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General relations between partially ordered multisets and their chains and antichains

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Abstract. In this paper we begin with the basics of multisets and their operations introduced in [5, 22] and define a multiset relation, an equivalence multiset relation and explore some of their basic properties. We also define a partially ordered multiset as a multiset relation being reflexive, antisymmetric and transitive, chains and antichains of a partially ordered multiset, and extend Dilworth's Theorems for partially ordered sets in the context of partially ordered multisets.

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1. Introduction

A multiset (mset for short) is a set with the additional feature that elements have multiple occurrences. A finite mset over a set X is an mset M formed with finitely many elements from X such that each element has a finite multiplicity of occurrence in M.

We begin with the definition of the notion of a subset of an mset called a submset, and the operations between msets [5, 22]. In 1950 R. P. Dilworth proved the theorem: Every partially ordered set (poset) can be partitioned into w-chains, where w is the width of the poset. Later in 1971 Mirsky proved its dual in the context of h-antichains, where h is the height of the poset. In this paper we will define an mset relation, a partially ordered mset (pomset), chains and antichains of pomsets and prove some theorems related to msets and pomsets. Finally, we will obtain the analogous of Dilworth's theorem and its dual for pomsets.

This paper is organized as follows. In section 2 we collect preliminaries and basic definitions based on msets. In sections 3, 4 and 5 we extend set theoretic results to msets, pomsets, chains and antichains of pomsets and a pomset version of Dilworth's theorem and its dual.

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2. Preliminaries and basic definitions

In this section we will collect basic definitions and notations as introduced by Cerf et al. [4] in 1972, Peterson [16] in 1976 and Yager [22] in 1986.

Definition 1 (see [5]). *A collection of elements which may contain duplicates is called a multiset. Formally if X is a set of elements, a multiset M drawn from the set X is represented by a function count M or C_M defined as $C_M : X \to N$ where N represents the set of non-negative integers.*

For each $x \in X$, $C_M(x)$ is the characteristic value of x in M and it indicates the number of occurrences of the elements x in M. A multiset M is a set if $C_M(x) = 0$ or 1 for all $x \in X$.

Definition 2 (see [5]). *Let M_1 and M_2 be two msets selected from a set X, then M_1 is a sub mset of M_2 ($M_1 \subseteq M_2$) if $C_{M1}(x) \leq C_{M2}(x)$ for all $x \in X$. M_1 is a proper sub mset of M_2 ($M_1 \subset M_2$) if $C_{M1}(x) \leq C_{M2}(x)$ for all $x \in X$ and there exists at least one $x \in X$ such that $C_{M1}(x) < C_{M2}(x)$.*

Definition 3 (see [5]). *Two msets M_1 and M_2 are equal ($M_1 = M_2$) if ($M_1 \subseteq M_2$) and ($M_2 \subseteq M_1$).*

Definition 4 (see [5]). *An mset M is empty if $C_M(x) = 0$ for all $x \in X$.*

Definition 5 (see [5]). *The cardinality of an mset M drawn from a set X, Card $M = \sum_{x\in X} C_M(x)$. It is also denoted by $|M|$.*

Definition 6 (see [5]). *Insertion of an element x into an mset M results in a new mset M' denoted by $M' = M \oplus x$ such that $C_{M'}(x) = C_M(x) + 1$ and $C_{M'}(y) = C_M(y)$ for all $y \neq x$.*

Definition 7 (see [5]). *Addition of two msets M_1 and M_2 drawn from a set X results in a new mset $M = M_1 \oplus M_2$ such that for all $x \in X$, $C_M(x) = C_{M1}(x) + C_{M2}(x)$.*

Definition 8 (see [5]). *The removal of an element x from an mset M results in a new mset M' denoted by $M' = M \ominus x$ such that $C_{M'}(x) = \max\{C_M(x) - 1, 0\}$ and $C_{M'}(y) = C_M(y)$ for all $y \neq x$.*

Definition 9 (see [5]). *Subtraction of two msets M_1 and M_2 drawn from a set X results in a new mset M, denoted by $M = M_1 \ominus M_2$ such that*

$$C_M(x) = \max\{C_{M1}(x) - C_{M2}(x), 0\}.$$

Definition 10 (see [5]). *The union of two msets M_1 and M_2 drawn from a set X is an mset M denoted by $M = M_1 \cup M_2$ such that for all $x \in X$,*

$$C_M(x) = \max\{C_{M1}(x), C_{M2}(x)\}.$$

Definition 11 (see [5]). *The intersection of two msets M_1 and M_2 drawn from a set X is an mset M denoted by $M = M_1 \cap M_2$ such that for all $x \in X$,*

$$C_M(x) = \min\{C_{M1}(x), C_{M2}(x)\}.$$

2.1. Notation

Let M be an mset from X and let x appear n times in M. We denote it by $x \in^n M$. $M = \{k_1/x_1, k_2/x_2, \ldots, k_n/x_n\}$ also means that M is an mset with x_1 appearing k_1 times, x_2 appearing k_2 times and so on. $[M]_x$ denotes the element x belonging to the mset M and $|[M]_x|$ denotes the cardinality of an element x in M.

