Better butterfly theorem in the isotropic plane

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Abstract. *A real affine plane A_2 is called an isotropic plane I_2, if in A_2 a metric is induced by an absolute $\{f, F\}$, consisting of the line at infinity f of A_2 and a point $F \in f$.*

Better butterfly theorem is one of the generalisations of the well-known butterfly theorem ([1],[4]). In this paper the better butterfly theorem has been adapted for the isotropic plane and its validity in I_2 has been proved.

Key words: *isotropic plane, better butterfly theorem*

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1. Isotropic plane

Let $P_2(\mathbf{R})$ be a real projective plane, f a real line in P_2, and $A_2 = P_2 \backslash f$ the associated affine plane. The *isotropic plane* $I_2(\mathbf{R})$ is a real affine plane A_2 where the metric is introduced with a real line $f \subset P_2$ and a real point F incidental with it. The ordered pair $\{f, F\}$, $F \in f$ is called the *absolute figure* of the isotropic plane $I_2(\mathbf{R})$ ([2], [3]). In the affine model, where

$$x = x_1/x_0, \quad y = x_2/x_0, \tag{1}$$

the absolute figure is determined by the *absolute line* $f \equiv x_0 = 0$, and the *absolute point* F (0:0:1).

We will first define some terms and point out some properties of triangles and circles in I_2 that are going to be used further on. The geometry of I_2 could be seen for example in Sachs [2], or Strubecker [3].

All straight lines through the point F are called *isotropic straight lines.* A triangle in I_2 is called *allowable* if none of its sides is isotropic.

An *isotropic circle* (*parabolic circle* or simply *circle*) is a regular 2^{nd} order curve in $P_2(\mathbf{R})$ which touches the absolute line f in the absolute point F. In I_2 there exists a three parametric family of circles, given by $y = Rx^2 + \alpha x + \beta$, $R \neq 0$, $\alpha, \beta \in \mathbf{R}$. Each circle can be reduced to the normal form $y = Rx^2$. Two circles $k_i \equiv y = R_i x^2 + \alpha_i x + \beta_i, (i = 1, 2)$ are called *congruent* if $R_1 = R_2$; they are called *concentric* if $\alpha_1 = \alpha_2$.

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2. Better butterfly theorem

Euclidean version
Let there be 2 concentric circles with the common centre O. A line crosses the two circles at points P, Q and $P^{'}$, $Q^{'}$, M being the common midpoint of PQ and $P^{'}Q^{'}$. Through M, draw two lines $AA^{'}B^{'}B$ and $CC^{'}D^{'}D$ and connect $AD^{'}$, $A^{'}D$, $BC^{'}$, $B^{'}C$. Let X, Y, Z, W be the points of intersection of $PP^{'}Q^{'}Q$ with $AD^{'}$, $B^{'}C$, $A^{'}D$, and $BC^{'}$, respectively. Then

$$\frac{1}{MX}+\frac{1}{MZ}=\frac{1}{MY}+\frac{1}{MW}$$

The proof is to be found in [4].

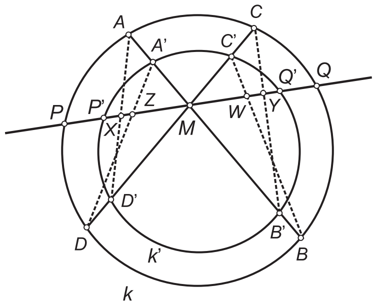

Figure 1. *Better butterfly theorem*

Isotropic version
This statement remains valid in the isotropic plane provided *concentric circles* are replaced by *congruent and concentric circles* and the corresponding equation for the signed lengths reads:

$$\frac{1}{d(M,X)}+\frac{1}{d(M,Z)}=-\frac{1}{d(M,Y)}-\frac{1}{d(M,W)} \tag{2}$$

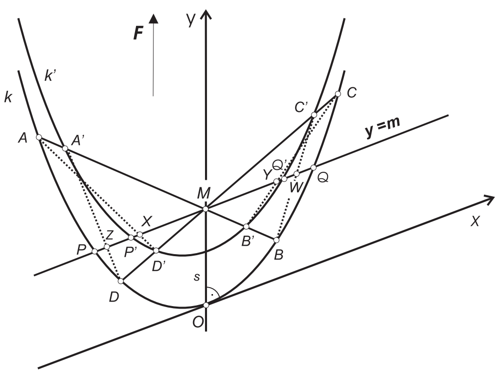

Figure 2. *Better butterfly theorem in I_2*

The proof depends on the following lemma:

Lemma 1. *In the allowable triangle* $\triangle RST$ *let* RU *be a non-isotropic straight line connecting the vertex* R *with some point* U *on the opposite side* ST *of* R. *Let's introduce angles* $\alpha = \angle(UR, RS)$, *and* $\beta = \angle(TR, RU)$. *Then*

$$\frac{\alpha+\beta}{d(U,R)} = \frac{\alpha}{d(T,R)} - \frac{\beta}{d(R,S)} \tag{3}$$

Proof. Without loos of generality, we can assume that the vertex coordinates are as follows: $S(0,0)$, $T(t_1,0)$, $R(r_1,r_2)$, and $U(u_1,0)$, with $t_1 \neq r_1 \neq u_1$ (see *Figure 3*).

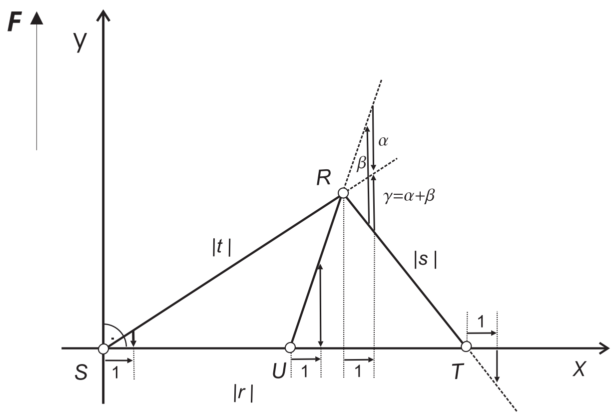

Figure 3.

For angles α, β and $\alpha+\beta$ we have:

$$\alpha = \angle(UR, RS) = u(RS) - u(UR) = \frac{s_2 - r_2}{s_1 - r_1} - \frac{r_2 - u_2}{r_1 - u_1}, \tag{4}$$

$$\beta = \angle(TR, RU) = u(RU) - u(TR) = \frac{u_2 - r_2}{u_1 - r_1} - \frac{r_2 - t_2}{r_1 - t_1}, \tag{5}$$

and

$$\alpha+\beta = \angle(TR, RS) = u(RS) - u(TR) = \frac{s_2 - r_2}{s_1 - r_1} - \frac{r_2 - t_2}{r_1 - t_1}. \tag{6}$$

Inserting (4), (5), and (6) in (3), together with $d(U,R) = r_1 - u_1$, $d(T,R) = r_1 - t_1$, $d(R,S) = -r_1$ an equality is obtained. □

Proof of the theorem. Let k and k' be two congruent and concentric circles in I_2, $k \equiv y = Rx^2$, $k' \equiv y = Rx^2 + s$, $s \neq 0$ and let M be the midpoint of the chord $\overrightarrow{PQ}$ of k. Let us choose the coordinate system as shown (in the affine model) in *Figure 2*, i.e. the tangent on the circle k parallel to the chord $\overrightarrow{PQ}$ as the x-axis, and the isotropic straight line through M as the y-axis.

Choosing $M(0,m)$, for the chord $\overrightarrow{PQ}$ we have $\overrightarrow{PQ} \equiv y = m$, and $P(p_1,m)$, $Q(q_1,m)$, $P'(p_1',m)$, $Q'(q_1',m)$. Note that $p_1^2 = q_1^2 = \frac{m}{R}$, and $p_1'^2 = q_1'^2 = \frac{m-s}{R}$.

