Unique representation $d = 4k(k^2 - 1)$ in $D(4)$-quadruples $\{k-2, k+2, 4k, d\}$

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Abstract. *Let $k \geq 3$ be an integer. We show that if d is a positive integer such that the product of any two distinct elements of the set $\{k-2, k+2, 4k, d\}$ increased by 4 is a square, then d must be $4k(k^2-1)$.*

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1. Introduction

Let n be a nonzero integer. A set of m positive integers $\{a_1, \ldots, a_m\}$ is called a $D(n)$-m-tuple if $a_i a_j + n$ is a square for all i and j with $1 \leq i < j \leq m$. Diophantus found a $D(256)$-quadruple $\{1, 33, 68, 105\}$, and Fermat found a $D(1)$-quadruple $\{1, 3, 8, 120\}$ (cf. [5]).

In 1969, Baker and Davenport ([2]) showed that if the set $\{1, 3, 8, d\}$ is a $D(1)$-quadruple, then $d = 120$. This result has been generalized in three directions: first, Dujella ([7]) showed that if $\{k-1, k+1, 4k, d\}$ is a $D(1)$-quadruple with an integer $k \geq 2$, then $d = 4k(4k^2 - 1)$; secondly, Dujella and Pethő ([10]) showed that if $\{1, 3, c, d\}$ is a $D(1)$-quadruple with $3 < c < d$, then $d = 7c + 4 + 4\sqrt{(c+1)(3c+1)}$; and thirdly, Dujella ([8]) showed that if $\{F_{2k}, F_{2k+2}, F_{2k+4}, d\}$ is a $D(1)$-quadruple (where F_ν is the ν-th Fibonacci number), then $d = 4F_{2k+1}F_{2k+2}F_{2k+3}$. These results lead us to the following.

Conjecture 1 [[1]]. *If $\{a, b, c, d\}$ is a $D(1)$-quadruple with $a < b < c < d$, then $d = a + b + c + 2abc + 2rst$, where r, s, t are positive integers given by $ab + 1 = r^2$, $ac + 1 = s^2$, $bc + 1 = t^2$.*

Note that this conjecture immediately implies that there does not exist a $D(1)$-quintuple, which is a longstanding conjecture. It has been known that there does not exist a $D(1)$-sextuple and that there exist only finitely many $D(1)$-quintuples ([9]).

As for $D(4)$-quadruples, Mohanty and Ramasamy ([13]) showed that the $D(4)$-quadruple $\{1, 5, 12, 96\}$ cannot be extended to a $D(4)$-quintuple, and Kedlaya ([12])

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showed that if $\{1,5,12,d\}$ is a $D(4)$-quadruple, then $d=96$. This result also has been generalized by Dujella and Ramasamy ([11]) as follows: if $\{F_{2k},5F_{2k},4F_{2k+2},d\}$ is a $D(4)$-quadruple, then $d=4L_{2k}F_{4k+2}$, where L_ν is the ν-th Lucas number.

In this paper, we ameliorate the result of Kedlaya in another direction.

Theorem 1. *Let $k\geq 3$ be an integer. If $\{k-2,k+2,4k,d\}$ is a $D(4)$-quadruple, then d must be $4k(k^2-1)$.*

It is easy to check that $\{k-2,k+2,4k,4k(k^2-1)\}$ is a $D(4)$-quadruple for $k\geq 3$ (cf. [6, Section 4]). We will prove this theorem on similar lines to Theorem 1 in [7].

These results lead us to the following.

Conjecture 2 [[11]]. *If $\{a,b,c,d\}$ is a $D(4)$-quadruple with $a<b<c<d$, then $d=a+b+c+(abc+rst)/2$, where r,s,t are positive integers given by $ab+4=r^2,\ ac+4=s^2,\ bc+4=t^2$.*

Note that this immediately implies that there does not exist a $D(4)$-quintuple. It has been known that there does not exist a $D(4)$-8-tuple and that there exist only finitely many $D(4)$-7-tuples ([11]).

In case $k=3$, *Theorem 1* is valid because of the result of Kedlaya; in case k is even, say $k=2k'$, *Theorem 1* follows from the result on the $D(1)$-triple $\{k'-1,k'+1,4k'\}$ ([7]). Hence, it suffices to show *Theorem 1* on the assumption that $k\geq 5$ is an odd integer.

