Lacunary statistical cluster points of sequences

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Abstract. *In this note we introduce the concept of a lacunary statistical cluster (l.s.c.) point and prove some of its properties in finite dimensional Banach spaces. We develop the method suggested by S. Pehlivan and M.A. Mamedov [20] where it was proved that under some conditions optimal paths have the same unique stationary limit point and stationary cluster point. We also show that the set* Γ_x^θ *of l.s.c. points is nonempty and compact.*

Key words: *natural density, lacunary statistical cluster points, statistically bounded sequence, statistical cluster points*

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1. Introduction and background

The theory about the notion of statistical convergence, which is a generalization of the usual notion of convergence for real valued sequences, follows parallel lines of development to that of the usual theory of convergence. Recall that a real-valued sequence (x_k) is said to be statistically convergent to a number l provided that the set $\{k : |x_k - l| \geq \epsilon\}$ has natural density 0 (see [9] and [6]). If $x = (x_k)$ is statistically convergent to l, then there is a convergent sequence $y = (y_k)$ such that y is convergent to l and the set $\{k : x_k \neq y_k\}$ has natural density 0, i.e., $x_k = y_k$ "for almost all" k [10]. Statistical convergence of a sequence x has been studied by many authors including Fast [7], Fridy [10], Fridy and Miller [11], Connor [2], Connor and Kline [3], Salat [22], Pehlivan and Fisher [19], Maddox [16] and Kolk [14].

Fridy [12] introduced the concept of statistical limit points and statistical cluster points of real number sequences and studied some properties of the sets of statistical limit and cluster points. A number $l \in \mathbb{R}$ is called a statistical limit point of a sequence $x = (x_n)$ if there is a set $\{n_1 < n_2 < \ldots < n_k < \ldots\} \subseteq \mathbb{N}$, the asymptotic

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density of which is not zero (i.e., it is greater than zero or does not exist), such that $\lim_{k\to\infty} x_{n_k} = l$. Let Λ_x denote the set of statistical limit points of x. This concept of statistical limit points is extended in [18] to the concept of $T-$statistical limit points, T being a non-negative regular matrix. Pehlivan and Mamedov [20] have proved that all optimal paths have the same unique statistical cluster point which is also a statistical limit point. This notion turns out to be a very useful and interesting tool in turnpike theory [17]. Recently, Aytar et al. introduced and discussed the concept of statistical limit inferior and limit superior for sequences of fuzzy numbers [1] .

In [15] topological properties of the set Λ_x of all statistical limit points of x are investigated and the relation of Λ_x to distribution functions of x is established. The set Λ_x is equal to the set of discontinuity points of a distribution function of x.

By a lacunary sequence we mean an increasing sequence of positive integers $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. We write $I_r = (k_{r-1}, k_r]$ [8]. Let θ be a lacunary sequence. The sequence x is said to be lacunary statistically convergent to l provided that for every $\epsilon > 0$

$$\lim_r {h_r}^{-1} |\{k \in I_r : |x_k - l| \geq \epsilon\}| \;=\; 0,$$

where the vertical bars indicate the number of elements in the enclosed set. In this case we write $S_\theta - \lim x = l$ or $x_n \to l(S_\theta)$. Fridy and Orhan [13], give some examples to illustrate the difference between lacunary statistical convergence and statistical convergence. A subset K of $\mathbb{N}$ has $\theta-$density if

$$\delta_\theta(K) = \lim_r \frac{|K \cap I_r|}{h_r}$$

exits.

The concept of a lacunary statistical cluster point is a special case of a $T-$statistical cluster point as introduced by J. Connor and J. Kline [3]; for T one has to take the matrix (t_{nk}) with $t_{nk} = 1/h_n$ if $k_{n-1} < k \leq k_n$ and 0 else; see also in [5] comparisons are made between the set Λ_x of statistical limit points of x and the set Λ_x^θ of lacunary statistical limit points of x.

Miller [18] has studied a measure theoretical subsequence characterization of lacunary statistical convergence and gave an example of lacunary sequence θ and a sequence x such that the analogue of the characterization of statistical convergence does not carry over to lacunary statistical convergence.

Let X be a finite dimensional Banach space, let $x = (x_k)$ be an $X-$valued sequence, and $\gamma \in X$. The sequence (x_k) is norm statistically convergent to γ provided that $\delta(\{k : \|x_k - \gamma\| \geq \epsilon\}) = \lim_{n\to\infty} n^{-1} |\{k \leq n \,:\, \|x_k - \gamma\| \geq \epsilon\} = 0$ for all $\epsilon > 0$ [4].

