Partitions of sets and the Riemann integral

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Abstract. *We will discuss the definition of the Riemann integral using general partitions and give an elementary explication, without resorting to nets, generalized sequences and such, of what is meant by saying that "the Riemann integral is the limit of Darboux sums when the mesh of the partition approaches zero".*

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1. Introduction

To define the Riemann integral one traditionally starts with a bounded real-valued function $f\colon I \to \mathbb{R}$, where $I \in \mathbb{R}^n$ is an n-dimensional parallelepiped, and using the Darboux sums corresponding to partitions of I, one defines Riemann integrability and the Riemann integral for such a function (e.g. [Spivak]). Next, one defines the Jordan measurable subsets of $\mathbb{R}^n$ as those bounded sets S for which the characteristic function is Riemann integrable on some parallelepiped I containing S. Now one defines the Riemann integral for a bounded real-valued function defined on a Jordan measurable subsets $S \subseteq \mathbb{R}^n$, by extending the given function f over some parallelepiped I containing S by setting it equal to zero outside S, and taking its integral over I to be the integral of f over S. Without using any measure theory, one can define what is meant by a set of (Lebesgue's) measure zero, and prove the Lebesgue's criterion for Riemann integrability, acquiring a powerful tool for proving basic properties of the Riemann integral as well as for recognizing Jordan measurable sets and Riemann integrable functions defined on such sets. In this way one obtains a sound theory of the Riemann integral.

One thing that can make one unhappy with this definition is the fact that for the integral over a parallelepiped one uses partitions, whereas for the integral over more general sets the definition is not so natural.

Another thing that one would like, even in the case of the Riemann integral over a rectangle, is to make precise — in an elementary way, without using e.g. generalized

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sequences over partially ordered sets — the meaning of the phrase saying that *the Riemann integral is the limit of Darboux (or integral) sums when the mesh of the partition tends to zero.*

2. Preliminaries

In order to fix the notations, recall the definition of the Riemann integral. For simplicity of both notation and terminology we are going to consider the case $n = 2$. It will be clear that all our arguments are valid in the general case as well.

By a *partition of the rectangle* $I = [a, b] \times [c, d] \subseteq \mathbb{R}^2$ we mean a pair $\rho = (\rho_x, \rho_y)$ of partitions, i.e. finite ordered subsets containing the end-points, $\rho_x := \{a = x_0 < x_1 < \cdots < x_k = b\}$ and $\rho_y := \{c = y_0 < y_1 < \cdots < y_\ell = d\}$ of the segments $[a, b]$ and $[c, d]$, respectively. Sometimes, to avoid confusion, we will call such a partition *rectangular.* Given a bounded function $f\colon I \to \mathbb{R}$, denote $m_{ij} := \inf f(I_{ij})$ and $M_{ij} := \sup f(I_{ij})$, where $I_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ are the rectangles defined by the partition ρ. Denoting by $\pi(I_{ij})$ the area of the rectangle I_{ij}, one defines the lower and the upper Darboux sums as $s(f, \rho) := \sum_{i,j} m_{ij}\, \pi(I_{ij})$ and $S(f, \rho) := \sum_{i,j} M_{ij}\, \pi(I_{ij})$ respectively. Function f is said to be *Riemann integrable* if $\sup_{\rho\in\rho(I)} s(f, \rho) = \inf_{\rho\in\rho(I)} S(f, \rho)$, where $\rho(I)$ denotes the set of all partitions of the rectangle I, and this common value is called the Riemann integral of f, denoted by $\int_I f$. Obviously, f is Riemann integrable if and only if for every $\varepsilon > 0$ there exists a partition ρ such that $S(f, \rho) - s(f, \rho) < \varepsilon$.

A bounded set $S \subseteq \mathbb{R}^2$ is said to be *Jordan measurable* if its characteristic function χ_S is Riemann integrable over some, and hence over every rectangle I containing S. The integral $\int_I \chi_S$, denoted by $\pi(S)$, is called the *content* or the *Jordan measure* of the set S. The content of a rectangle is just its area. Jordan measurable sets having content equal to zero will be called *content-zero sets.* They are characterized by the property that for every positive ε they can be covered by finitely many rectangles of their total area smaller than ε. For compact sets, being a content-zero set is the same as having the Lebesgue's measure equal to zero. Therefore a bounded set is Jordan measurable if and only if its boundary is a content-zero set.