2. Fundamental solutions of simultaneous Diophantine equations

In this section we translate the assumption of *Theorem 1* into simultaneous Diophantine equations and determine their fundamental solutions.

Suppose that $\{k-2,k+2,4k,d\}$ is a $D(4)$-quadruple. Then there exist integers x,y,z such that

$$(k-2)d+4=x^2,\ (k+2)d+4=y^2,\ 4kd+4=4z^2.$$

Eliminating d, we obtain simultaneous Diophantine equations:

$$(k-2)y^2-(k+2)x^2=-16, \tag{1}$$

$$(k-2)z^2-kx^2=-3k-2, \tag{2}$$

$$(k+2)z^2-ky^2=-3k+2. \tag{3}$$

We describe the solutions of equations (1) and (2).

Lemma 1 [(cf. [11, Lemma 2])]. *Let $\{a,b\}$ be a $D(4)$-pair with $0<a<b$ and let r be a positive integer such that $ab+4=r^2$. There exist a positive integer i_0 and integers $y_0^{(i)}$, $x_0^{(i)}$, $i=1,\dots,i_0$, with the following properties:*

(i) $(y_0^{(i)},x_0^{(i)})$ *is a solution of*

$$ay^2-bx^2=4(a-b). \tag{4}$$

(ii) $y_0^{(i)}$ *and* $x_0^{(i)}$ *satisfy the following inequalities*

$$1 \leq x_0^{(i)} \leq \sqrt{\frac{a(b-a)}{r-2}}, \quad |y_0^{(i)}| \leq \sqrt{\frac{(r-2)(b-a)}{a}}.$$

(iii) If (y, x) *is a positive solution of* (4)*, then there exist* $i \in \{1, \ldots, i_0\}$ *and an integer* $m \geq 0$ *such that*

$$y\sqrt{a} + x\sqrt{b} = (y_0^{(i)}\sqrt{a} + x_0^{(i)}\sqrt{b})\left(\frac{r+\sqrt{ab}}{2}\right)^m.$$

Proof. Although [11, Lemma 2] is concerned with a $D(4)$-triple $\{a, b, c\}$ and the attached equations

$$az^2 - cx^2 = 4(a-c), \tag{5}$$
$$bz^2 - cy^2 = 4(b-c), \tag{6}$$

one can show the statements for the equations (5) and (6) independently (see the proof of [11, Lemma 2]). Thus, *Lemma 1* follows. □

Lemma 2. *Let* $k \geq 5$ *be an odd integer.*

(i) If (y, x) *is a positive solution of* (1)*, then there exists an integer* $m \geq 0$ *such that*

$$y\sqrt{k-2} + x\sqrt{k+2} = 2(\sqrt{k-2} + \sqrt{k+2})\left(\frac{k+\sqrt{k^2-4}}{2}\right)^m. \tag{7}$$

(ii) If (z, x) *is a positive solution of* (2)*, then there exist an integer* $n \geq 0$ *and a solution* (z_0, x_0) *of* (2) *with*

$$1 \leq x_0 < k-2 \tag{8}$$

such that

$$z\sqrt{k-2} + x\sqrt{k} = (z_0\sqrt{k-2} + x_0\sqrt{k})\left(k-1+\sqrt{k(k-2)}\right)^n. \tag{9}$$

Proof. (i) Let (y, x) be a positive solution of (1). Then, replacing a, b, r in *Lemma 1* by $k-2, k+2, k$, respectively, we see that there exist an integer $m \geq 0$ and a solution (y_1, x_1) of (1) with

$$1 \leq x_1 \leq \sqrt{\frac{(k-2)(k+2-(k-2))}{k-2}} = 2 \tag{10}$$

such that

$$y\sqrt{k-2} + x\sqrt{k+2} = (y_1\sqrt{k-2} + x_1\sqrt{k+2})\left(\frac{k+\sqrt{k^2-4}}{2}\right)^m.$$

If $x_1 = 1$, then

$$y_1 = \pm\sqrt{\frac{k-14}{k-2}},$$

which cannot be an integer for odd k. Hence we have $x_1 = 2$ and $y_1 = \pm 2$. However $y > 0$ and

$$(-2\sqrt{k-2} + 2\sqrt{k+2})\left(\frac{k+\sqrt{k^2-4}}{2}\right) = 2\sqrt{k-2} + 2\sqrt{k+2};$$

hence we have $y_1 = 2$. Therefore we obtain (7).