An entry of the form $(m/x, n/y)/k$ means that the pair (x, y) is repeated with x m-times, y n-times and the pair occurring k-times. $C_1(x, y)$ denotes the count of the first coordinate in the ordered pair (x, y) and $C_2(x, y)$ denotes the count of the second coordinate in the ordered pair (x, y).

The mset order relations and quasi-mset order relations are denoted by the symbol $\leq:\leq$ and strict mset orders by $<:<$. We write $m/x \leq:\leq n/y$ in R which means x and y are comparable and irrespective of the counts of x and y, $m/x \leq:\leq: n/y$ in R which means x and y are incomparable and irrespective of the counts of x and y and $m/x \leq\leq n/y$ which means that x and y are comparable and the corresponding counts are also comparable. A strict mset order is denoted analogously. Also $m/x == n/y$ in R if and only if $m = n$ and $x = y$, $m/x =\neq n/y$ in R if and only if $m = n$ and $x \neq y$, $m/x \neq= n/y$ in R if and only if $m \neq n$ and $x = y$, and $m/x \neq\neq n/y$ if and only if $m \neq n$ and $x \neq y$. The notation $m/x <:< n/y$ in R means that $m/x \neq\neq n/y$ and $m/x \leq:\leq n/y$.

Definition 12 (see [5]). *The mset space X^n is the set of all msets whose elements are in X such that no element in an mset occurs more than n times. The set X^∞ is the set of all msets over a domain X such that there is no limit on the number of occurrences of an element in an mset. If $X = \{x_1, x_2, \ldots, x_k\}$, then*

$$X^n = \{\{n_1/x_1, n_2/x_2, \ldots, n_k/x_k\} \ \ \textit{for } i = 1, 2, \ldots, k, \ \ n_i \in \{0, 1, 2, \ldots, n\}\}$$

Definition 13 (see [5]). *Let M be an mset drawn from a set X. The support set of M denoted by M^* is a subset of X and $M^* = \{x \in X : C_M(x) > 0\}$, i.e., M^* is an ordinary set. M^* is also called a root set.*

Definition 14 (see [5]). *Let X be a support set and X^n the mset space defined over X. Then for any mset $M \in X^n$, the complement M^c of M in X^n is an element of X^n such that $C_{M^c}(x) = n - C_M(x)$ for all $x \in X$.*

Remark 1. *Using Definition 12, the mset sum can be modified as follows:*

$$C_{M1 \oplus M2}(x) = \min\{n, C_{M1}(x) + C_{M2}(x)\} \ \textit{for all } x \in X.$$

If X is a set with n distinct elements, then a power set of $X, P(X)$ contains exactly 2^n distinct elements. If X is an mset with n-elements (repetitions counted), then $P(X)$ contains strictly less than 2^n elements because singleton submsets do not repeat in $P(X)$. Unlike classical set theory, Cantor's power set theorem fails for msets. It is possible to formulate a reasonable definition of a power mset of X for finite msets X that preserves Cantor's theorem.

Definition 15 (see [6]). *A power mset, denoted by $\tilde{P}(X)$, of mset X is defined as follows:*

$Y \in \tilde{P}(X)$ iff $Y \subseteq X$, if $Y = \varphi$, then $Y \in^1 \tilde{P}(X)$, and if $Y \neq \varphi$, then $Y \in^k \tilde{P}(X)$ where $k = \prod_z \binom{|[X]_z|}{|[Y_z]|}$, the product $\prod_z$ is taken over distinct elements of z of the mset Y and $|[X]_z| = m$ iff $z \in^m X$,$|[Y]_z| = n$ iff $z \in^n Y$, then

$$\binom{|[X]_z|}{|[Y]_z|} = \binom{m}{n} = \frac{m!}{n!(m-n)!}.$$

Example 1. *Let $M = \{6/x, 3/y\}$ be an mset and let $\tilde{P}(M)$ denote the power mset, if $\{3/x\}$ is a member of $\tilde{P}(M)$, then $\{3/x\}$ repeats $k = \binom{6}{3} = 20$ times. Also, if $\{4/x, 2/y\}$ is a member of $\tilde{P}(M)$, then $\{4/x, 2/y\}$ repeats $k = \binom{6}{4}\binom{3}{2} = 45$ times.*

Example 2. *Let $M = \{1, 2, 2\} = \{1/1, 2/2\}$ be an mset. Then the power mset of M*

$$\tilde{P}(M) = \{1/\phi, 1/M, 1/\{1/1\}, 2/\{1/2\}, 1/\{2/2\}, 2/\{1/1, 1/2\}\}.$$

3. Multiset relations

The concept of a multirelation was introduced by Winskel [21] in 1987, and those are structures similar to Multiset relations. But the way in which Multiset relations are defined in this paper is entirely different from multirelations.