A bounded real-valued function f defined on a Jordan measurable set S is Riemann integrable if its zero-extension $\tilde f\colon I \to \mathbb{R}$ defined by $\tilde f(\mathbf{x}) := f(\mathbf{x})$ for $\mathbf{x} \in S$ and $\tilde f(\mathbf{x}) := 0$ for $\mathbf{x} \in I \setminus S$, is Riemann integrable for some, and hence for every rectangle I containing S, and the integral of f is defined as $\int_S f := \int_I \tilde f$.

3. Jordan partitions of sets and the Riemann integral

Let $S \subseteq \mathbb{R}^2$ be a Jordan measurable set. A *partition* σ of the set S is a family $\sigma = \{S_\alpha\}_{\alpha=1}^k$ of Jordan measurable subsets $S_\alpha \subseteq S$ such that $S = \bigcup_{\alpha=1}^k S_\alpha$ and for $\alpha \neq \beta$ the intersections $S_\alpha \cap S_\beta$ are content-zero sets. The set of all partitions of the set S will be denoted by $\sigma(S)$. The partition $\sigma' = \{S'_\beta\}_\beta$ is said to *refine* the partition $\sigma = \{S_\alpha\}_\alpha$, denoted $\sigma' \geq \sigma$, if for each β there exists an α such that $S'_\beta \subseteq S_\alpha$.

Let $f\colon S \to \mathbb{R}$ be a bounded function and let $\sigma = \{S_\alpha\}_\alpha$ be a partition of the Jordan measurable set S. Denote $m_\alpha := \inf f(S_\alpha)$ and $M_\alpha := \sup f(S_\alpha)$, and in analogy with the Darboux sums, define $s(f,\sigma,S) := \sum_{\alpha=1}^k m_\alpha\, \pi(S_\alpha)$ and $S(f,\sigma,S) := \sum_{\alpha=1}^k M_\alpha\, \pi(S_\alpha)$. If the partition σ' refines σ then

$$s(f,\sigma,S) \le s(f,\sigma',S) \le S(f,\sigma',S) \le S(f,\sigma,S)\,.$$

The following proposition proves our first objective — that the Riemann integral over a Jordan measurable set can be defined using partitions in the same spirit as in the case of a rectangle.

Proposition 1. *Let $f\colon S \to \mathbb{R}$ be a Riemann integrable function on the Jordan measurable set $S \subseteq \mathbb{R}^2$. Then*

$$\int_S f = \sup_{\sigma\in\sigma(S)} s(f,\sigma,S) = \inf_{\sigma\in\sigma(S)} S(f,\sigma,S)\,. \tag{1}$$

Moreover, if $\pi(S) \neq 0$, the above supremum and infimum can be taken over only those partitions for which $\pi(S_\alpha) > 0$ for all α.

Therefore for every $\varepsilon > 0$ there exists a partition σ of the set S such that $S(f,\sigma,S) - s(f,\sigma,S) < \varepsilon$.

Proof. For every partition $\sigma = \{S_\alpha\}_\alpha$ of S the function f is Riemann integrable over every S_α, and

$$\int_S f = \sum_\alpha \int_{S_\alpha} f \ge \sum_\alpha \int_{S_\alpha} m_\alpha = \sum_\alpha m_\alpha\, \pi(S_\alpha)\,.$$

In order to prove the first equality in (1) it suffices to show that for every $\varepsilon > 0$ there exists a partition σ of the set S such that

$$\int_S f - \sum_\alpha m_\alpha\, \pi(S_\alpha) < \varepsilon\,. \tag{2}$$

Let $I \supseteq S$ be a rectangle containing S and let $\tilde f\colon I \to \mathbb{R}$ be the zero-extension of f. Since $\tilde f$ is Riemann integrable over I, for the given $\varepsilon > 0$ there exists a rectangular subdivision ρ of I such that

$$\int_I \tilde f - \sum_{i,j} m_{ij}(\tilde f)\, \pi(I_{ij}) < \frac{\varepsilon}{2}\,, \tag{3}$$

