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On an extension of a quadratic transformation formula due to Kummer

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Abstract. The aim of this research note is to prove the following new transformation formula

$$(1-x)^{-2a}\ {}_3F_2\left[\begin{matrix} a,\ a+\frac{1}{2},\ d+1 \\ c+\frac{3}{2},\quad d \end{matrix}\ ;\ \frac{x^2}{(1-x)^2}\right]$$
$$= {}_4F_3\left[\begin{matrix} 2a,\quad c,\quad 2d+\frac{1}{2}A+\frac{1}{2},\ 2d-\frac{1}{2}A+\frac{1}{2} \\ 2c+2,\ 2d+\frac{1}{2}A-\frac{1}{2},\ 2d-\frac{1}{2}A-\frac{1}{2} \end{matrix}\ ;\ 2x\right]$$

valid for $|x| < \frac{1}{2}$ and if $|x| = \frac{1}{2}$, then $\mathrm{Re}(c-2a) > 0$, where $A = \sqrt{16d^2 - 16cd - 8d + 1}$. For $d = c + \frac{1}{2}$, we get quadratic transformations due to Kummer. The result is derived with the help of the generalized Gauss's summation theorem available in the literature.

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Key words: quadratic transformation formula, Kummer transformation, Gauss summation theorem

1. Introduction and results required

We start with the following very interesting and useful quadratic transformation formula due to Kummer [3] or [5, eq. 1, p. 65] viz.

$$(1-x)^{-2a}\,{}_2F_1\left[\begin{matrix} a,\ a+\frac{1}{2}, \\ c+\frac{1}{2}, \end{matrix}\ ;\ \frac{x^2}{(1-x)^2}\right] = {}_2F_1\left[\begin{matrix} 2a,\ \ c, \\ 2c, \end{matrix}\ ;\ 2x\right]. \tag{1}$$

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The aim of this research note is to provide an extension of (1) by employing the following summation formula [4, eq. 10, p. 534]

$$ {}_3F_2\left[\begin{matrix} f,\ a,\ c+1 \\ b,\quad c \end{matrix};1\right] = \frac{(c-a)(\alpha-f)}{c}\,\frac{\Gamma(b)\Gamma(b-a-f-1)}{\Gamma(b-a)\Gamma(b-f)} \tag{2} $$

provided $Re(b-a-f)>1, c\neq 0,-1,-2,\ldots$ and α is given by

$$ \alpha = \frac{c(1+a-b)}{a-c}. $$

2. Main result

The extension of the Kummer's quadratic transformation formula (1) to be established is given by the following theorem.

Theorem 1. *For* $|x|<\frac{1}{2}$ *or if* $|x|=\frac{1}{2}$*, then* $\mathrm{Re}(c-2a)>0$*, we have for* $d\neq 0,-1,-2,\ldots$

$$ \begin{aligned} (1-x)^{-2a}\,{}_3F_2&\left[\begin{matrix} a,\ a+\frac{1}{2},\ d+1 \\ c+\frac{3}{2},\quad d \end{matrix};\frac{x^2}{(x-1)^2}\right] \\ &= {}_4F_3\left[\begin{matrix} 2a, & c, & 2d+\frac{1}{2}A+\frac{1}{2}, & 2d-\frac{1}{2}A+\frac{1}{2} \\ & 2c+2, & 2d+\frac{1}{2}A-\frac{1}{2}, & 2d-\frac{1}{2}A-\frac{1}{2} \end{matrix};2x\right], \end{aligned} \tag{3} $$

where

$$ A=\sqrt{16d^2-16cd-8d+1}. \tag{4} $$

Proof. In order to derive (3), we proceed as follows. Denoting the left-hand side of (3) by $S(x)$, we have

$$ \begin{aligned} S(x) &= (1-x)^{-2a}\,{}_3F_2\left[\begin{matrix} a,\ a+\frac{1}{2},\ d+1 \\ c+\frac{3}{2},\quad d \end{matrix};\frac{x^2}{(1-x)^2}\right] \\ &= \sum_{m=0}^{\infty}\frac{(a)_m(a+\frac{1}{2})_m(d+1)_m}{(c+\frac{3}{2})_m(d)_m m!}x^{2m}(1-x)^{-(2a+2m)} \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m(a+\frac{1}{2})_m(d+1)_m}{(c+\frac{3}{2})_m(d)_m m!}x^{2m}\frac{(2a+2m)_n}{n!}x^n. \end{aligned} $$

Using now Bailey's transform of the double series, the appropriate Pochhammer symbol transformation formula and summing up the resulting series, we get

$$\begin{aligned} S(x) &= \sum_{n=0}^{\infty} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(a)_m (a+\frac{1}{2})_m (d+1)_m (2a+2m)_{n-2m}}{(c+\frac{3}{2})_m (d)_m m! (n-2m)!} x^n \\ &= \sum_{n=0}^{\infty} \frac{(2a)_n}{n!} x^n \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-\frac{1}{2}n)_m (-\frac{1}{2}n+\frac{1}{2})_m (d+1)_m}{(c+\frac{3}{2})_m (d)_m m!} \\ &= \sum_{n=0}^{\infty} \frac{(2a)_n}{n!} x^n \, {}_3F_2 \left[\begin{matrix} -\frac{1}{2}n, \, -\frac{1}{2}n+\frac{1}{2}, \, d+1 \\ c+\frac{3}{2}, \quad d \end{matrix} ; 1 \right]. \end{aligned}$$

The inner ${}_3F_2$ can be evaluated using (2) by taking $f = -\frac{1}{2}n, a = -\frac{1}{2}n + \frac{1}{2}, b = c + \frac{3}{2}$ and $c = d$. After simplification we get

$$\begin{aligned} S(x) &= \sum_{n=0}^{\infty} \frac{(2a)_n (c)_n 2^n x^n}{(2c+2)_n n!} \frac{(2d+\frac{1}{2}A+\frac{1}{2})_n (2d-\frac{1}{2}A+\frac{1}{2})_n}{(2d+\frac{1}{2}A-\frac{1}{2})_n (2d-\frac{1}{2}A-\frac{1}{2})_n} \\ &= {}_4F_3 \left[\begin{matrix} 2a, \quad c, \quad 2d+\frac{1}{2}A+\frac{1}{2}, \, 2d-\frac{1}{2}A+\frac{1}{2} \\ 2c+2, \, 2d+\frac{1}{2}A-\frac{1}{2}, \, 2d-\frac{1}{2}A-\frac{1}{2} \end{matrix} ; 2x \right] \end{aligned}$$

where A is the same as in (4). This completes the proof of (3). □

Corollary 1. *In (3), if we take $d = c + \frac{1}{2}$, then since $A = 1$, we get at once the Kummer's result (1) which was rederived by Bailey [1, p. 243] by employing Gauss's summation theorem [2, eq. 1, p. 2]. Hence (3) can be regarded as an extension of (1).*

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