ (1)]$ is not one-to-one. Then there exist a continuum C_a in X_a and two continua C, D in X such that $p_a(C) = p_a(D) = C_a$. It is impossible that $C \subset D$ or $D \subset C$ since p_a is hereditarily irreducible. Otherwise, if $C \cap D \neq \emptyset$, then for the continuum $Y = C \cup D$ we have that C and D are subcontinua of Y and $p_a(Y) = p_a(C) = p_a(D) = C_a$ which is impossible since p_a is hereditarily irreducible. We infer that $C \cap D = \emptyset$. There exists a non-degenerate subcontinuum $E \subset \lim \mathbf{X}$ such that $C \cap E \neq \emptyset \neq D \cap E$ and $(C \cup D) \setminus E \neq \emptyset$ since $\lim X$ is semi-aposyndetic (Step 3). Moreover, we may assume that $E \cap C \neq C$ and $E \cap D \neq D$. Now $p_a(E \cup D) = p_a(E)$ which is impossible since p_a is hereditarily irreducible. It follows that the restriction $P_a = C(p_a)|(C(X) \setminus X(1))$ is one-to-one and closed [2, p. 95, Proposition 2.1.4].

Step 5. *$C(X) \setminus X(1)$ is metrizable and $w(C(X) \setminus X(1)) \leq \aleph_0$.* From Step 4 it follows that P_a is a homeomorphism and $C(X) \setminus X(1)$ is metrizable. Moreover, $w(C(X) \setminus X(1)) \leq \aleph_0$ since Y_a as a compact metrizable space is separable and, consequently, second-countable [2, p. 320].

Step 6. *X is metrizable.* Apply *Theorem 3.*

Step 6 contradicts the assumption that X is non-metrizable. The proof is completed. □

Let us observe that in the proof of *Theorem 4* the semi-aposyndesis is used only in Step 3 to ensure, for every pair C, D of disjoint non-degenerate subcontinua of $\lim \mathbf{X}$, the existence of a non-degenerate subcontinuum $E \subset \lim X$ such that $C \cap E \neq \emptyset \neq D \cap E$ and $(C \cup D) \setminus E \neq \emptyset$. The existence of such continuum can be ensured in other classes of continua.

An easy proof of the following lemma is left to the reader.

Lemma 3. *If X is an arcwise connected continuum, then for every pair C, D of disjoint non-degenerate subcontinua of X there exists a non-degenerate subcontinuum $E \subset X$ such that $C \cap E \neq \emptyset \neq D \cap E$ and $(C \cup D) \setminus E \neq \emptyset$.*

Theorem 5. *An arcwise connected continuum X is metrizable if and only if it admits a Whitney map for $C(X)$.*

Proof. Repeat the proof of *Theorem 4* replacing b) in Step 3 by *Lemma 3.* □

An *arboroid* is a hereditarily unicoherent arcwise connected continuum. A metrizable arboroid is a *dendroid*.

Corollary 2. *Let X be an arboroid. Then X is metrizable if and only if it admits a Whitney map for $C(X)$.*

Proof. Apply *Theorem 5.* □

We say that a continuum X admits a Whitney map for $C^2(X)$ if $C(X)$ admits a Whitney map for $C(C(X))$. It is known that if X is a continuum, then $C(X)$ is arcwise connected [8, p. 1209, Theorem]. Hence, using *Theorem 5*, we obtain the following corollary.

Corollary 3. *A continuum X is metrizable if and only if it admits a Whitney map for $C^2(X)$.*

Proof. If X admits a Whitney map for $C^2(X) = C(C(X))$, then $C(X)$ admits a Whitney map for $C(C(X) = C^2(X)$. From *Theorem 5* it follows that $C(X)$ is metrizable. Hence, X is metrizable. □

It is known [2, p. 171, Corollary 3.1.20] that if a compact space X is the countable union of its subspaces $X_n, n \in \mathbb{N}$, such that $w(X_n) \leq \aleph_0$, then $w(X) \leq \aleph_0$. Using this fact and theorems proved in the previous section we obtain the following theorems.

Theorem 6. *Let a continuum X be the countable union of its semi-aposyndetic subcontinua. Then X is metrizable if and only if it admits a Whitney map for $C(X)$.*

Theorem 7. *If a continuum X is the countable union of its arcwise connected subcontinua, then X is metrizable if and only if it admits a Whitney map for $C(X)$.*

A continuum X is said to be *σ-rim-semi-aposyndetic* provided for each $x \in X$ and for each open set U containing x there exists an open set V such that $x \in V \subset U$ and the boundary $\mathrm{Bd}(V)$ is the countable union of its semi-aposyndetic subcontinua.

Theorem 8. *If a continuum X is σ-rim-semi-aposyndetic, then it is metrizable if and only if it admits a Whitney map for $C(X)$.*

Proof. It is known that if X is metrizable, then it admits a Whitney map for $C(X)$ [10, pp. 24-26], [3, p. 106]. Conversely, let X be a σ-rim-semi-aposyndetic continuum which admits a Whitney map for $C(X)$. We shall prove that X is rim-metrizable. Let $x \in X$ be a point of X and let U be an open set which contains x. There exists an open set V such that $x \in V \subset U$ and the boundary $\mathrm{Bd}(V) = \cup\{C_i : i \in \mathbb{N}\}$ of semi-aposyndetic continua C_i. If $\mu : C(X) \to [0, \infty)$ is a Whitney map, then the restriction $\mu|C(C_i)$ is a Whitney map. From *Theorem 4* it follows that every C_i is metrizable since every C_i is a semi-aposyndetic continuum. Using [2, p. 171, Corollary 3.1.20] we conclude that $\mathrm{Bd}(V)$ is metrizable. Finally, from [6, p. 5, Theorem 11] it follows that X is metrizable. □

References

[1] J. J. Charatonik, W. J. Charatonik, *Whitney maps—a non-metric case*, Colloq. Math. **83**(2000), 305-307.

[2] R. Engelking, *General Topology*, PWN, Warszawa, 1977.

[3] A. Illanes, S. B. Nadler, Jr., *Hyperspaces: Fundamentals and Recent advances*, Marcel Dekker, New York-Basel, 1999.

[4] F. B. Jones, *Aposyndetic continua and certain boundary problems*, Amer. J. Math. **63**(1941), 545-553.

[5] Y. Kodama, S. Spież, T. Watanabe, *On shape of hyperspaces*, Fund. Math. **100**(1979), 59-67.

[6] I. Lončar, *A note on a Whitney map for continua*, Math. Commun. **6**(2001), 1-9.

[7] I. Lončar, *A fan X admits a Whitney map for $C(X)$ iff it is metrizable*, Glas. Mat. Ser. III 38 (58) (2003), 395-411.

[8] M. M. McWaters, *Arcs, semigroups and hyperspaces*, Canad. J. Math. **20** (1968), 1207-1210.

[9] E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc. **71**(1951), 152-182.

[10] S. B. Nadler, *Hyperspaces of Sets*, Marcel Dekker, Inc., New York, 1978.

[11] S. B. NADLER, *Continuum Theory*, Marcel Dekker, Inc., New York, 1992.

[12] G. T. WHYBURN, *Analytic Topology*, Vol. 28, Amer. Math. Soc., Providence, R.I., 1963.