where $m_{ij}(\tilde f) := \inf \tilde f(I_{ij})$. Since the set S is Jordan measurable, its boundary ∂S is a content-zero set, hence we can choose a partition ρ such that, in addition to (3), it also satisfies

$$\sum_{I_{ij}\cap\partial S\neq\emptyset} \pi(I_{ij}) < \frac{\varepsilon}{4M}\,, \tag{4}$$

where $M := \sup_{\mathbf{x}\in S} |f(\mathbf{x})| = \sup_{\mathbf{x}\in I} |\tilde f(\mathbf{x})|$. (If $f(\mathbf{x}) = 0$ for all $\mathbf{x}$ everything is zero, so there is nothing to prove.)

Let σ be the partition of the set S consisting of all intersections $S_{ij} := S \cap I_{ij}$, which are easily shown to be Jordan measurable sets. Let us split the lower Darboux sum as follows:

$$\sum_{i,j} m_{ij}(\tilde{f})\,\pi(I_{ij}) = \sum_{I_{ij}\subseteq \operatorname{Int} S} m_{ij}(\tilde{f})\,\pi(I_{ij}) + \sum_{I_{ij}\cap\partial S\neq 0} m_{ij}(\tilde{f})\,\pi(I_{ij}) + \sum_{I_{ij}\subseteq I\setminus\overline{S}} m_{ij}(\tilde{f})\,\pi(I_{ij})\,. \tag{5}$$

The first sum on the right-hand side equals $\sum_{I_{ij}\subseteq \operatorname{Int} S} m_{ij}\,\pi(S_{ij})$, since for $I_{ij} \subseteq \operatorname{Int} S$ we have $S_{ij} = I_{ij}$ and $m_{ij} := \inf f(S_{ij}) = \inf f(I_{ij}) = \inf \tilde{f}(I_{ij})$. The third sum equals zero because $\tilde{f}(\mathbf{x}) = 0$ for all $\mathbf{x} \in I \setminus \overline{S} \subseteq I \setminus S$.

Since $m_{ij}(\tilde{f}) \leq M$ for all i, j, (4) implies that the second sum on the right-hand side in (5) is smaller than $\varepsilon/4$, and therefore (3) implies

$$\int_I \tilde{f} < \sum_{I_{ij}\subseteq \operatorname{Int} S} m_{ij}\,\pi(S_{ij}) + \frac{3}{4}\varepsilon\,. \tag{6}$$

Furthermore, since $m_{ij} \leq M$ and $\pi(S_{ij}) \leq \pi(I_{ij})$ for all i, j, we have

$$\sum_{I_{ij}\cap\partial S\neq\emptyset} m_{ij}\,\pi(S_{ij}) \leq \sum_{I_{ij}\cap\partial S\neq\emptyset} M\,\pi(I_{ij}) \overset{(4)}{<} \frac{\varepsilon}{4}\,. \tag{7}$$

Since $\int_I \tilde{f} = \int_S f$, and $S_{ij} = \emptyset$ for $I_{ij} \subseteq \overline{S}$, from (6) we obtain

$$\int_S f < \sum_{i,j} m_{ij}\,\pi(S_{ij}) + \varepsilon\,,$$

which shows (2), and proves the first equality in (1).

The second equality is proved similarly. □

Given a partition $\sigma = \{S_\alpha\}_\alpha$ of the set S, define its *mesh* to be the number $\mu(\sigma) := \max_\alpha \operatorname{diam} S_\alpha$.

Proposition 2. *Let $f\colon I := [a,b]\times[c,d] \to \mathbb{R}$ be a Riemann integrable function. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every partition $\sigma = \{S_\alpha\}_\alpha$ of the set I with mesh $\mu(\sigma) < \delta$ we have*

$$S(f,\sigma,I) - s(f,\sigma,I) < \varepsilon\,,$$

i.e.