Definition 16. *Let M_1 and M_2 be two msets drawn from a set X; then the Cartesian product of M_1 and M_2 is defined as*

$$M_1 \times M_2 = \{(m/x, n/y)/mn : x \in^m M_1, y \in^n M_2\}.$$

We now define the Cartesian product of three or more nonempty msets by generalizing the definition of the Cartesian product of two msets. That is, the Cartesian product $M_1 \times M_2 \times \cdots \times M_n$ of nonempty msets $M_1, M_2, \ldots, M_n$ is the mset of all ordered n-tuples $(m_1, m_2, \ldots, m_n)$ where $m_i \in^{ri} M_i$, $i = 1, 2, \ldots, n$ and $(m_1, m_2, \ldots, m_n) \in^p M_1 \times M_2 \times \ldots \times M_n$ with $p = \prod r_i$, $r_i = C_{M_i}(m_i)$, $i = 1, 2, \ldots, n$.

Example 3. *Let $M_1 = \{1/1, 2/2\}$ and $M_2 = \{4/3\}$ be two msets; then $M_1 \times M_2 = \{(1/1, 4/3)/4, (2/2, 4/3)/8\}$.*

Theorem 1. *For any two nonempty msets M_1 and M_2,*

$$C_{M_1 \times M_2}[(x, y)] = C_{M_1}(x) \cdot C_{M_2}(y) \text{ and } |M_1 \times M_2| = |M_1| \cdot |M_2|.$$

In general, $|M_1 \times M_2 \times M_3 \times \cdots \times M_n| = |M_1| \cdot |M_2| \cdot |M_3| \ldots |M_n|$.

Definition 17. *A sub mset R of $M \times M$ is said to be an mset relation on M if every member $(m/x, n/y)$ of R has count $C_1(x, y) \cdot C_2(x, y)$. We denote m/x related to n/y by $m/xRn/y$.*

Definition 18. *Domain and range of the mset relation R on M is defined as follows:*
$Dom\, R = \{x \in^r M : \exists y \in^s M$ such that $r/xRs/y\}$, where $C_{Dom\, R}(x) = \sup\{C_1(x, y) : x \in^r M\}$.
$Ran\, R = \{y \in^s M : \exists x \in^r M$ such that $r/xRs/y\}$, where $C_{Ran\, R}(x) = \sup\{C_2(x, y) : y \in^s M\}$.

Example 4. *Let $M = \{8/x, 11/y, 15/z\}$ be an mset. Then*

$$R = \{(2/x, 4/y)/8, (5/x, 3/x)/15, (7/x, 11/z)/77, (8/y, 6/x)/48, (11/y, 13/z)/143, (7/z, 7/z)/49, (12/z, 10/y)/120, (14/z, 5/x)/70\}$$

is an mset relation defined on M. Here Dom $R = \{7/x, 11/y, 14/z\}$ and Ran $R = \{6/x, 10/y, 13/z\}$.

Definition 19. *It holds:*

(i) An mset relation R on an mset M is reflexive iff $m/xRm/x$ for all m/x in M, irreflexive iff $m/xRm/x$ never holds.

(ii) An mset relation R on an mset M is symmetric iff $m/xRn/y$ implies $n/yRm/x$, antisymmetric iff $m/xRn/y$ and $n/yRm/x$ implies $m/x == n/y$ for all m/x, n/y in M.

(iii) An mset relation R on an mset M is transitive if $m/xRn/y$, $n/yRk/z$, then $m/xRk/z$.

Definition 20. *An mset relation R on an mset M is called an equivalence mset relation if it is reflexive, symmetric and transitive.*

Example 5. *Let $M = \{3/x, 5/y, 3/z, 7/r\}$. Then the mset relation given by*

$$R = \{(3/x, 3/x)/9, (3/x, 3/z)/9, (3/x, 7/r)/21, (7/r, 3/x)/21, (5/y, 5/y)/25, (3/z, 3/z)/9, (7/r, 7/r)/49, (3/z, 3/x)/9, (3/z, 7/r)/21, (7/r, 3/z)/21\}$$

is an equivalence mset relation.

Definition 21. *The identity mset relation in any mset M is the set of all pairs in $M \times M$ with equal co-ordinates and it is denoted by I_M.*

Example 6. *Let $M = \{2/x, 3/y, 2/z\}$ be an mset. Then the identity mset relation on M*

$$I_M = \{(2/x, 2/x)/4, (3/y, 3/y)/9, (2/z, 2/z)/4\}.$$

Definition 22. *A partition of a nonempty mset M is a collection P of nonempty sub msets of M such that*

(1) Each element of M belongs to one of the msets in P.

