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Fixed points of strip φ-contractions

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Abstract. In this paper, we introduce strip φ-contraction, where φ is an altering distance function, and obtain sufficient conditions for the existence of fixed points for such maps. Further, we extend it to a pair of selfmaps. These results improve and generalize the results of Khan, Swaleh and Sessa [1], Sastry and Babu [5] and Park [4] to strip φ-contractions.
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1. Introduction

Throughout this paper we assume that $(X,\ d)$ is a metric space denoted simply by X and T a selfmap of X, $R^+ = [0,\ \infty)$, N denotes the set of all natural numbers. For $x \in X$, $O_T(x) = \{x,\ Tx,\ T^2x, \dots\}$ denotes the *orbit of* x with respect to T. We denote the closure of $O_T(x)$ by $\overline{O_T(x)}$.

We say that T is *orbitally continuous* at a point $z \in X$ with respect to $x \in X$ if for any sequence $\{x_n\} \subset O_T(x)$, with $x_n \to z$ as $n \to \infty$ implies $Tx_n \to Tz$ as $n \to \infty$. Here we note that any continuous selfmap of a metric space is orbitally continuous, but an orbitally continuous map may not be continuous. For more details and examples, see Turkoglu *et al.* [6].

We write
$\Phi = \{\varphi : R^+ \to R^+ :\ \varphi$ is continuous and $\varphi(t) = 0$ if and only if $t = 0\}$.

We call an element $\varphi \in \Phi$ an "altering distance function".

Park [4] proved the following theorem.

Theorem 1 (see [4])**.** *Let T be a selfmap of X.*

$$\text{Suppose that for some } x_0 \in X, O_T(x_0) \ \text{ has a cluster point } z \text{ in } X. \tag{1}$$

If T is orbitally continuous at z and Tz and T satisfies

$$d(Tx,\ Ty) < d(x,\ y) \tag{2}$$

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By using an altering distance function $\varphi \in \Phi$, Sastry and Babu [5] proved the following theorem.

Theorem 2 (see [5]). *Let* T *be a selfmap of* X. *Suppose that* T *satisfies* (1). *If* T *is orbitally continuous at* z *and* Tz, *and if there exists* $\varphi \in \Phi$ *such that*

$$\varphi(d(Tx,\ Ty)) < \varphi(d(x,\ y)) \tag{3}$$

for each $x,\ y \in \overline{O_T(x_0)},\ x \neq y,\ y = Tx$, *then* z *is a fixed point of* T.

Remark 1. *Theorem 1 follows by choosing* $\varphi(t) = t,\ t \geq 0$, *in Theorem 2.*

Theorem 3 (see [4]). *Let* T *be a selfmap of a metric space* X. *Assume that for some positive integer* m, *there exists a point* $x_0 \in X$ *such that*

$$O_{T^m}(x_0) \text{ has a cluster point } z \text{ in } X, \tag{4}$$

and

$$d(T^m x,\ T^m y) < d(x,\ y) \tag{5}$$

for all $x,\ y \in X,\ x \neq y$. *Then* z *is a unique fixed point of* T *in* X.

The study of fixed points of Meir-Keeler type contractions in the presence of an altering distance function is an interesting and open area. Thus the purpose of this paper is to introduce strip φ- contraction for $\varphi \in \Phi$, which is more general than Meir-Keeler type contraction (Example 1), and obtain sufficient conditions for the existence of fixed points for such maps. Further, it is extended to a pair of selfmaps. These results improve and generalize the theorems of Khan, Swaleh and Sessa [1], Sastry and Babu [5] and Park [4] to strip φ-contractions.

2. Preliminaries

Meir and Keeler [3] established a fixed point theorem for selfmaps satisfying the following $(\epsilon,\ \delta)$ - contraction, which is known as Meir-Keeler type contraction.

Definition 1. *Given* $\epsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\epsilon \leq d(x,\ y) < \epsilon + \delta \text{ implies } d(Tx,\ Ty) < \epsilon \tag{6}$$

for all $x,\ y$ *in* X.