$$\sum_\alpha (M_\alpha - m_\alpha)\,\pi(S_\alpha) < \varepsilon\,.$$

Proof. Let us first prove the special case when the function f is continuous. By uniform continuity, given $\varepsilon > 0$ let $\delta > 0$ be such that

$$\text{for all } \mathbf{x}, \mathbf{x}' \in I,\ \|\mathbf{x} - \mathbf{x}'\| < \delta \text{ implies } |f(\mathbf{x}) - f(\mathbf{x}')| < \frac{\varepsilon}{\pi(I)}\,. \tag{8}$$

Let $\sigma = \{S_\alpha\}_\alpha$ be a partition of the set I such that $\mu(\sigma) < \delta$. The sets S_α as well as their closures $\overline{S}_\alpha = S_\alpha \cup \partial S_\alpha$ are Jordan measurable, and $\pi(\overline{S}_\alpha) = \pi(S_\alpha)$. By compactness, $\overline{m}_\alpha := \inf f(\overline{S_\alpha}) = \min f(S_\alpha)$ and $\overline{M}_\alpha := \sup f(\overline{S}_\alpha) = \max f(\overline{S}_\alpha)$, so by uniform continuity (8), $\overline{M}_\alpha - \overline{m}_\alpha < \frac{\varepsilon}{\pi(I)}$. Therefore

$$\sum_\alpha (M_\alpha - m_\alpha)\,\pi(S_\alpha) \le \sum_\alpha (\overline{M}_\alpha - \overline{m}_\alpha)\,\pi(S_\alpha) < \varepsilon\,.$$

In the general case, the set $D(f)$, the set of points where f is not continuous, is a set of (Lebesgue) measure zero. Denote $M := \sup_{\mathbf{x}\in I} |f(\mathbf{x})|$. Given $\varepsilon > 0$ choose some rectangles I_j, $j \in \mathbb{N}$, such that

$$D(f) \subseteq \bigcup_{j\in\mathbb{N}} \mathring{I}_j \quad\text{and}\quad \sum_{j=1}^{\infty} \pi(I_j) < \frac{\varepsilon}{8M}\,. \tag{9}$$

Let I'_j, $j \in \mathbb{N}$, be rectangles concentric with I_j and of twice their area, and let $U := \bigcup_{j\in\mathbb{N}} \mathring{I}_j$ and $V := \bigcup_{j\in\mathbb{N}} \mathring{I}'_j$. Note that the restriction map $f|_{I\setminus U}\colon I\setminus U \to [-M, M]$ is continuous. Since the set $I \setminus U$ is closed in I, by the Tietze's extension theorem (e.g. [Munkres]), there exists a continuous function $g\colon I \to [-M, M]$ extending $f|_{I\setminus U}$. By the special case, there exists a $\delta > 0$ such that for every partition σ of the set I for which $\mu(\sigma) < \delta$, we have

$$S(g, \sigma, I) - s(g, \sigma, I) < \frac{\varepsilon}{2}\,. \tag{10}$$

Let $\delta' < \delta$ be a Lebesgue's number for the open cover $\{B(\mathbf{x}, \frac{\delta}{2}) \cap (I \setminus \overline{U}) : \mathbf{x} \in I \setminus V\} \cup \{V\}$ of the rectangle I. Then for every partition σ of the set I for which $\mu(\sigma) < \delta'$, we have

$$\begin{aligned}\sum_\alpha S(f, \sigma, I) - s(f, \sigma, I) &= \sum_\alpha \big(M_\alpha(f) - m_\alpha(f)\big)\,\pi(S_\alpha)\\ &\le \sum_{S_\alpha \subseteq I\setminus\overline{U}} \big(M_\alpha(f) - m_\alpha(f)\big)\,\pi(S_\alpha)\\ &\quad + \sum_{S_\alpha\subseteq V} \big(M_\alpha(f) - m_\alpha(f)\big)\,\pi(S_\alpha) < \varepsilon\,.\end{aligned}$$

Indeed, outside of the set U, f equals g, hence for $S_\alpha \subseteq I \setminus \overline{U}$ we have $M_\alpha(f) = M_\alpha(g)$ and $m_\alpha(f) = m_\alpha(g)$. Therefore, by (10), the first sum in the second line is smaller then $\varepsilon/2$. The second sum is smaller than $\varepsilon/2$ because $M_\alpha(f) - m_\alpha(f) \le 2M$, and for $\alpha \ne \beta$ the sets $S_\alpha \cap S_\beta$ are content-zero sets, and therefore $\sum\limits_{S_\alpha\subseteq V} \pi(S_\alpha) \le \sum\limits_{j=1}^{\infty} \pi(I'_j) < \frac{\varepsilon}{4M}$. (The inequality between the first and the second line comes from the fact that some sets S_α may lie in both $I \setminus \overline{U}$ and V.) □