(ii) Let (z, x) be a positive solution of (2). Then, replacing a, b, r, y in *Lemma 1* by $k-2$, $4k$, $2(k-1)$, $2z$, respectively, we see that there exist an integer $n \geq 0$ and a solution (z_0, x_0) of (2) with

$$1 \leq x_0 \leq \sqrt{\frac{(k-2)(4k-(k-2))}{2(k-1)-2}} = \sqrt{\frac{3k+2}{2}} < k-2$$

such that (9) holds (the last inequality holds because of $k \geq 5$). This completes the proof of *Lemma 2*. □

If we express a positive solution (y, x) of (1) as $y = v'_m$, $x = v_m$ with an integer m in (7), then v'_m and v_m satisfy the following relation

$$v'_{m+1}\sqrt{k-2} + v_{m+1}\sqrt{k+2} = (v'_m\sqrt{k-2} + v_m\sqrt{k+2}) \cdot \frac{k+\sqrt{k^2-4}}{2},$$

that is,

$$\begin{aligned} v'_{m+1} &= \frac{1}{2}(kv'_m + (k+2)v_m), \\ v_{m+1} &= \frac{1}{2}(kv_m + (k-2)v'_m), \end{aligned}$$

which, together with (7), implies

$$v_0 = 2,\ v_1 = 2(k-1),\ v_{m+2} = kv_{m+1} - v_m. \tag{11}$$

Similarly, if we express a positive solution (z, x) of (2) as $z = w'_n$, $x = w_n$ with an integer n in (9), then w'_n and w_n satisfy the following relation

$$w'_{n+1}\sqrt{k-2} + w_{n+1}\sqrt{k} = (w'_n\sqrt{k-2} + w_n\sqrt{k})(k-1+\sqrt{k(k-2)}),$$

that is,

$$\begin{aligned} w'_{n+1} &= (k-1)w'_n + kw_n, \\ w_{n+1} &= (k-1)w_n + kw'_n, \end{aligned}$$

which, together with (9), implies

$$w_0 = x_0,\ w_1 = (k-1)x_0 + (k-2)z_0,\ w_{n+2} = 2(k-1)w_{n+1} - w_n. \tag{12}$$

By induction we see from (11) that $v_m \equiv 2 \pmod{(k-2)}$ for all $m \geq 0$ and from (12) that $w_n \equiv x_0 \pmod{(k-2)}$ for all $n \geq 0$. Hence if $v_m = w_n$, then we have $x_0 \equiv 2 \pmod{(k-2)}$. It follows from (8) that $x_0 = 2$, and that $z_0 = \pm 1$. Hence by (12) we have

$$w_0 = 2,\ w_1 = 2(k-1) \pm (k-2),\ w_{n+2} = 2(k-1)w_{n+1} - w_n. \tag{13}$$

If we define $w_{-n} = 2(k-1)w_{-n+1} - w_{-n+2}$ for $n \geq 1$ recursively, we may rephrase (13) in terms of the two-sided sequence $\{w_n\}$ ($n \in \mathbf{Z}$) as

$$w_0 = 2,\ w_1 = 3k-4,\ w_{n+2} = 2(k-1)w_{n+1} - w_n. \tag{14}$$

To sum up, we obtain the following.

Lemma 3. *Let $k \geq 5$ be an odd integer. Let (x, y, z) be a positive solution of the simultaneous Diophantine equations* (1) *and* (2). *Then, there exist integers $m \geq 0$ and n such that $x = v_m = w_n$, where the sequence $\{v_m\}$ is given by* (11) *and the two-sided sequence $\{w_n\}$ is given by* (14).

3. A lower bound for $\log z$

In this section, we give a lower bound for $\log z$ in terms of k.

Lemma 4. *Let $k \geq 5$ be an integer. If $v_m = w_n$, then we have*

$$n \equiv 0 \ \text{or} \ -2 \pmod{2k}.$$

Proof. We see from (11) and (14) that

$$\begin{aligned}
(v_m \mod (2k-2))_{m\geq 0} &= (2, 0, -2, -2, 0, 2, 2, 0, \ldots),\\
(w_n \mod (2k-2))_{n\geq 0} &= (2, -k, -2, k, 2, -k, \ldots),\\
(w_n \mod (2k-2))_{n\leq 0} &= (2, k, -2, -k, 2, k, \ldots).
\end{aligned}$$