(2) If M_1 and M_2 are distinct elements of P, then $M_1 \cap M_2 = \emptyset$.

4. Partially ordered multisets

Definition 23. *Let R be an mset relation on an mset M. Then R is called a quasi-mset order (or a pre-mset order) if it is reflexive and transitive. In addition, if R is antisymmetric it is also called an mset order relation (or in short, an mset order) or a partially ordered mset relation. The pair (M, R) is called an ordered mset or partially ordered multiset (pomset) and it is denoted by M_P.*

Definition 24. *The mset relation R is called a linear mset order (or a total mset order) on M, if R is an mset order and if for every two elements $m/x \neq\neq n/y$ of M either $m/xRn/y$ or $n/yRm/x$ holds. (Both cannot hold since this would imply $m/x == n/y$ because of antisymmetry of R).*

Definition 25. *The mset relation R is called a strict mset order if R is irreflexive and transitive, and R is called a strict linear mset order if R is a strict mset order which satisfies for every two distinct elements $m/x, n/y$ of M either $m/xRn/y$ or $n/yRm/x$ hold.*

Remark 2. *A strict mset order R is antisymmetric since m/x and n/y would be elements with $m/xRn/y$ and $n/yRm/x$, the transitivity of R would imply $m/xRm/x$ contradiction to the irreflexivity of R.*

Theorem 2. *Let $<:<$ be a strict mset order on M. Then the mset relation*

$$\leq:\leq \quad = \quad <:< \cup\, I_M$$

is an mset order on M.

Proof. $\leq:\leq$ is reflexive by definition. Let $m/x, n/y$ be in M and $m/x \leq:\leq n/y$ and $n/y \leq:\leq m/x$. If $m/x \neq\neq n/y$ holds, we have $m/x <:< n/y$ and $n/y <:< m/x$, hence $m/x <:< m/x$ which is a contradiction to the irreflexivity of $<:<$. Thus $\leq:\leq$ is antisymmetric.

Let $m/x, n/y, p/z$ be in M and $m/x \leq:\leq n/y$ and $n/y \leq:\leq p/z$. If $m/x == n/y$ or $n/y == p/z$ holds, then $m/x \leq:\leq p/z$ is trivial. Otherwise, $m/x <:< n/y$ and $n/y <:< p/z$, then $m/x <:< p/z$ and $m/x \leq:\leq p/z$. □

Theorem 3. *Let R be a reflexive and antisymmetric relation on an mset M. Then the following statements are equivalent.*
(a) R is a linear mset order on M.
(b) R and its complementary mset relation R^c are both transitive.

Proof. (a) ⇒ (b): Clearly R is transitive. Let $m/x, n/y, p/z$ be in M and $m/xR^cn/y$ and $n/yR^cp/z$. Then neither $m/xRn/y$ nor $n/yRp/z$ would hold. Therefore m/x is not R-related to p/z. Thus $m/xR^cp/z$ and R^c is transitive.
(b) ⇒ (a): Suppose R and its complementary mset relation R^c are both transitive. If $m/x, n/y$ are distinct elements of M, then either $m/xRn/y$ or $n/yRm/x$ must hold. Otherwise, we would have $m/xR^cn/y$ and $n/yR^cm/x$, hence $m/xR^cm/x$ since R^c is transitive. But this contradicts $m/xRm/x$. And so R is a linear mset order. □

Remark 3. *For a linear mset order R on M the inverse mset relation R^{-1} and the complementary mset relation R^c "nearly" coincide. They differ only in the identity mset relation I_M.*

Theorem 4. *Let R be an mset order relation on M. Then the following two statements are equivalent.*
(a) R is a linear mset ordering.
(b) $R^c = R^{-1} \setminus\ I_M$.

Proof. (a) $\Rightarrow$ (b): Trivial.
(b) $\Rightarrow$ (a): Let $m/x, n/y$ be distinct elements of M. If neither $m/xRn/y$ nor $n/yRm/x$ held, we would have $m/xR^c n/y$ and $n/yR^c m/x$ and therefore by (b), $m/xR^{-1}n/y$ and $n/yR^{-1}m/x$. This implies $n/yRm/x$ and $m/xRn/y$, so that $m/x == n/y$ a contradiction to our assumption. Thus R is a linear mset ordering. $\square$

Remark 4. *If we have a reflexive and antisymmetric mset relation $\leq:\leq$ on an mset M and if we want to show that it is also transitive, it suffices evidently to prove that the mset relation $<:<$, which is defined by $<:<=\leq:\leq \setminus I_M$, is transitive.*