Maiti and Pal [2] improved condition (6) in the following way and obtained fixed points: given $\epsilon > 0$, there is a $\delta > 0$, such that

$$\epsilon \leq max\{d(x,\ y),\ d(x,\ Tx),\ d(y,\ Ty)\} < \epsilon + \delta$$

implies

$$d(Tx,\ Ty) < \epsilon$$

for all $x,\ y$ in X.

We now introduce *"strip φ-contraction"* as follows:

Definition 2. *Let (X,d) be a metric space and T a selfmap on X. Let $\varphi \in \Phi$. We say that T is a strip φ-contraction if for a given $\epsilon > 0$, there is a $\delta > 0$, such that*

$$\epsilon \leq \varphi(d(x,\ y)) < \epsilon + \delta \text{ implies } \varphi(d(Tx,\ Ty)) < \epsilon \tag{7}$$

for all $x,\ y$ in X.

Here we observe that every strip φ-contraction is a Meir-Keeler type contraction when φ is the identity map of R^+. The following example shows that the class of all strip φ-contractions is larger than the class of all Meir-Keeler type contractions.

Example 1. *Let $X = N$ with the usual metric. Define $T : X \to X$ by $Tx = x^3$. Define $\varphi : R^+ \to R^+$ by*

$$\varphi(t) = \begin{cases} \dfrac{t^2}{2}, & \text{if } 0 \leq t \leq 1 \\ \dfrac{1}{2t^2}, & \text{if } t \geq 1. \end{cases}$$

Then clearly $\varphi \in \Phi$.
We now show that T is a strip φ-contraction. Let $0 < \epsilon < 1$. For any $l,\ m \in X$, with $l \neq m$,

$$0 < \epsilon = \varphi(|l-m|) = \frac{1}{2(l-m)^2} < \epsilon + \delta \text{ with } \delta = min\{\epsilon,\ 1-\epsilon\}.$$

Then we have

$$\varphi(|Tl - Tm|) = \varphi(|l^3 - m^3|) = \frac{1}{2((l^3-m^3)^2)} < \frac{1}{4(l-m)^2} < \frac{1}{2}(\epsilon+\delta) \leq \epsilon,$$

so that T satisfies the strip φ-contraction condition. The case when $\epsilon \geq 1$ is trivial.
But for $x = 1$, $y = 5$, with φ the identity map of R^+, choosing $\epsilon = 4$ and for any $\delta > 0$, we have

$$\epsilon \leq |x-y| = 4 < \epsilon + \delta \text{ and } |Tx - Ty| = |T1 - T5| = 124 \nleq \epsilon,$$

so that T is not a Meir-Keeler type contraction.

The following example shows that the orbital continuity of T at z may not imply the orbital continuity of T at Tz, where z is as in (1).

Example 2. *Let $X = \{\frac{1}{n},\ n \in N\} \cup \{1 - \frac{1}{n},\ n \in N\}$ with the usual metric. We define $T : X \to X$ by*

$$T(0) = 1,\ T(1) = 1,\ T\left(\frac{1}{n}\right) = 1 - \frac{1}{n} \text{ for } n = 2,3,\ldots$$

and

$$T\left(1 - \frac{1}{n}\right) = \frac{1}{n+1} \text{ for } n = 3,4,\ldots.$$

First we show that T is orbitally continuous at 0. *Let $x \in X$. If $\{x_n\} \subseteq O_T(x)$ such that $x_n \to 0$, then $\{x_n\}$ is a subsequence of $\{\frac{1}{k}\}$ and hence $Tx_n = 1 - x_n \to 1 = T(0)$.*

But T is not orbitally continuous at $T(0)$, since $1 - \frac{1}{n} \in O_T(\frac{1}{3})$ for $n \geq 3$, $1 - \frac{1}{n} \to 1 = T(0)$ as $n \to \infty$ and $T(1 - \frac{1}{n}) = \frac{1}{n+1} \to 0 \neq T(T(0)) = 1$ as $n \to \infty$.