Note that by the recursive formula (11) the values $v_m \mod (2k-2)$ and $v_{m+1} \mod (2k-2)$ determine the value $v_{m+2} \mod (2k-2)$, whence the sequence $(v_m \mod (2k-2))_{m\geq 0}$ is periodic with period 6, and similarly that the sequences $(w_n \mod (2k-2))_{n\geq 0}$ and $(w_n \mod (2k-2))_{n\leq 0}$ are periodic with period 4. Hence, if $v_m = w_n$, then we may write $n = 2l$ for some integer l. We then have

$$\begin{aligned}
(v_m \mod 2k(k-2))_{m\geq 0} &= (2, 2k-2, 2k-2, 2, 2, 2k-2, \ldots),\\
(w_{2l} \mod 2k(k-2))_{l\geq 0} &= (2, -2k+6, -4k+10, -6k+14, \ldots),\\
(w_{2l} \mod 2k(k-2))_{l\leq 0} &= (2, 2k-2, 4k-6, 6k-10, \ldots).
\end{aligned}$$

We can prove by induction that for all integers l,

$$w_{2l} \equiv -2lk + 2(2l+1) \pmod{2k(k-2)}.$$

Hence we have

$$-2lk + 2(2l+1) \equiv 2 \ \text{or} \ 2k-2 \pmod{2k(k-2)}.$$

If $-2lk+2(2l+1)\equiv 2 \pmod{2k(k-2)}$, then we have $2l(k-2)\equiv 0 \pmod{2k(k-2)}$, that is, $n=2l\equiv 0 \pmod{2k}$. If $-2lk+2(2l+1)\equiv 2k-2 \pmod{2k(k-2)}$, then we have $2(l+1)(k-2)\equiv 0 \pmod{2k(k-2)}$, that is, $n=2l\equiv -2 \pmod{2k}$. This completes the proof of *Lemma 4*. $\square$

Lemma 5. *Let $k\geq 5$ be an integer. Let (x,y,z) be a positive solution of the simultaneous Diophantine equations* (1) *and* (2) *with $z\notin\{1,2k^2-1\}$. Then we have*

$$\log z > 2(k-1)\log(2k-3).$$

Proof. Note that if $z=1$ (resp. $2k^2-1$), then $d=0$ (resp. $4k(k^2-1)$). By (9) and (14), we may write $z=|s_n|$ for some integer n, where

$$s_0=1,\ s_1=3k-1,\ s_{n+2}=2(k-1)s_{n+1}-s_n,$$

that is,

$$s_n=\frac{2\sqrt{k}+\sqrt{k-2}}{2\sqrt{k-2}}(k-1+\sqrt{k(k-2)})^n-\frac{2\sqrt{k}-\sqrt{k-2}}{2\sqrt{k-2}}(k-1-\sqrt{k(k-2)})^n.$$

If $n\geq 0$, then by $k\geq 5$ we have

$$\begin{aligned}s_n&>\left(1+\frac{1}{2}\right)(k-1+\sqrt{k(k-2)})^n-(k-1-\sqrt{k(k-2)})^n\\&>(k-1+\sqrt{k(k-2)})^n>(2k-3)^n;\end{aligned}$$

and if $n<0$, then we have

$$\begin{aligned}|s_n|&>\left(\frac{1}{2}+\frac{2}{3k-2}\right)(k-1+\sqrt{k(k-2)})^{-n}-2(k-1-\sqrt{k(k-2)})^{-n}\\&>\frac{1}{2}(k-1+\sqrt{k(k-2)})^{-n}>\frac{1}{2}(2k-3)^{-n}.\end{aligned}$$

Hence, if $n\geq 0$, then *Lemma 4* and $z\neq 1=s_0$ imply that

$$z=s_n>(2k-3)^{2k-2};$$

if $n<0$, then *Lemma 4* and $z\neq 2k^2-1=|s_{-2}|$ imply that

$$z=|s_n|>\frac{1}{2}(2k-3)^{2k}>(2k-3)^{2k-2}.$$

In any case, we obtain

$$\log z > 2(k-1)\log(2k-3).$$

$\square$

4. Application of a theorem of Rickert

In this section, we show that *Theorem 1* holds for odd $k \geq 63$, combining the results in *Section 3.* with a slight modification of a theorem of Rickert (or of Bennett).