Example 7. *Let M be an mset and $P(M)$ the set of all submsets of M. For submsets M_1, M_2 of M we put $M_1 \leq\leq M_2$ if and only if $M_1 \subseteq M_2$. Then $\leq\leq$ is an mset order relation.*

Example 8. *The set of n-tuples, from $\prod M_i$, $i = 1, 2, \ldots, n$ and M_i's are msets and partially ordered by*

$$(m_1/x_1, m_2/x_2, \ldots, m_n/x_n) \leq:\leq (k_1/y_1, k_2/y_2, \ldots, k_n/y_n)$$

if and only if $m_i/x_i \leq:\leq k_i/y_i$ for $i = 1, 2, \ldots, n$. Then $\leq\leq$ is an mset order relation.

Remark 5. *Next statements also holds:*

1. *Let R be an mset relation which is symmetric and transitive. If m/x is R-related to n/y, then $m/xRm/x$ because $m/xRn/y$ implies $n/yRm/x$. Then, we have $M = S \cup (M \setminus S)$, where S is the mset of elements of M which are not R-related to any element of M, and $R/(M \setminus S)$ is an equivalence mset relation on $M \setminus S$.*

2. *If the mset relation R is antisymmetric and transitive, then $R \cup I_M$ is an mset order relation $\leq:\leq$ on M. Its corresponding strict mset order $<:<$ is a sub mset order relation R.*

3. *The mset relation R is irreflexive, symmetric and transitive only if $R = \emptyset$. If there existed elements $m/x, n/y$ in M with $m/xRn/y$, we would also have $n/yRm/x$ and then $m/xRm/x$, which contradicts irreflexivity.*

5. Chains and antichains in pomsets

Definition 26. *Let $M_P = (M, \leq:\leq)$ be a pomset, $m/x, n/y$ in M. If $m/x <:< n/y$ holds, m/x is said to be a predecessor of n/y and n/y a successor of m/x.*

If $m/x <:< n/y$ holds and if there is no element k/z in M which satisfies $m/x <:< k/z <:< n/y$, then m/x is called an immediate predecessor of n/y or n/y is called an immediate successor of m/x.

Definition 27. *A submset C of $M_P = (M, \leq:\leq)$ is called a chain in a pomset if every distinct pair of points from C is comparable in M_P, i.e., for all distinct m/x, n/y in C, then $m/x <:< n/y$ in M_P.*

A pomset $M_P = (M, \leq:\leq)$ itself is called a chain if every distinct pair of points from M is comparable in M_P. When $(M, \leq:\leq)$ is a chain, we call M_P a linear mset order (also a total mset order) on an mset M.

Definition 28. *A submset A of $M_P = (M, \leq:\leq)$ is called an antichain if every distinct pair of points from A is incomparable in M_P, i.e., for all distinct m/x, n/y in A, then $m/x <:<: n/y$ in M_P.*

A pomset $M_P = (M, \leq:\leq)$ itself is called an antichain if every distinct pair of points from M is incomparable in M_P.

Definition 29. *Let $M_P = (M, \leq:\leq)$ be a pomset and N a nonempty submset of M, the restriction of M_P to N, denoted by $M_P(N)$ (or M_P/N) is a partial mset order on N and we call $(N, M_P(N))$ a sub pomset of $M_P = (M, \leq:\leq)$.*

Definition 30. *A nonempty submset $N \subseteq M$ is called a chain (resp. antichain) if the sub pomset $(N, M_P(N))$ is a chain (resp. antichain). One element (element with some multiplicity) of M is both a chain and an antichain and it is said to be trivial. Chains and antichains of two or more points are non trivial.*

Definition 31. *If the cardinality of an mset M is m, then the corresponding chain (resp. antichain) $M_P = (M, \leq:\leq)$ is called an m-chain (resp. m-antichain) on M.*

Definition 32. *A point m/x in M is called a maximal element (resp. minimal element) if there is no element n/y in M with $m/x <:< n/y$ in M_P (resp. $m/x :>> n/y$ in M_P). We denote the mset of all maximal elements of a pomset $M_P = (M, \leq:\leq)$ by Max (M_P), while Min(M_P) denotes the mset of all minimal elements.*

Definition 33. *An element m/x in M is called the greatest element of M_P if $n/y \leq:\leq m/x$ for every n/y in M. Similarly, m/x is called a least element of M_P if $n/y :\leq\leq m/x$ for every n/y in M.*

Remark 6. *If a pomset has a least element, then this is also the minimal element, analogously the greatest element is the maximal element. In a linearly ordered mset the notions "minimal element" and "least element" evidently coincide, as well as "maximal" and "greatest". But this is not true in general.*

Example 9. *Consider an mset $\{3/x_1, 4/x_2, 2/x_3, 3/x_4, 5/x_5\}$ whose order $\leq:\leq$ is given by*

$$\begin{aligned} &3/x_1 <:< 2/x_3 <:< 5/x_5, 4/x_2 <:< 2/x_3 <:< 3/x_4,\\ &3/x_1 <:<: 4/x_2, 3/x_4 <:<: 5/x_5. \end{aligned}$$

In this mset $3/x_1$ and $4/x_2$ are minimal elements, $3/x_4$ and $5/x_5$ are maximal elements. This mset has neither a greatest nor a least element.