Let X be a finite dimensional space and let Y be any closed subset of X. Let $\rho(Y, \gamma)$ stand for the distance from a point γ to the closed set Y, where

$$\rho(Y, \gamma) = \min_{y \in Y} \|y - \gamma\|.$$

Throughout this paper, let X be a finite dimensional Banach space. The purpose of this note is to introduce the concept of a lacunary statistical cluster (l.s.c.) point and to prove some properties of the set of l.s.c. points in finite dimensional spaces.

2. Statistical cluster points in the finite dimensional spaces

In this section we give some properties of the set of statistical cluster points in finite dimensional spaces. We will use some notions from analysis.

Definition 1. *A point $\gamma \in X$ is called a statistical cluster point(s.c.p.) if for every $\alpha > 0$*

$$\limsup_{n\to\infty} \frac{1}{n}|\{k \le n : \|x_k - \gamma\| < \alpha\}| > 0. \tag{1}$$

We will denote the set of s.c.p. of the sequence $x = (x_k)$ for every $k \in \mathbb{N}$ by $\Gamma_x = \{\gamma \in X \,:\, \gamma\, s.c.p.\}$. The concept of a statistical cluster point in $\mathbb{R}$ (or $\mathbb{C}$) is given by Fridy [12], and extended to finite dimensional spaces by Pehlivan et al. [21].

An $X-$valued sequence (x_k) is said to be statistically convergent to a closed subset Y, provided that the set $\{k \in \mathbb{N} \,:\, \rho(Y, x_k) \ge \alpha\}$ for every $\alpha > 0$, instead of being finite, has natural density 0.

We say that the set Y is a minimal closed set satisfying the following property

$$\lim_{n\to\infty} \frac{1}{n}|\{k \le n \,:\, \rho(Y, x_k) \ge \alpha\}| = 0 \text{ for every } \alpha > 0. \tag{2}$$

The set $G = \cap_\sigma Y_\sigma$ is the smallest nonempty closed set for which holds (2), [21]. To see this, we define a sequence (x_k) in $X = \mathbb{R}$, the set of real numbers with the usual norm, as follows: $x_k = 1$ if k is square and $x = 0$ otherwise. Now we take the closed set $Y = [0, 1]$. Observe that for all k

$$\rho(Y, x_k) = \min_{y\in Y} |y - x_k|.$$

Hence for any $\epsilon > 0$ we have the empty set

$$\{k \in \mathbb{N} \,:\, \rho(Y, x_k) \ge \epsilon\} = \emptyset.$$

Since $\delta(\emptyset) = 0$, we immediately conclude that x is statistically convergent to Y. Now we take another closed set $G = [-1, 1]$. Then we get

$$\{k \in \mathbb{N} \,:\, \rho(G, x_k) \ge \epsilon\} = \emptyset.$$

Since $\delta(\emptyset) = 0$, we say x is also statistically convergent to G. We could find infinitely many closed sets which x is statistically convergent to each of them. Consequently, the intersection of the members of an arbitrary family of closed sets is closed. We know that $\Gamma_x = \{0\}$ and the set of limit points $L_x = \{0, 1\}$ of this sequence. Hence $\Gamma_x = \{0\}$ is the smallest nonempty closed set for this example.

The sequence $x = (x_k)$ is said to be statistically bounded if

$$\delta(\{k \in \mathbb{N} \,:\, \|x_k\| > M\}) = 0$$

for some $M > 0$.

The following theorem gives a relationship between Γ_x and the smallest nonempty closed set which satisfies (2).

Theorem 1. *Let $x = (x_k)$ be a statistically bounded sequence. The smallest nonempty closed set G which satisfies (2) is Γ_x, that is $G = \Gamma_x$.*

Proof. Suppose that $\gamma \in G$ and $\gamma \notin \Gamma_x$ then given $\epsilon > 0$

$$\limsup_{n\to\infty} \frac{1}{n}|\{k \le n : \|x_k - \gamma\| < \epsilon\}| = 0.$$

Define $\overset{\circ}{S}_{\frac{\epsilon}{2}}(\gamma) = \{y : \|y - \gamma\| < \frac{\epsilon}{2}\}$ for $0 < \alpha < \frac{\epsilon}{2}$. Consider the set $\tilde{G} = G \setminus \overset{\circ}{S}_{\frac{\epsilon}{2}}(\gamma)$. It is clear that $\tilde{G}$ is a compact set.

$$\{k \le n : \rho(\tilde{G}, x_k) \ge \alpha\} \subset \{k \le n : \rho(G, x_k) \ge \alpha\} \cup \{k \le n : \|x_k - \gamma\| < \epsilon\}.$$