Theorem 2 [(cf. [4, Theorem 3.2], [14, Theorem] or [15, Theorem])]. *Let $N \geq 63$ be an integer. Then the numbers*

$$\theta_1 := \sqrt{\frac{N-2}{N}} \quad \text{and} \quad \theta_2 := \sqrt{\frac{N+2}{N}}$$

satisfy

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \left|\theta_2 - \frac{p_2}{q}\right|\right\} > (22.6N)^{-1} q^{-1-\lambda}$$

for all integers p_1, p_2, q with $q > 0$, where

$$\lambda := \frac{\log(11.2N)}{\log(0.197N^2)} < 1.$$

Proof. Note that the assumption $N \geq 63$ implies $\lambda < 1$. All we have to do is find those real numbers satisfying the assumption in the following lemma.

Lemma 6 [(cf. [4, Lemma 3.1], [14, Lemma 2.1])]. *Let $\theta_1, \ldots, \theta_m$ be arbitrary real numbers and $\theta_0 = 1$. Assume that there exist positive real numbers l, p, L, P and positive integers D, f with f dividing D and with $L > D$, having the following property. For each positive integer κ, we can find rational numbers $p_{ij\kappa}$ $(0 \leq i, j \leq m)$ with a nonzero determinant such that $f^{-1}D^{\kappa}p_{ij\kappa}$ $(0 \leq i, j \leq m)$ are integers and*

$$|p_{ij\kappa}| \leq pP^{\kappa} \ \ (0 \leq i, j \leq m), \qquad \left|\sum_{j=0}^{m} p_{ij\kappa}\theta_j\right| \leq lL^{-\kappa} \ \ (0 \leq i \leq m).$$

Then

$$\max\left\{\left|\theta_1 - \frac{p_1}{q}\right|, \ldots, \left|\theta_m - \frac{p_m}{q}\right|\right\} > cq^{-1-\lambda}$$

holds for all integers $p_1, \ldots, p_m, q$ with $q > 0$, where

$$\lambda = \frac{\log(DP)}{\log(L/D)} \quad \text{and} \quad c^{-1} = 2mf^{-1}pDP\left(\max\{1, 2f^{-1}l\}\right)^{\lambda}.$$

Here, we used "κ" instead of "k" which is used in [4] and [14]. Note that l, p, L, P, p_{ijk} in [4, Lemma 3.1] denote $f^{-1}l$, $f^{-1}p$, L/D, DP, $f^{-1}D^{\kappa}p_{ij\kappa}$ in the lemma above, respectively. In our situation, we take $m = 2$ and θ_1, θ_2 as in *Theorem 2.* The only difference from Theorem 3.2 in [4] is that we may take $f = 2$ and $D = 32N$, whereas in [4] $f = 1$ and $D = 64N$ are taken (note that C_k in [4] denotes $f^{-1}D^{\kappa}$ in our notation). The validity of this substitution follows from the fact that

$$\prod_{0 \leq i < j \leq 2} (a_i - a_j) = 16$$

is even, where $a_0 = -2$, $a_1 = 0$, $a_2 = 2$. Indeed, let $p_{ij}(x)$ be those polynomials appearing in [14, Lemma 3.3], which have rational coefficients of degree at most κ ([14, (3.7)]). Following [14], we take $p_{ij\kappa} = p_{ij}(1/N)$ for varying values of κ. Then we see from the expression (3.7) in [14] of $p_{ij}(1/N)$ that

$$2^l N^\kappa p_{ij}(1/N) \in \mathbf{Z}$$

for some integer l; we may take $l = 5\kappa - 1$ by a consideration similar to the proof of Lemma 4.3 in [14]. Hence we obtain

$$2^{-1}(2^5N)^\kappa p_{ij}(1/N) \in \mathbf{Z}.$$

Thus, by exactly the same arguments as the ones following Lemma 3.1 in [4] (with $a_0 = -2$, $a_1 = 0$, $a_2 = 2$), the numbers

$$p = \left(1 + \frac{1}{N-2}\right)^{1/2}, \quad P = \frac{1}{3} + \frac{1}{N}, \quad l = \frac{27}{64}\left(1 - \frac{2}{N}\right)^{-1}, \quad L = \frac{27}{4}\left(1 - \frac{2}{N}\right)^2 N^3$$

and $f = 2$, $D = 32N$, $p_{ij\kappa} = p_{ij}(1/N)$ satisfy the assumption in *Lemma 6*. Since $N \geq 63$, we have

$$DP < 11.2N, \quad 2pDP < 22.6, \quad \frac{L}{D} > 0.197N^2.$$

Therefore, *Theorem 2* immediately follows from *Lemma 6*. $\square$

Lemma 7. *Let $N = k \geq 63$ be an integer and let θ_1, θ_2 be as in* Theorem 2. *Then all positive solutions (x, y, z) of the simultaneous Diophantine equations* (2) *and* (3) *satisfy*

$$\max\left\{\left|\theta_1 - \frac{x}{z}\right|, \left|\theta_2 - \frac{y}{z}\right|\right\} < 1.55z^{-2}.$$