The mset of all chains in a pomset M_P is partially ordered by mset inclusion and the maximal elements in this pomset are called maximal chains. A chain C is a maximum chain if no other chain contains more points than C. Maximal and maximum antichains are defined analogously. Both Max (M_P) and Min (M_P) are maximal antichains.

Theorem 5 (see [19], Pigeonhole principle for msets). *Suppose that M_1, M_2, ..., M_n are submsets of an mset M satisfying $M_i \cap M_j = \emptyset$ for all i and j. Then $M_1 \cup M_2 \cup \cdots \cup M_n \subseteq M$.*

Definition 34. *Let $M_P = (M, \leq:\leq)$ be a finite pomset, m/x an element of M_P. The height $h(m/x)$ is the greatest non-negative integer h, so that there exists a chain $\{m_1/x_1, m_2/x_2, \ldots, m_t/x_t\}$, where $m_1/x_1 <:< m_2/x_2 <:< \cdots <:< m_t/x_t = m/x$ and $h = \sum m_i$, $i = 1, 2, \ldots, t$.*

For $t \in N$, let $C_t = \{m_1/x_1, m_2/x_2, \ldots, m_t/x_t\}$ be the chain, then

$$\begin{aligned} h(m_1/x_1) &= m_1, \\ h(m_2/x_2) &= m_1 + m_2, \\ &\vdots \\ h(m_t/x_t) &= m_1 + m_2 + \cdots + m_t, \end{aligned}$$

where m_t is the count of the element x_t.

For $n \in N$, let L_n denote the mset of all elements of M which have height m_n. It is called the n-level or height-n-mset of M. Thus L_n is the mset of all last elements of the chains of height m_n.

Definition 35. *The height of a pomset $M_P = (M, \leq:\leq)$, denoted by height (M_P), is the largest integer h for which there exists a chain $\{m_1/x_1, m_2/x_2, \ldots, m_t/x_t\}$ of height $h = \sum m_i$, $i = 1, 2, \ldots, t$, i.e., height (M_P) is the cardinality of a maximum chain.*

From Definition 34 it follows that height $(M_P) = \text{Max}\ \{h(m/x) : x \in^m M\}$.

Theorem 6. *Let $M_P = (M, \leq:\leq)$ be a finite pomset.*

(1) For two non-negative integers $\sum_i m_i$, $\sum_j m_j$ less than height (M_P), $m_i \leq m_j$, $x \in^{mi} L_i$, $y \in^{m_j} L_j$, we have $m_i/x \leq\leq m_j/y$ or $m_i/x \leq:\leq m_j/y$.

(2) For every non-negative integer $\sum_n m_n <$ Height (M_P) the level L_n is an antichain. Levels L_n are pairwise disjoint and they form a partition of M in antichains.

Proof. Every chain among those chains which have m_j/y in L_j as the last element contains $\sum_j m_j$ elements. Otherwise we would have $m_j/y << m_i/x$, one could extend these chains by adjoining m_i/x and obtain a chain of cardinality $\sum_j m_{j+1} > \sum_i m_i$ which has m_i/x as the last element. But this contradicts m_i/x in L_i, so part (1) is proved. Proof of (2) follows from (1) by putting $m_i = m_j$. □

Theorem 7. *Let h be the height of a nonempty finite pomset M_P. For $\sum_i m_i < h$, then L_i is the mset of all minimal elements of $M_P \backslash \cup \{L_v : \sum_v m_v < \sum_i m_i\}$. In particular, L_1 is the mset of all minimal elements of M_P.*

Proof. Let m_i/x_i be in L_i. If m_i/x_i were minimal in the mset $\cup\{L_v : \sum_v m_v \geq \sum_i mi\}$, then this mset would contain an element $m_v/y < m_i/x$, and we would have m_v/y in L_v for an index $v \geq i$. Hence there would exists a chain of cardinality

$\sum_v m_v$ with m_v/y as the last element and then also a chain of cardinality $\sum_v m_{v+1}$ with m_i/x as the last element. This would yield $h(m_i/x) \geq \sum_v m_v > \sum_i m_i$ contradicting m_i/x in L_i. $\square$

Theorem 8. *Let m/x, n/y be elements of a finite pomset $M_P = (M, \leq:\leq)$ with m/x as the immediate predecessor of n/y. Then we have $h(n/y) \geq h(m/x)$.*