3. Fixed point theorems using strip φ-contractions

Theorem 4. *Let T be a selfmap of X. Suppose that T satisfies* (1). *Further, assume that*

$$\begin{gathered} \text{given } \epsilon > 0, \text{ there exist } \varphi \in \Phi \text{ and } \delta > 0, \\ \text{such that } \epsilon \leq \varphi(d(x,\ y)) < \epsilon + \delta \text{ implies } \varphi(d(Tx,\ Ty)) < \epsilon \end{gathered} \tag{8}$$

for all $x,\ y \in \overline{O_T(x_0)}$, $x \neq y$, $y = Tx$. Then z is a fixed point of T in $\overline{O_T(x_0)}$ provided T is orbitally continuous at z. This z is unique in the sense that $\overline{O_T(x_0)}$ contains one and only one fixed point z of T.

Proof. We define the sequence $\{x_n\} \subseteq X$ by $\{x_n\} = T^n x_0$, for $n = 1, 2, \ldots$. Let $\alpha_n = \varphi(d(x_n,\ x_{n+1}))$. If $x_n = x_{n+1}$ for some $n \in N$, then the conclusion of the theorem trivially holds.

Suppose $x_n \neq x_{n+1}$ for all n. Then from (8), we have

$$\alpha_{n+1} = \varphi(d(Tx_n, Tx_{n+1})) < \varphi(d(x_n, x_{n+1})) = \alpha_n.$$

Similarly $\alpha_n < \alpha_{n-1}$.

Therefore $\{\alpha_n\}$ is a decreasing sequence of non-negative reals and hence it converges to a nonnegative real number α (e.g.).

From (1), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \to z$ *as* $k \to \infty$. Hence

$$\begin{aligned} \alpha &= \lim_{k\to\infty} \alpha_{n(k)} \\ &= \lim_{k\to\infty} \varphi(d(x_{n(k)},\ x_{n(k)+1})) \\ &= \varphi(\lim_{k\to\infty} d(x_{n(k)},\ x_{n(k)+1})) \\ &= \varphi(d(z,\ Tz)) \end{aligned}$$

(since T is orbitally continuous at z). Now, we claim that $\alpha = 0$. Suppose $\alpha > 0$. Then

$$\alpha = \inf_{n\geq 1} \varphi(d(x_n,\ x_{n+1})).$$

Also, for any $\delta > 0$ there exists m in N such that

$$\alpha \leq \varphi(d(x_n,\ x_{n+1})) < \alpha + \delta \ for\ all\ n \geq m. \tag{9}$$

In particular

$$\alpha \le \varphi(d(x_m,\ x_{m+1})) < \alpha + \delta.$$

Hence from (8) and (9), we have

$$\alpha \le \varphi(d(x_{m+1},\ x_{m+2})) < \alpha,$$

a contradiction.

Therefore $\alpha = 0$ so that $\lim_{n\to\infty} \varphi(d(x_n,\ x_{n+1})) = 0$ and since φ is an element of Φ, it follows that $\lim_{n\to\infty} d(x_n,\ x_{n+1}) = 0$ so that $d(z,\ Tz) = 0$. Hence $Tz = z$. □

Theorem 5. *Let T be a selfmap of X. Assume that T satisfies* (1)*. Further, assume that given $\epsilon > 0$ there exist $\varphi \in \Phi$ and $\delta > 0$, such that*

$$\epsilon \le max\{\varphi(d(x,\ y)),\ \varphi(d(x,\ Tx)),\ \varphi(d(y,\ Ty))\} < \epsilon + \delta \tag{10}$$

implies

$$\varphi(d(Tx,\ Ty)) < \epsilon$$

for all $x,\ y$ in X. Then z of (1) *is a unique fixed point of T.*

Proof. Follows as a corollary to Theorem 4, in the sense that condition (10) implies (8). □

Theorem 6. *Let T be a selfmap of X. Assume that for some $x_0 \in X$ and for some positive integer m*

$$O_{T^m}(x_0) \text{ has a cluster point } z \text{ in } X. \tag{11}$$

Further, assume that for a given $\epsilon > 0$ there exist $\varphi \in \Phi$ and $\delta > 0$ such that

$$\epsilon \le \varphi(d(x,\ y)) < \epsilon + \delta \ \text{ implies } \ \varphi(d(T^m x,\ T^m y)) < \epsilon \tag{12}$$

for all $x \ne y,\ x,\ y \in X$, i.e. T^m is a strip φ-contraction. Then z is the unique fixed point of T, provided T^m is orbitally continuous at z.