Proof. We have

$$\begin{aligned}\left|\sqrt{\frac{k-2}{k}} - \frac{x}{z}\right| &= \left|\frac{k-2}{k} - \frac{x^2}{z^2}\right| \left|\sqrt{\frac{k-2}{k}} + \frac{x}{z}\right|^{-1} \\ &< \frac{1}{kz^2}|-3k-2| \left(2\sqrt{1 - \frac{2}{k}}\right)^{-1} < 1.55z^{-2}\end{aligned}$$

and

$$\left|\sqrt{\frac{k+2}{k}} - \frac{y}{z}\right| < \frac{1}{kz^2}|-3k+2| \left(2\sqrt{1 + \frac{2}{k}}\right)^{-1} < 1.5z^{-2}.$$

$\square$

Proposition 1. *Let $k \geq 63$ be an odd integer. If $\{k-2, k+2, 4k, d\}$ is a $D(4)$-quadruple, then we have $d = 4k(k^2-1)$.*

Proof. Suppose that $d \neq 4k(k^2-1)$. Since this implies $z \neq 2k^2 - 1$, we may apply *Lemma 5*. *Theorem 2* (with $N = k$) and *Lemma 7* (with $p_1 = x$, $p_2 = y$, $q = z$) together imply that

$$(22.6k)^{-1} z^{-1-\lambda} < 1.55z^{-2}.$$

Since $\lambda < 1$, we have $z^{1-\lambda} < 35.03k$ and

$$\log z < \frac{\log(35.1k)}{1-\lambda}. \tag{15}$$

Since

$$\frac{1}{1-\lambda} < \frac{\log(0.197k^2)}{\log(0.0175k)} < \frac{2\log(0.444k)}{\log(0.0175k)},$$

we see from *Lemma 5* and (15) that

$$k-1 < \frac{\log(0.444k)\log(35.1k)}{\log(2k-3)\log(0.0175k)} =: f(k).$$

It is easy to see from

$$2k-3 < 35.1k \ \text{ and } \ 0.0175k < 0.444k$$

that $f(k)$ is decreasing. Since $f(63) < 55$, we must have $k < 63$, which is a contradiction. Therefore we obtain $d = 4k(k^2-1)$. □

5. Completion of the proof of Theorem 1

In this section, we complete the proof of *Theorem 1* using the reduction method of Dujella and Pethő (based on that of Baker and Davenport). On account of Proposition 1, it suffices to show *Theorem 1* for odd integers k with $5 \le k \le 61$. Throughout this section, let k be such an integer and assume that $\{k-2, k+2, 4k, d\}$ is a $D(4)$-quadruple with $d \ne 4k(k^2-1)$, which implies that $v_m = w_n$ for some integers $m \ge 1$ and $n \notin \{0, -2\}$.

Lemma 8. *Let $k \ge 5$ be an integer. If $v_m = w_n$ for some nonzero integers m and n, then we have*

$$0 < \Lambda := m\log\alpha_1 - |n|\log\alpha_2 + \log\alpha_3 < 0.8\alpha_1^{-2m}, \tag{16}$$

where

$$\alpha_1 := \frac{k+\sqrt{k^2-4}}{2}, \ \alpha_2 := k-1+\sqrt{k(k-2)}, \ \alpha_3 := \frac{2(\sqrt{k-2}+\sqrt{k+2})\sqrt{k}}{(\pm\sqrt{k-2}+2\sqrt{k})\sqrt{k+2}}.$$