Therefore,

$$\frac{1}{n}|\{k \le n : \rho(\tilde{G}, x_k) \ge \alpha\}| \le \frac{1}{n}|\{k \le n : \rho(G, x_k) \ge \alpha\}| + \frac{1}{n}|\{k \le n : \|x_k - \gamma\| < \epsilon\}|.$$

for infinitely many n. Hence $\lim_{n\to\infty} \frac{1}{n}|\{k \le n : \rho(\tilde{G}, x_k) \ge \alpha\}| = 0$. Then for this $\tilde{G}$, we obtain $\tilde{G} \subset G$. This contradicts that G is the smallest closed set. Thus we have $\gamma \in \Gamma_x$. This means that $G \subset \Gamma_x$. Conversely, suppose that $\gamma \in \Gamma_x$ and $\gamma \notin G$. By $S_\epsilon(G) = \{y \in X : \rho(G, y) \le \epsilon\}$ we will denote a closed $\epsilon-$neighbourhood of the closed set G. There exists $\epsilon > 0$ such that $S_\epsilon(\gamma) \cap S_\epsilon(G) = \emptyset$. Then $\{k \le n : \|x_k - \gamma\| < \epsilon\} \subset \{k \le n : \rho(G, x_k) \ge \epsilon\}$ since density of the right-hand set is zero,

$$\lim_{n\to\infty} \frac{1}{n}|\{k \le n : \|x_k - \gamma\| < \epsilon\} = 0.$$

It follows that γ is not in Γ_x and this contradiction shows that we have $\gamma \in G$. Hence $\Gamma_x \subset G$. Consequently, $\Gamma_x = G$. □

Corollary 1. *For every $\alpha > 0$*

$$\lim_{n\to\infty} \frac{1}{n}|\{k \le n : \rho(\Gamma_x, x_k) \ge \alpha\}| = 0.$$

We note that if the sequence x is not statistically bounded then *Corollary 1* is not true.

We can give an example in $\mathbb{R}$. Define the sequence $x = (x_k)$ by $x_k = 1$ if $k = 2n - 1, n = 1, 2, \ldots$ and $x_k = k$ otherwise. $\Gamma_x = \{1\}$ but, for every $\alpha > 0$,

$$\delta\{k \le n : \rho(\Gamma_x, x_k) \ge \alpha\} = \frac{1}{2} > 0.$$

Definition 2. *An $X-$valued sequence (x_k) is said to be $\Gamma-$ statistically convergent to the closed subset $\Gamma_x \subset X$ if for every $\alpha > 0$,*

$$\lim_{n\to\infty} \frac{1}{n}|\{k \le n : \rho(Y, x_k) \ge \alpha\}| = 0. \tag{3}$$

3. Lacunary statistical cluster points in the finite dimensional spaces

In this section we give some properties of the set of l.s.c points in finite dimensional spaces. Now we give a definition of l.s.c points in finite dimensional spaces.

Definition 3. *A point γ is a lacunary statistical cluster (l.s.c) point of the $X-$valued sequence (x_k), if for every $\epsilon > 0$*

$$\limsup_{r\to\infty} {h_r}^{-1}|\{k \in I_r : \|x_k - \gamma\| < \epsilon\}| > 0.$$

By Γ_x^θ we denote the set of all l.s.c. points of the X-valued sequence x.

Lemma 1. *Let θ be lacunary sequence and $\Gamma_x^\theta \neq \emptyset$. Then Γ_x^θ is a closed set.*

Proof. Let $\gamma_n \in \Gamma_x^\theta$ and $\gamma_n \to \gamma$. We will show that $\gamma \in \Gamma_x^\theta$. Let $\epsilon > 0$ be a given positive number. There exists a number n' such that $\|\gamma_{n'} - \gamma\| < \epsilon/2$. Since $\gamma_{n'} \in \Gamma_x^\theta$ by *Definition 3*, we have

$$\limsup_{r\to\infty} {h_r}^{-1}|\{k \in I_r : \|x_k - \gamma_{n'}\| < \epsilon/2\}| > 0$$

and so

$$\{k \in I_r : \|x_k - \gamma\| < \epsilon\} \supset \{k \in I_r : \|x_k - \gamma_{n'}\| < \epsilon/2\}.$$

Therefore

$$\limsup_{r\to\infty} {h_r}^{-1}|\{k \in I_r : \|x_k - \gamma\| < \epsilon\}| > 0$$

which implies $\gamma \in \Gamma_x^\theta$. □

Lemma 2. *Let θ be lacunary sequence and Γ_x^θ a set of l.s.c. points of the X-valued sequence x. Let $M \subset X$ be a bounded closed set. If*

$$\limsup_{r\to\infty} {h_r}^{-1}|\{k \in I_r : x_k \in M\}| > 0 \tag{4}$$

then $\Gamma_x^\theta \cap M \neq \emptyset$.