Proof. By replacing T by T^m in Theorem 4, T^m has a unique fixed point z in X. Therefore $T^m z = z$. Now

$$Tz = T(T^m z) = T^{m+1} z = T^m(Tz).$$

Therefore Tz is also a fixed point of T^m. We now show that $Tz = z$. Suppose $Tz \ne z$. Then from (12) for

$$\epsilon = \varphi(d(z, Tz)) < \epsilon + \delta$$

implies

$$\varphi(d(T^m z, T^m(Tz))) = \varphi(d(z, Tz)) < \epsilon = \varphi(d(z, Tz)),$$

a contradiction. Therefore $Tz = z$. □

Remark 2. *Strip φ-contraction is actually stronger than* (3)*, since condition* (8) *implies* (3)*. Hence some condition(s) in the hypotheses of Theorem 2, namely T is orbitally continuous at Tz, may be relaxed under strip φ-contraction in obtaining fixed points, which is established in our results (Theorem 4 and Theorem 6).*

Remark 3. *In Theorem 4 we need not assume that strip φ-contraction condition* (8) *holds on the whole space X. The following example gives its justification.*

Example 3. *Let $X = N \cup \{0,\ 2^{-1}, 2^{-2}, ...\}$ with the usual metric. We define $T : X \to X$ by*

$$T(0) = 0,\ T(n) = n + 1,\ T(2^{-n}) = 2^{-(n+1)},\ n = 1, 2, 3, \ldots.$$

Here $X = O_T(1) \cup O_T(2^{-1}) \cup \{0\}$. At $x = 1,\ y = 2$, condition (7) *fails to hold for any $\varphi \in \Phi$, since $\varphi(d(x,\ y)) = \varphi(d(1,\ 2)) = \varphi(1)$, and*

$$\varphi(d(Tx,\ Ty)) = \varphi(d(T1,\ T2)) = \varphi(d(2,\ 3)) = \varphi(1).$$

Therefore for $\epsilon = \varphi(1)$, strip φ-contraction condition (8) *fails to hold in $O_T(1)$ for any $\varphi \in \Phi$ and has no fixed point in $O_T(1)$.*

But strip φ-contraction holds on the closure of the orbit of 2^{-1}, where

$$\overline{O_T(2^{-1})} = \{0,\ 2^{-1},\ 2^{-2}, ...\} \text{ with } \varphi(t) = t^2,\ t \geq 0 \text{ and } \delta = min\{\epsilon, 1 - \epsilon\}$$

when $0 < \epsilon < 1$; T satisfies all the hypotheses of Theorem 4, with 0 as the cluster point of $O_T(2^{-1})$; and T has the unique fixed point 0.

Thus, condition (8) *is more general than condition* (7).

Remark 4. *The following two examples show that*

(1) every strip φ-contraction need not be a contraction, and

(2) an operator satisfying strip φ-contraction may not have a fixed point if T does not satisfy orbital continuity at z of (1) *in X.*

Example 4. *Let $X = \{1 + 2^{-n} : n = \ 1,\ 2,\ 3,\ ...\} \cup \{1\}$ with the usual metric. We define T on X by*

$$T(1) = 1 + 2^{-1} \text{ and } T(1 + 2^{-n}) = 1 + 2^{-(n+1)},\ \ n = \ 1,\ 2,\ 3, \ldots.$$

For $x_0 = 1$; $O_T(x_0) = \{1 + 2^{-n} : n = \ 1,\ 2,\ 3,\ ....\}$, $\overline{O_T(x_0)} = O_T(x_0) \cup \{1\}$.

Then T satisfies all the conditions of Theorem 4 with φ, the identity map of R^+ with $\delta = min\{\epsilon, 1 - \epsilon\}$ for $0 \leq \epsilon < 1$, but T is not orbitally continuous at $z(= 1)$ and it has no fixed point.