Let $A(a_1, Ra_1^2)$, $B(b_1, Rb_1^2)$, with $a_1 \neq b_1$, and $C(c_1, Rc_1^2)$, $D(d_1, Rd_1^2)$, with $c_1 \neq d_1$, be the four points on the circle k, and $A^{'}(a_1^{'}, R{a_1^{'}}^2 + s)$, $B^{'}(b_1^{'}, R{b_1^{'}}^2 + s)$, with $a_1^{'} \neq b_1^{'}$, $C^{'}(c_1^{'}, R{c_1^{'}}^2 + s)$, $D^{'}(d_1^{'}, R{d_1^{'}}^2 + s)$, with $c_1^{'} \neq d_1^{'}$, the four points on the circle $k^{'}$. Let us introduce angles $\alpha = \angle(PM, MA) = \angle(QM, MB)$ and $\beta = \angle(DM, MP) = \angle(CM, MQ)$. Applying *Lemma 2* to allowable triangles $\triangle AD'M$, $\triangle A'DM$, $\triangle B'CM$, and $\triangle BC'M$ successively one gets

$$\frac{\alpha+\beta}{d(X,M)} = \frac{\alpha}{d(D',M)} - \frac{\beta}{d(M,A)} \quad (7)_1, \qquad \frac{\alpha+\beta}{d(Z,M)} = \frac{\alpha}{d(D,M)} - \frac{\beta}{d(M,A')} \quad (7)_2,$$

$$\frac{\alpha+\beta}{d(Y,M)} = \frac{\alpha}{d(C,M)} - \frac{\beta}{d(M,B')} \quad (7)_3, \qquad \frac{\alpha+\beta}{d(W,M)} = \frac{\alpha}{d(C',M)} - \frac{\beta}{d(M,B)} \quad (7)_4.$$

From $(7)_1$ and $(7)_2$ we obtain:

$$(\alpha+\beta)\Big(\frac{1}{d(X,M)} + \frac{1}{d(Z,M)}\Big) = \alpha\Big(\frac{1}{d(D',M)} + \frac{1}{d(D,M)}\Big) - \beta\Big(\frac{1}{d(M,A)} + \frac{1}{d(M,A')}\Big). \tag{8}$$

Analogously, $(7)_3$ and $(7)_4$ yield that

$$(\alpha+\beta)\Big(\frac{1}{d(Y,M)} + \frac{1}{d(W,M)}\Big) = \alpha\Big(\frac{1}{d(C',M)} + \frac{1}{d(C,M)}\Big) - \beta\Big(\frac{1}{d(M,B)} + \frac{1}{d(M,B')}\Big). \tag{9}$$

Using $d(Y,M) = -d(M,Y)$ and $d(W,M) = -d(M,W)$, the latter becomes

$$(\alpha+\beta)\Big(\frac{1}{d(M,Y)} + \frac{1}{d(M,W)}\Big) = -\alpha\Big(\frac{1}{d(C',M)} + \frac{1}{d(C,M)}\Big) + \beta\Big(\frac{1}{d(M,B)} + \frac{1}{d(M,B')}\Big). \tag{10}$$

Showing that the right-hand sides in (8) and (10) are equal, i.e.

$$\alpha\Big(\frac{1}{d(D',M)} + \frac{1}{d(D,M)}\Big) - \beta\Big(\frac{1}{d(M,A)} + \frac{1}{d(M,A')}\Big) = -\alpha\Big(\frac{1}{d(C',M)} + \frac{1}{d(C,M)}\Big) + \beta\Big(\frac{1}{d(M,B)} + \frac{1}{d(M,B')}\Big) \tag{11}$$

the theorem will be proved.

Using the point coordinates we can rewrite the identity given in (11) to the following form

$$\beta\Big(\frac{1}{a_1} + \frac{1}{a_1^{'}}\Big) + \beta\Big(\frac{1}{b_1} + \frac{1}{b_1^{'}}\Big) = -\alpha\Big(\frac{1}{c_1} + \frac{1}{c_1^{'}}\Big) - \alpha\Big(\frac{1}{d_1} + \frac{1}{d_1^{'}}\Big),$$

which is equivalent to

$$\beta\Big(\frac{a_1+b_1}{a_1 b_1}\Big) + \beta\Big(\frac{a_1^{'}+b_1^{'}}{a_1^{'} b_1^{'}}\Big) = -\alpha\Big(\frac{c_1+d_1}{c_1 d_1}\Big) - \alpha\Big(\frac{c_1^{'}+d_1^{'}}{c_1^{'} d_1^{'}}\Big). \tag{12}$$