Proof. We know by (11) and (14) that

$$\begin{aligned} v_m = \frac{1}{\sqrt{k+2}} \Bigg\{ & (\sqrt{k-2}+\sqrt{k+2})\left(\frac{k+\sqrt{k^2-4}}{2}\right)^m \\ & -(\sqrt{k-2}-\sqrt{k+2})\left(\frac{k-\sqrt{k^2-4}}{2}\right)^m \Bigg\} \end{aligned}$$

and

$$\begin{aligned} w_n = \frac{1}{2\sqrt{k}} \Big\{ & (\pm\sqrt{k-2}+2\sqrt{k})(k-1+\sqrt{k(k-2)})^n \\ & -(\pm\sqrt{k-2}-2\sqrt{k})(k-1-\sqrt{k(k-2)})^n \Big\}, \end{aligned}$$

where the plus (resp. minus) sign corresponds to the case $n > 0$ (resp. $n < 0$). Putting

$$P := \frac{\sqrt{k-2}+\sqrt{k+2}}{\sqrt{k+2}}\left(\frac{k+\sqrt{k^2-4}}{2}\right)^m, \; Q := \frac{\sqrt{k-2}+2\sqrt{k}}{2\sqrt{k}}(k-1\pm\sqrt{k(k-2)})^n,$$

we see from $v_m = w_n$ that

$$P + \frac{4}{k+2}P^{-1} = Q + \frac{3k+2}{4k}Q^{-1}. \tag{17}$$

Since $4/(k+2) < 1$, $P > 1$, $Q > 1$ and

$$\begin{aligned} P - Q &= \frac{3k+2}{4k}Q^{-1} - \frac{4}{k+2}P^{-1} \\ &> \frac{4}{k+2}(Q^{-1} - P^{-1}) = \frac{4}{k+2}(P-Q)P^{-1}Q^{-1}, \end{aligned}$$

we have $P > Q$. The assumption $m \geq 1$ implies that

$$P \geq \frac{\sqrt{k-2}+\sqrt{k+2}}{\sqrt{k+2}} \cdot \frac{k+\sqrt{k^2-4}}{2} > \frac{2\sqrt{k-2}(k-1)}{\sqrt{k+2}} > k,$$

and the relation (17) implies that

$$Q > P - \frac{3k+2}{4k}Q^{-1} > P - \frac{3k+2}{4k}.$$

Hence by $k \geq 5$ we have

$$\begin{aligned} P - Q &= \frac{3k+2}{4k}Q^{-1} - \frac{4}{k+2}P^{-1} \\ &< \frac{3k+2}{4k}\left(1 - \frac{3k+2}{4k}P^{-1}\right)^{-1} P^{-1} - \frac{4}{k+2}P^{-1} \\ &< \left(\frac{3k+2}{4k}\left(1 - \frac{3k+2}{4k^2}\right)^{-1} - \frac{4}{k+2}\right)P^{-1} \\ &< \frac{3k^3 - (8k^2 - 16k - 8)}{4k^3 + (5k^2 - 8k - 4)}P^{-1} < \frac{3}{4}P^{-1}. \end{aligned}$$

It follows from

$$0 < \frac{P-Q}{P} < \frac{3}{4}P^{-2} < \frac{3}{4}k^{-2} < 0.03$$

that

$$\begin{aligned} 0 < \log\frac{P}{Q} &= -\log\left(1 - \frac{P-Q}{P}\right) \\ &< \frac{3}{4}P^{-2} + \left(\frac{3}{4}P^{-2}\right)^2 \\ &< \frac{3}{4}P^{-2}\left(1 + \frac{3}{4}k^{-2}\right) < 0.8P^{-2}. \end{aligned}$$

Since

$$P^{-2} < \left(\frac{k+\sqrt{k^2-4}}{2}\right)^{-2m},$$

we obtain (16). □

The first inequality of (16) immediately implies that

$$m \geq |n|. \tag{18}$$

Indeed, if $m \leq |n| - 1$, then we would have

$$\begin{aligned}
\Lambda &\leq |n| \log\left(\frac{k+\sqrt{k^2-4}}{2} \cdot \frac{1}{k-1+\sqrt{k(k-2)}}\right) \\
&\quad + \log\left(\frac{2(\sqrt{k-2}+\sqrt{k+2})\sqrt{k}}{(\pm\sqrt{k-2}+2\sqrt{k})\sqrt{k+2}} \cdot \frac{2}{k+\sqrt{k^2-4}}\right) \\
&< \log\left(\frac{1}{k-1+\sqrt{k(k-2)}} \cdot \frac{2\sqrt{k(k-2)}+2\sqrt{k(k+2)}}{\sqrt{k(k+2)}}\right) \\
&< \log\frac{2\sqrt{k(k+2)}+2\sqrt{k(k-2)}}{k(k-1)+k\sqrt{k(k-2)}} < 0,
\end{aligned}$$

which is a contradiction.