Example 5. *Let*

$$X = \left\{ \sum_{i=0}^{n} 2^{-i} : n \in N \right\} \cup \{1, 2\},$$

with the usual metric. Define T on X by

$$T2 = 1,\ T1 = 1 + 2^{-1},\ T\left(\sum_{i=0}^{n} 2^{-i}\right) = \sum_{i=0}^{n+1} 2^{-i},\ for\ n \in N.$$

If $x_0 = 1 + 2^{-1}$, then

$$O_T(1 + 2^{-1}) = \{\sum_{i=0}^{n} 2^{-i} : n \in N\}\ and\ \overline{O_T(1 + 2^{-1})} = O_T(1 + 2^{-1}) \cup \{2\}.$$

Also, T satisfies all the hypotheses of Theorem 4, with $\varphi(t) = \frac{t^2}{2},\ t > 0$, with

$$\delta = min\{\epsilon, 1 - \epsilon\}$$

but T is not orbitally continuous at $z(= 2)$ and it has no fixed point.

Remark 5. *Let us mention:*

(i) *In Theorem 4 we do not assume the orbital continuity of T at Tz. Hence Theorem 4 improves the results of Sastry and Babu [5] and hence also Park [4], which in turn improves the results of Khan, Swaleh and Sessa [1].*

(ii) *By strengthening condition (3) by (8), the orbital continuity at Tz is relaxed.*

4. Common fixed points for a pair of strip φ-contractions

We now extend Theorem 4 and Theorem 5 to a pair of selfmaps.

Theorem 7. *Let S and T be selfmaps of X such that for some $x_0 \in X$ we define the sequence $\{x_n\}$ by $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$, $n = 0,\ 1,\ 2, \ldots$. Assume that either (a) or (b) of the following holds:*

(a) *$\{x_{2n}\}$ has a cluster point z in X, S and TS are orbitally continuous at z,*

(b) *$\{x_{2n+1}\}$ has a cluster point z in X, T and ST are orbitally continuous at z.*

Further, assume that given $\epsilon > 0$ there exist $\varphi \in \Phi$ and $\delta > 0$ such that

$$\epsilon \le \varphi(d(x,\ y)) < \epsilon + \delta\ implies\ \varphi(d(Sx,\ Ty)) < \epsilon \tag{13}$$

for all $x,\ y$ in $\overline{\{x_n\}}, x \neq y$ satisfying either $x = Ty$ or $y = Sx$. Then either (i) or (ii) of the following is true:

(i) *either S or T has a fixed point in X,*

(ii) *z is a unique common fixed point of S and T in $\overline{\{x_n\}}$.*

Proof.

Suppose that $x_{2n} = x_{2n+1}$ for some n in N. Then S has a fixed point in X. (14)

If $x_{2n+1} = x_{2n+2}$ for some n in $N,$ then T has a fixed point in X. (15)

(14) and (15) together imply that conclusion (i) holds. Now assume that $x_n \neq x_{n+1}$ for all n. Write $\beta_n = \varphi(d(x_n,\ x_{n+1}))$. From (13) we have

$$\beta_{2n} = \varphi(d(x_{2n}, x_{2n+1})) = \varphi(d(Tx_{2n-1}, Sx_{2n})) < \varphi(d(x_{2n-1}, x_{2n})) = \beta_{2n-1}.$$

Therefore

$$\beta_{2n} < \beta_{2n-1}. \tag{16}$$

Similarly,

$$\beta_{2n+1} < \beta_{2n}. \tag{17}$$

Hence from (16) and (17) it follows that $\{\beta_n\}$ is a decreasing sequence of non-negative reals and it converges to a real number β (e.g.).