Besides, knowing that $\overrightarrow{AB}$ is a chord through M, the following relations are obtained:

$$\begin{aligned} M, A, B \quad \text{collinear points} &\Leftrightarrow \det \begin{pmatrix} 0 & m & 1 \\ a_1 & Ra_1^2 & 1 \\ b_1 & Rb_1^2 & 1 \end{pmatrix} = 0 \\ &\Leftrightarrow -m(a_1 - b_1) - Ra_1 b_1 (a_1 - b_1) = 0 \\ &\Leftrightarrow a_1 b_1 = -\frac{m}{R}. \end{aligned} \tag{13}$$

Analogously, for $\overrightarrow{CD}$ being a chord through M, we get that

$$c_1 d_1 = -\frac{m}{R}. \tag{14}$$

Relations given in (13) and (14) can be reached using the following lemma:

Lemma 2. *Let k be a circle in I_2, a point $P \in I_2$, $P \notin k$, and S_1, S_2 two points of intersection of a non-isotropic straight line g through P with k. The product $f(P) := d(P, S_1) \cdot d(P, S_2)$ does not depend on the line g, but only on k and P.*
The proof is given in [2, p. 32].

So,

$$a_1 b_1 = d(M, A) \cdot d(M, B) = d(M, P) \cdot d(M, Q) = p_1 q_1 = p_1(-p_1) = -p_1^2 = -\frac{m}{R},$$

and

$$c_1 d_1 = d(M, C) \cdot d(M, D) = d(M, P) \cdot d(M, Q) = p_1 q_1 = p_1(-p_1) = -p_1^2 = -\frac{m}{R}.$$

Analogously,

$$\begin{aligned} a_1' b_1' &= d\ (M, A') \cdot d(M, B') = d(M, P') \cdot d(M, Q') \\ &= p_1' q_1' = p_1'(-p_1') = -p_1'^2 = -\frac{m - s}{R}, \end{aligned} \tag{15}$$

$$\begin{aligned} c_1' d_1' &= d(M, C') \cdot d(M, D') = d(M, P') \cdot d(M, Q') \\ &= p_1' q_1' = p_1'(-p_1') = -p_1'^2 = -\frac{m - s}{R}. \end{aligned} \tag{16}$$

Since A, A', and M as well as A, M, and B' are collinear points the relations $-m(a_1 - a_1') - a_1 a_1' R(a_1 - a_1') + a_1 s = 0$, $-m(a_1 - b_1') - a_1 b_1' R(a_1 - b_1') + a_1 s = 0$ respectively, are valid. Subtracting these relations we get

$$m(a_1' - b_1') + a_1 R(a_1'^2 - b_1'^2) - a_1^2 R(a_1' - b_1') = 0.$$

The chord $\overrightarrow{A'B'}$ being a non-isotropic line allows us to rewrite the latter equation as $m + a_1 R(a_1' + b_1') - a_1^2 R = 0$, wherefrom, using (13), we finally obtain that

$$a_1 + b_1 = a_1' + b_1'. \tag{17}$$

Following the similar procedure, it can be shown that

$$c_1 + d_1 = c_1^{'} + d_1^{'} \tag{18}$$

holds as well. For the oriented angles α, β, introduced at the beginning, we have as follows:

$$\alpha = \angle(PM, MA) = u(MA) - u(PM) = \frac{a_2 - m_2}{a_1 - m_1} - \frac{m_2 - p_2}{m_1 - p_1} = \frac{Ra_1^2 - m}{a_1}, \tag{19}$$

$$\beta = \angle(CM, MQ) = u(MQ) - u(CM) = \frac{q_2 - m_2}{q_1 - m_1} - \frac{m_2 - c_2}{m_1 - c_1} = \frac{m - Rc_1^2}{c_1}. \tag{20}$$

Finally, using the relations given in (13), (14),..., and (20) in (12) one gets that

$$\begin{aligned}(12) &\Leftrightarrow \beta(a_1 + b_1) = -\alpha(c_1 + d_1)\\ &\Leftrightarrow \Big(\frac{m - Rc_1^2}{c_1}\Big)\Big(a_1 - \frac{m}{Ra_1}\Big) = \Big(\frac{Ra_1^2 - m}{a_1}\Big)\Big(c_1 - \frac{m}{Rc_1}\Big)\\ &\Leftrightarrow \frac{(m - Rc_1^2)(Ra_1^2 - m)}{Ra_1c_1} = \frac{(m - Rc_1^2)(Ra_1^2 - m)}{Ra_1c_1}.\end{aligned}$$

□

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