In order to bound m above, we need the following theorem due to Baker and Wüstholz.

Theorem 3 [[3, Theorem]]. *For a linear form $\Lambda \neq 0$ in logarithms of l algebraic numbers $\alpha_1, \dots, \alpha_l$ with rational integer coefficients $\beta_1, \dots, \beta_l$, we have*

$$\log|\Lambda| \geq -18(l+1)!\, l^{l+1}(32d)^{l+2} h'(\alpha_1)\cdots h'(\alpha_l)\log(2ld)\log\beta,$$

where $\beta := \max\{|\beta_1|, \dots, |\beta_l|\}$, $d := [\mathbf{Q}(\alpha_1, \dots, \alpha_l) : \mathbf{Q}]$ and

$$h'(\alpha) := \frac{1}{d}\max\{h(\alpha), |\log\alpha|, 1\}$$

with the standard logarithmic Weil height $h(\alpha)$ of α.

Let α_3' be the "conjugate" of α_3:

$$\alpha_3' := \frac{2(\sqrt{k-2}+\sqrt{k+2})\sqrt{k}}{(\mp\sqrt{k-2}+2\sqrt{k})\sqrt{k+2}}.$$

Applying *Theorem 3* with $l = 3$, $d = 4$, $\beta = m$ and

$$\begin{aligned}
h'(\alpha_1) &= \frac{1}{2}\log\alpha_1, \\
h'(\alpha_2) &= \frac{1}{2}\log\alpha_2, \\
h'(\alpha_3) &\leq \frac{1}{4}\left\{\log\left((3k+2)^2(k+2)^2\right) + \log(\alpha_3\alpha_3')\right\} \\
&< \frac{1}{4}\log(16k^2(3k+2)(k+2)) < \frac{1}{4}\log(77k^4),
\end{aligned}$$

we have

$$\log\Lambda > -18\cdot 4!\cdot 3^4(32\cdot 4)^5\cdot \frac{1}{2}\log\alpha_1\cdot\frac{1}{2}\log\alpha_2\cdot\frac{1}{4}\log(77k^4)\cdot\log 24\cdot\log m.$$

Since $\alpha_2 < 2k-1$, we see from (16) that

$$\frac{m}{\log m} < 1.2\cdot 10^{14}\log(2k-1)\log(77k^4).$$

It follows from $k \le 61$ that

$$m < 5\cdot 10^{17}.$$

The following is based on the Baker-Davenport lemma ([2, Lemma]).

Lemma 9 [[10, Lemma 5 a)]]. *Let M be a positive integer. Let p/q be the convergent of the continued fraction expansion of κ such that $q > 6M$. Put $\epsilon := ||\mu q|| - M\,||\kappa q||$, where $||\cdot||$ denotes the distance from the nearest integer. If $\epsilon > 0$, then the inequality*

$$0 < m\kappa - n + \mu < AB^{-m}$$

has no solution in the range

$$\frac{\log(Aq/\epsilon)}{\log B} \le m < M.$$

Now dividing (16) by $\log\alpha_2$ leads us to the inequality

$$0 < m\kappa - |n| + \mu < AB^{-m}, \tag{19}$$

where

$$\kappa := \frac{\log\alpha_1}{\log\alpha_2},\ \ \mu := \frac{\log\alpha_3}{\log\alpha_2},\ \ A := \frac{0.8}{\log\alpha_2},\ \ B := \alpha_1^2.$$

We apply *Lemma 9* to the inequality (19) with $M = 5\cdot 10^{17}$. Note that (18), $n \notin \{0,-2\}$ and *Lemma 4* together imply that if $k \ge 7$ (resp. $k = 5$), then

$$m \ge |n| \ge 2k-2 \ge 12\ \text{ (resp. } m \ge 8).$$

We have to examine $29\cdot 2 = 58$ cases (the doubling comes from the signs "$\pm$" in α_3), of which the second convergent of κ with $q > 6M$ is needed only in two cases. Thus, in case $k \ge 7$, we obtain $m < 12$, which is a contradiction; in case $k = 5$, we obtain $m < 14$, in which case the second step of reduction with $M = 13$ gives $m < 4$, which is a contradiction. This completes the proof of *Theorem 1.*

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