Now assume (a). Then there exists a sequence $\{n(k)\}$ of positive integers such that

$$x_{2n(k)} \to z, Sx_{2n(k)} \to Sz, T(Sx_{2n(k)}) \to TSz. \tag{18}$$

From the continuity of φ, we have

$$\beta = lim_{k\to\infty}\beta_{2n(k)} = lim_{k\to\infty}\varphi(d(x_{2n(k)}, x_{2n(k)+1})) = \varphi(d(z,\ Sz)).$$

We now claim that $\beta = 0$. Suppose that $\beta > 0$, then

$$\beta = \inf_{n\geq 1} \varphi(d(x_n,\ x_{n+1})).$$

Then for any $\delta > 0$, there exists $m \in N$ such that

$$\beta \leq \varphi(d(x_n,\ x_{n+1})) < \beta + \delta \ for\ all\ n \geq m. \tag{19}$$

In particular, writing $n = 2m$ and using (13), we have

$$\beta \leq \varphi(d(x_{2m},\ x_{2m+1})) < \beta + \delta$$

which implies

$$\varphi(d(Sx_{2m},\ Tx_{2m+1})) = \varphi(d(x_{2m+1},\ x_{2m+2})) < \beta,$$

a contradiction to (19). Hence $\beta = 0$ and it implies that $Sz = z$.

Now we prove that $Tz = z$. From (13) we have

$$\varphi(d(Sx_{2n(k)},\ Tx_{2n(k)+1})) < \varphi(d(x_{2n(k)},\ Sx_{2n(k)})).$$

Now by taking limits as $k \to \infty$, by using (18) and continuity of φ, it follows that $\varphi(d(Sz,\ T(Sz)) \leq \varphi(d(z,\ Sz)) = 0$. Thus $TSz = Sz$. Since $z = Sz$, it follows that $Tz = z$; and hence z is a common fixed point of S and T.

Similarly, when (b) holds, then it follows that z is a common fixed point of S and T. Uniqueness of a fixed point trivially follows from (13). Thus S and T have a unique common fixed point z in $\overline{\{x_n\}}$.

Hence conclusion (ii) follows. □

Theorem 8. *Let S and T be selfmaps of X such that for some $x_0 \in X$ the sequence $\{(TS)^n x_0\}$ has a convergent subsequence, which converges to a point z in X and S, and let TS be orbitally continuous at z. Further assume that S and T satisfy the following condition: given $\epsilon > 0$, there exist $\varphi \in \Phi$ and $\delta > 0$, such that*

$$\epsilon \leq max\{\varphi(d(x,\ y)),\ \varphi(d(x,\ Sx)),\ \varphi(d(y,\ Ty))\} < \epsilon + \delta \tag{20}$$

implies

$$\varphi(d(Sx,\ Ty)) < \epsilon \tag{21}$$

for all $x,\ y \in X$, $x \neq y$ satisfying either $x = Ty$ or $y = Sx$. Then, either (i) or (ii) of the following is true:

(i) either S or T has a fixed point in X,

(ii) S and T have a unique common fixed point in $\overline{\{(TS)^n x_0\}}$.

Proof. Follows as a corollary to Theorem 7, since (20) implies (13). □

The following is an example in support of Theorem 7.

Example 6. *Let $X = [0,\ 2)$ with the usual metric. We define $S,\ T:\ X \to X$ by*

$$Sx = \begin{cases} \frac{x}{2}, & if\ \ x \in [0,1) \\ \frac{x^2}{8}, & if\ \ x \in [1,\ 2), \end{cases} \quad and \quad Tx = \begin{cases} \frac{x}{2}, & if\ \ x \in [0,\ 1) \\ \frac{x^2}{16}, & if\ \ x \in [1,\ 2). \end{cases}$$

For any $x_0 \in [0,\ 1)$, the sequence $\{x_n\}$ defined in Theorem 7 is given by $x_n = \frac{x_0}{2^n}$, $n = 0, 1, 2, 3, \ldots$ and $\overline{\{x_n\}} = \{x_n\}_{n=0}^{\infty} \cup \{0\}$. Now for the case when $x_0 \in [1,\ 2)$, the sequence $\{x_n\}$ is given by

$$\{x_n\} = \{x_0\} \cup \{\frac{x_0^2}{2^{n+2}} : n = 1, 2, 3, ...\}\ and\ \overline{\{x_n\}} = \{x_0\} \cup \{x_n : n = 1, 2, 3, ...\} \cup \{0\}.$$