Proof. Every chain of cardinality $h(m/x)$ which ends in m/x yields a chain of cardinality $h(m/x) + n$ $(\leq h(n/y))$ which ends in n/y by attaching the element n/y. $\square$

Remark 7. *From Theorem 8 one cannot conclude that $h(n/y) = h(m/x) + n$ always holds. For example, let M be an mset with $3/x_1, 2/x_2, 4/x_3, 3/x_4$ as elements, where $3/x_1 <:< 2/x_2 <:< 4/x_3$ and $3/x_4 <:< 4/x_3$ hold and $3/x_4$ is incomparable with $3/x_1$ and $2/x_2$, then $h(4/x_3) = 9$, $h(3/x_4) = 3$, but $3/x_4$ the immediate predecessor of $4/x_3$ holds. Thus $h(4/x_3) \neq h(3/x_4) + 4$.*

Theorem 9. *Let the assumptions of Theorem 7 be fulfilled and m_n/x in L_n for a non negative integer $\sum_n m_n < h$. Further, let $m_1/x_1 <:< m_2/x_2 <:< \cdots <:< m_n/x_n$ be the longest chain among the chains which have $m_n/x == m_n/x_n$ as the last element. Then m_v/x_v is in L_v, $v = 1, 2, \ldots, n$.*

Proof. The msets $L_1, L_2, \ldots$ are antichains and therefore the elements $m_1/x_1, m_2/x_2, \ldots, m_n/x_n$ are all in different levels. Then by Theorem 8 we have $h(m_1/x_1) < h(m_2/x_2) < \cdots < h(m_n/x_n) = \sum_n m_n$. This is possible only if $h(m_v/x_v) = \sum_v m_v$, $v = 1, 2, \ldots, n$. $\square$

Theorem 10. *Every mset order $\leq:\leq$ on a finite mset M is a submset of a linear mset order on this mset. In other words: Every mset order relation on a finite mset is extensible to a linear mset order.*

Proof. Let $M_P = (M, \leq:\leq)$ be a non-empty finite pomset. We consider the levels $L_1, L_2, \ldots, L_n$ of M_P, where $\sum_n m_n$ is the height (M_P). Then we take an arbitrary linear mset order in each of these levels, and if m_i, m_j are elements of $\{m_1, m_2, \ldots, m_n\}$ with $m_i < m_j$, we put $m_i/x << m_j/y$ for all m_i/x in L_i and all m_j/y in L_j. Together we obtain a linear mset order in M_P, which extends the original mset order $\leq:\leq$. $\square$

Theorem 11. *Let $M_P = (M, \leq:\leq)$ be a pomset and let height $(M_P) = h$, then there exists a partition $M = M_1 \cup M_2 \cup \cdots \cup M_t$ where M_i's are antichains for $i = 1, 2, \cdots, t$ and $t \leq h$. Furthermore, there is no partition using a fewer number of antichains.*

Proof. For each m/x in M, let height (m/x) be the largest integer s for which there exists a chain $m_1/x_1 <:< m_2/x_2 <:< \cdots . <:< m_r/x_r$ with $s = \sum m_i$, $i = 1, 2, \ldots, r$ and $m/x == m_r/x_r$. Evidently, height $(m/x) \leq h$, i.e., $s \leq h$ for all m/x in M. Then for each $i = 1, 2, \ldots, t$, let $M_i = \{x \in^m M :$ height $(m/x) = \sum_i m_i\}$. It is easy to see that each M_i is an antichain, as if m/x, n/y in M_i are such that $m/x <:< n/y$, then there is a chain

$$m_1/x_1 <:< m_2/x_2 <:< \cdots <:< m_i/x_i == m/x <:< m_{i+1}/x_{i+1} == n/y.$$

So height

$$(n/y) = \sum m_p > \sum m_p = \text{ height } (m/x),$$

which is not possible. So each M_i is an antichain. Since height $(M_P) = h$, there is a maximum chain $C = \{m_1/x_1, m_2/x_2, \ldots, m_t/x_t\}$ with $h = \sum m_i/x_i$, $i = 1, 2, \ldots, t$. If it were possible to partition M_P into r antichains with $r < t$, then by the Pigeonhole principle one of the antichain would contain two elements with some multiplicity or one element with some multiplicity from C, but which is absurd. $\square$

Definition 36. *The width of a pomset $M_P = (M, \leq:\leq)$ denoted by width (M_P) is the largest integer w for which there exists an antichain $\{m_1/x_1,\ m_2/x_2,\ \ldots,\ m_t/x_t\}$ with $w = \sum m_i$, $i = 1, 2, \ldots, t$, i.e., width (M_P) is the cardinality of the largest antichain.*