Corollary 1. *Let $S \subseteq \mathbb{R}^2$ be a Jordan measurable set and let $f\colon S \to \mathbb{R}$ a Riemann integrable function. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for* *every partition $\sigma = \{S_\alpha\}_\alpha$ of the set S for which $\mu(\sigma) < \delta$, $S(f,\sigma,S) - s(f,\sigma,S) < \varepsilon$. In particular, $\int_S f - s(f\sigma,S) < \varepsilon$.*

Proof. Let $I \supseteq S$ be a rectangle containing S and let $\tilde{f}\colon I \to \mathbb{R}$ be the zero-extension of f. Riemann integrability of f on S means Riemann integrability of $\tilde{f}$ on I. By the previous proposition, given $\varepsilon > 0$ there exists a $\delta > 0$ such that for every partition $\tilde{\sigma}$ of the set I for which $\mu(\tilde{\sigma}) < \delta$ we have $S(\tilde{f},\tilde{\sigma},I) - s(\tilde{f},\tilde{\sigma},I) < \varepsilon$. Let $\sigma = \{S_\alpha\}_\alpha$ be a partition of the set S such that $\mu(\sigma) < \delta$, and let $\rho = \{I_{ij}\}_{i,j}$ be a rectangular partition of the rectangle I such that $\mu(\rho) < \delta$. Denote by $\tilde{\sigma}$ the partition of the set I consisting of all sets S_α belonging to partition σ together with all intersections $I_{ij} \cap (I \setminus S) = I_{ij} \setminus S$. Then $\mu(\tilde{\sigma}) < \delta$, and since $\tilde{f}$ equals zero on $I_{ij} \setminus S$, we have

$$\begin{aligned} S(f,\sigma,S) - s(f,\sigma,S) &= \sum_\alpha \big(M_\alpha(f) - m_\alpha(f)\big)\,\pi(S_\alpha) \\ &\quad + \sum_{I_{ij}\setminus S \neq \emptyset} \big(\sup \tilde{f}(I_{ij} \setminus S) - \inf \tilde{f}(I_{ij} \setminus S)\big)\,\pi(I_{ij} \setminus S) \\ &= S(\tilde{f},\tilde{\sigma},I) - s(\tilde{f},\tilde{\sigma},I) < \varepsilon\,. \end{aligned}$$

□

The following Corollary gives an elementary explication, without resorting to nets, generalized sequences and such, of what is meant by saying that the Riemann integral is the limit of Darboux (or similarly integral) sums when the mesh of the partition approaches zero.

Corollary 2. *Let $S \subseteq \mathbb{R}^2$ be a Jordan measurable set and let $f\colon S \to \mathbb{R}$ be a Riemann integrable function. Suppose for every $\delta > 0$ we are given a partition σ_δ of the set S with mesh $\mu(\sigma_\delta) < \delta$. Then*

$$\int_S f = \lim_{\delta\to 0} S(f,\sigma_\delta,S) = \lim_{\delta\to 0} s(f,\sigma_\delta,S)\,.$$

Proof. By the previous Corollary, for every $\varepsilon > 0$ there exists a $\delta' > 0$ such that for every partition σ of the set S for which $\mu(\sigma) < \delta'$, we have $\int_S f - s(f,\sigma,S) < \varepsilon$. Therefore, for every $\delta < \delta'$ we have

$$0 \le \int_S f - s(f,\sigma_\delta,S) < \varepsilon$$

i.e.

$$\int_S f \ge s(f,\sigma_\delta,S) > \int_S f - \varepsilon$$

and therefore

$$\int_S f = \sup_\delta s(f,\sigma_\delta,S) = \lim_{\delta\to 0} s(f,\sigma_\delta,S)\,.$$

The other equality is proved similarly. □

References

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