Case (i): Let $x_0 \in [0, 1)$. Let $0 < \epsilon < 1$ with $\delta = min\{\epsilon,\ 1 - \epsilon\}$. Define φ on R^+ by $\varphi(t) = t^2, t \geq 0$. For $x = \frac{x_0}{2^n}$ and $y = Sx = \frac{x_0}{2^{n+1}}$, $n = 0, 1, 2, \ldots$, we have

$$\begin{aligned} \varphi(d(x,\ y)) &= \varphi(|\frac{x_0}{2^n} - \frac{x_0}{2^{n+1}}|) = \varphi(\frac{x_0}{2^{n+1}}) = (\frac{x_0}{2^{n+1}})^2 < \epsilon + \delta, \\ \varphi(d(Sx,\ Ty)) &= \varphi(|\frac{x_0}{2^{n+1}} - \frac{x_0}{2^{n+2}}|) = \varphi(\frac{x_0}{2^{n+2}}) = (\frac{x_0}{2^{n+2}})^2 = (\frac{x_0}{2^{n+1}.2})^2 \\ &= \frac{1}{4}(\frac{x_0}{2^{n+1}})^2 < \frac{1}{4}(\epsilon + \delta) < \epsilon. \end{aligned}$$

Case (ii): Let $x_0 \in [1,\ 2)$. *Let* $0 < \epsilon < 1$, *with* $\delta = min\{\epsilon,\ 1-\epsilon\}$. *When* $x = x_0$; $y = Sx = \frac{x_0^2}{2^3}$, *we have*

$$\varphi(d(x,\ y)) = \varphi(|x_0 - \frac{x_0^2}{2^3}|) = \varphi(\frac{8x_0 - x_0^2}{2^3}) = (\frac{8x_0 - x_0^2}{2^3})^2 < \epsilon + \delta$$

and

$$\varphi(d(Sx,\ Ty)) = \varphi(|\frac{x_0^2}{2^3} - \frac{x_0^2}{2^4}|) = (\frac{x_0^2}{16})^2 < \frac{1}{2}(\epsilon + \delta) < \epsilon.$$

In general, when $x = \frac{x_0^2}{2^{n+2}}$; $y = Sx = \frac{x_0^2}{2^{n+3}}$, *we have*

$$\varphi(d(x,\ y)) = \varphi(|\frac{x_0^2}{2^{n+2}} - \frac{x_0^2}{2^{n+3}}|) = \varphi(\frac{x_0^2}{2^{n+3}}) = \frac{x_0^4}{(2^{n+3})^2} < \epsilon + \delta$$

and

$$\varphi(d(Sx,\ Ty)) = \varphi(|\frac{x_0^2}{2^{n+3}} - \frac{x_0^2}{2^{n+4}}|) = \frac{x_0^4}{(2^{n+3})^2.2^2} < \frac{1}{4}(\epsilon + \delta) < \epsilon.$$

Thus S *and* T *satisfy condition* (13) *with* $\varphi(t) = t^2,\ t \geq 0$.

Also, in any case $\{x_n\}$ *has a convergent subsequence which converges to the point 0; and satisfy all the hypotheses of Theorem 7; and 0 is the unique common fixed point of* S *and* T.

Remark 6. *The following is an example to show that conclusion* (i) *of Theorem 8 is valid.*

Example 7. *Let* $X = \{0,\ 1\}$. *Define selfmaps* S *and* T *on* X *by*

$$Sx = \begin{cases} 0, \text{ if } x = 0 \\ 1, \text{ if } x = 1, \end{cases} \quad \text{and} \quad Tx = \begin{cases} 1, \text{ if } x = 0 \\ 0, \text{ if } x = 1. \end{cases}$$

Then S *and* T *trivially satisfy strip* φ*-contraction for any* $\varphi \in \Phi$ *(in particular, we take* $\varphi(t) = \frac{t^2}{2}, t \geq 0$*) and they also satisfy all the conditions of Theorem 7. Observe that* S *has two fixed points 0 and 1 whereas* T *has no fixed points.*

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