Remark 8. *It is clear that for a finite pomset M_P there always exists an antichain of M_P whose cardinality is width (M_P). For an infinite mset order this is not always valid.*

Definition 37. *Let $M_P = (M, \leq:\leq)$ be a pomset and m/x in M. Then the following is defined.*

$$\begin{aligned}
D(m/x) &= \{y \in^n M : n/y <:< m/x\}\\
D[m/x] &= \{y \in^n M : n/y \leq:\leq m/x\}\\
U(m/x) &= \{y \in^n M : n/y :>> m/x\}\\
U[m/x] &= \{y \in^n M : n/y :\geq\geq m/x\}\\
I(m/x) &= \{y \in^n M\backslash\{m/x\} : m/x <:< n/y\}.
\end{aligned}$$

When $S \subseteq M$, $D(S) = \{y \in^n M : n/y <:< m/x$ for some m/x in $S\}$ and $D[S] = S \cup D(S)$. The submsets $U(S)$ and $U[S]$ are defined dually.

Theorem 12 (Dilworth Theorem for Pomsets)**.** *Let $M_P = (M, \leq:\leq)$ be a pomset, and width $(M_P) = w$, then there exists a partition $M = C_1 \cup C_2 \cup \cdots \cup C_t$, where C_i's are chains for $i = 1, 2, \cdots, t$ and $t \leq w$. Furthermore, there is no partition into a fewer number of chains.*

Proof. As the proof of Theorem 11, the Pigeonhole principle implies that we require at least $t \leq w$ chains. To prove this, we proceed by induction on t, a suffix of the chains of M. The result is trivial if $t = 1$ i.e., $|M| = m_1$. Assume validity for all pomsets with $t \leq r$, i.e., $|M| = m_1 + m_2 + \cdots + m_t \leq m_1 + m_2 + \cdots + m_t + m_r$ and suppose that M_P is a pomset with

$$|M| = m_1 + m_2 + \cdots + m_t + m_{t+1} = m_1 + m_2 + \cdots + m_t + m_r.$$

Without loss of generality, width $(M_P) > m_1$; else a trivial partition $M = C_1$ satisfies the conclusion of the theorem. Furthermore, we observe that if C is a nonempty chain in M_P, then we may assume that the sub pomset $(M\backslash C, M_P(M\backslash C))$ also has width w. To see this, observe that the theorem holds for the subpomset, so that if width $(M\backslash C, M_P(M\backslash C)) = w' < w$, then we can partition $M\backslash C$ as

$M \backslash C = C_1 \cup C_2 \cup \cdots \cup C_s$ with $s < t$, so that $M = C \cup C_1 \cup C_2 \cup \cdots \cup C_s$ is a partition into $s+1$ chains. Since $s < t$, we know $s+1 \leq t$, so that we have a partition of M into atmost t chains with $t \leq w$. Since any partition of M into chains must use at least t chains, this is exactly the partition we seek.

If m/x in M is a loose point in M_P, then $C = \{m/x\}$ is a chain and a subpomset $(M \backslash C, M_P(M \backslash C))$ has a width less than w as any maximal antichain contains every loose point. We may therefore assume that M_P has no loose points.

Choose a maximal point m/x and a minimal point n/y with $n/y <:< m/x$ in M_P; such a pair exists since there are no loose points. Then set $C = \{m/x, n/y\}$, $N = M \backslash C$ and $Q = M_P(N)$. Now, width $(N, Q) = w$, so (N, Q) contains a w-element antichain $A = \{m_1/x_1, m_2/x_2, \ldots, m_t/x_t\}$ with $w = \sum m_i/x_i$, $i = 1, 2, \ldots t$. Note that $U[A] \neq M$ since n/y does not belong to $U[A]$, and $D[A] \neq M$ since m/x does not belong to $D[A]$. Therefore, we may apply the inductive hypothesis to $D[A]$ and $U[A]$. Also note that $D[A] \neq U[A] = A$ since if there were $z \in^q (D[A] \cup U[A]) \backslash A$, then there would become $z' \in^p A$ such that $p/z' :>> q/z$ and $z'' \in^r A$ such that $q/z :>> r/z''$. However, these facts would combine to imply that $p/z' :>> r/z''$, contrary to the fact that A is an antichain.

By the inductive hypothesis, we know that we can partition each $U[A]$ and $D[A]$ into chains. Without loss of generality, we may label these partitions as $U[A] = C'_1 \cup C'_2 \cup \cdots \cup C'_t$ and $D[A] = C''_1 \cup C''_2 \cup \cdots \cup C''_t$, where $x_i \in^{mi} C'_i \cap C''_i$ for $i = 1, 2, \ldots, t$. However, this implies that $M = (C'_1 \cup C''_1) \cup (C'_2 \cup C''_2) \cup \cdots \cup (C_t \cup C''_t)$, which is the desired partition. $\square$

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