Proof. Let $\Gamma_x^\theta \cap M = \emptyset$. Then exists a positive number $\alpha > 0$ such that $S_\alpha(\Gamma_x^\theta) \cap S_\alpha(M) = \emptyset$. Let $\gamma \in M$. Since $\gamma \notin S_\alpha(\Gamma_x^\theta)$ by *Definition 3*, there is a number $\epsilon(\gamma) < \alpha$ such that

$$\lim_{r\to\infty} {h_r}^{-1}|\{k \in I_r \,:\, \|x_k - \gamma\| < \epsilon(\gamma)\}| = 0. \tag{5}$$

Clearly, $M \subset \cup_{\gamma\in M} S_{\epsilon(\gamma)}(\gamma)$. Since M is a compact set, we can choose a finite subcovering of sets $S_n = S_{\epsilon(\gamma_n)}(\gamma_n), n = 1, 2, \ldots, N$ such that $M \subset \bigcup_{n=1}^N S_n$. Then for every n we have

$$\{k \in I_r : x_k \in M\}| \leq \sum_{n=1}^{N} |\{k \in I_r \,:\, \|x_k - \gamma_n\| < \epsilon(\gamma_n)\}|.$$

Then from (4) it follows that

$$\limsup_{r\to\infty} {h_r}^{-1}|\{k \in I_r : x_k \in M\}| = 0,$$

which contradicts (3). $\square$

Theorem 2. *Let θ be a lacunary sequence and $x = \{x_k : x_k \in X\}$ a bounded sequence. Then*

(i) Γ_x^θ is a nonempty compact set,

(ii) $\lim_{r\to\infty} {h_r}^{-1}|\{k \in I_r : \rho(\Gamma_x^\theta, x_k) \geq \epsilon\}| = 0$ for every $\epsilon > 0$.

Proof. Let x be a bounded sequence and $M \subset X$ a compact set such that $x_k \in M$ for every k. We have

$$\limsup_{r\to\infty} {h_r}^{-1}|\{k \in I_r : x_k \in M\}| = 1 > 0.$$

Then from *Lemma 2*, it follows that $\Gamma_x^\theta \cap M \neq \emptyset$, i.e. the set Γ_x^θ is nonempty. In this case from *Lemma 1* we have that Γ_x^θ is also a compact set.

Now we prove (ii). If we take

$$\limsup_{r\to\infty} {h_r}^{-1}|\{k \in I_r : \rho(\Gamma_x^\theta, x_k) \geq \epsilon\}| > 0,$$

then for the set $\tilde{M} = M \setminus \overset{\circ}{S}_\epsilon(\Gamma_x^\theta)$ we have

$$\limsup_{r\to\infty} {h_r}^{-1}|\{k \in I_r : x_k \in \tilde{M} \subset X\}| > 0.$$

In this case, by *Lemma 2*, $\Gamma_x^\theta \cap \tilde{M} \neq \emptyset$, which is a contradiction. The theorem is proved. $\square$

Theorem 3. *Let θ be lacunary sequence and let $M \subset X$ be a compact set and $M \cap \Gamma_x^\theta = \emptyset$. Then the set $\{k \in I_r : x_k \in M\}$ has lacunary density zero i.e.*

$$\lim_{r\to\infty} {h_r}^{-1}|\{k \in I_r : x_k \in M\}| = 0.$$

Proof. . For every point $\gamma \in M$ there is a positive number $\epsilon = \epsilon(\gamma) > 0$ such that $\lim_{r\to\infty} {h_r}^{-1}|\{k \in I_r : x_k \in M\}| = 0$.

Let $S_\epsilon(\gamma) = \{y \in X : \|y - \gamma\| < \epsilon\}$. The open sets $S_\epsilon(\gamma)$, $\gamma \in M$, consists an open covering of M. But M is a compact set and so there exists a finite subcover of M, say $S_j = S_{\epsilon_j}(\gamma_j),\ \ j = 1, 2, ..., m$. Clearly, $M \subset \cup_j S_j$ and $\lim_{r\to\infty} {h_r}^{-1}|\{k \in I_r : \|x_k - \gamma_j\| < \epsilon_j\}| = 0$ for every j. We can write

$$|\{k \in I_r : x_k \in M\}| \leq \sum_{j=1}^{m} |\{k \in I_r : \|x_k - \gamma_j\| < \epsilon_j\}|$$

and therefore

$$\lim_{r\to\infty} {h_r}^{-1}|\{k \in I_r : x_k \in M\}| \leq \sum_{j=1}^{m} \lim_{r\to\infty} {h_r}^{-1}|\{k \in I_r : \|x_k - \gamma_j\| < \epsilon_j\}| = 0$$

This implies that $\lim_{r\to\infty} {h_r}^{-1}|\{k \in I_r : x_k \in M\}| = 0$ and The theorem is proved. $\square$

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