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Levi subgroups of p-adic $Spin(2n+1)$

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Abstract. We explicitly describe Levi subgroups of odd spin groups over algebraic closure of a p-adic field.

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1. Introduction

Let F be an algebraic closure of a p-adic field. For $n \in \mathbb{N}$, let $Spin(2n+1, F)$ be the split simply-connected algebraic group of type B_n. $Spin(2n+1, F)$ is a double covering, as algebraic groups, of the odd special orthogonal group $SO(2n+1, F)$. In the representation theory, it is very important to know what Levi subgroups look like in the considered group. In some other classical groups, such as already mentioned $SO(n, F)$, Levi subgroups are isomorphic to a product of some general linear groups and another $SO(m, F)$, where $m \leq n$, i.e. the product of some general linear groups and a classical group of a smaller rank and of the same type. But, this is not the case for spin groups, which implies that some different techniques for investigating these groups have to be used. Examples of Levi subgroups of $Spin(5, F)$ can be found in [1], so we assume $n > 2$. Examples of Siegel Levi subgroups can be found in [5].

Here is an outline of the paper. Section 2 presents some preliminaries, mainly from [3] and [6]. In the third section, we have a case-by-case consideration of Levi subgroups. The same method was used by Asgari in [2] to determine Levi subgroups of a simply-connected group of type F_4.

2. Preliminaries

Fix a maximal torus T of $Spin(2n+1, F)$ and a Borel subgroup B containing T. The based root system associated to $(Spin(2n+1, F), B, T)$, $(X, \Sigma, X^\vee, \Sigma^\vee)$, is given by

$$X = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \cdots \oplus \mathbb{Z}e_{n-1} \oplus \mathbb{Z}\frac{e_1 + \cdots + e_n}{2},$$
$$X^\vee = \mathbb{Z}(e_1^\vee - e_2^\vee) \oplus \mathbb{Z}(e_2^\vee - e_3^\vee) \oplus \cdots \oplus \mathbb{Z}(e_{n-1}^\vee - e_n^\vee) \oplus \mathbb{Z}2e_n^\vee$$

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Let $\Sigma = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a system of simple roots, where $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, $\dots$, $\alpha_{n-1} = e_{n-1} - e_n$, $\alpha_n = e_n$. We denote the associated coroots by $\Sigma^\vee = \{\alpha_1^\vee, \alpha_2^\vee, \dots, \alpha_n^\vee\}$, where

$$\alpha_1^\vee = e_1^\vee - e_2^\vee, \alpha_2^\vee = e_2^\vee - e_3^\vee, \dots, \alpha_{n-1}^\vee = e_{n-1}^\vee - e_n^\vee, \alpha_n^\vee = 2e_n^\vee$$

(observe that $e_1, \dots, e_n$ are chosen in the standard way, such that $\langle e_i, e_j^\vee \rangle = \delta_{i,j}$).

Every standard Levi subgroup corresponds to some subset θ of Σ. A subgroup corresponding to θ will be denoted by M_θ. Each M_θ is an almost direct product of a connected component of its center and its derived group. A connected component of the center of M_θ will be denoted by A_θ, while a derived group of M_θ will be denoted by M'_θ. In other words,

$$M_\theta \simeq \frac{A_\theta \times M'_\theta}{A_\theta \cap M'_\theta}$$

Since $Spin(2n+1, F)$ is a simply-connected group, the derived group of each M_θ is also simply-connected, so it can be obtained directly from θ, i.e. from its root system. It is well - known that

$$A_\theta = (\bigcap_{\beta \in \theta} ker\beta)^0$$

so A_θ can also be obtained from the set of simple roots θ. After obtaining A_θ and M'_θ (which will be considered case-by-case, depending on the type of θ), we can construct their almost direct product to finally obtain M_θ.

The maximal torus of $Spin(2n+1, F)$ will be denoted by T. We have the next proposition ([2, Proposition 3.1.2], or [4, p. 108]), which holds for simply-connected groups:

Proposition 1. *Each $t \in T$ can be written uniquely as*

$$t = \prod_{i=1}^{n} \alpha_i^\vee(t_i), t_i \in F^*.$$

Kernels of simple roots in Σ can now be described as follows:

Proposition 2. *Let $t \in ker\alpha_i$. Then*

$$\alpha_i(t) = \alpha_i(\prod_{j=1}^{n} \alpha_j^\vee(t_j)) = \prod_{j=1}^{n} t_j^{\langle \alpha_i, \alpha_j^\vee \rangle} = 1.$$

This implies:

- *if $i = 1$, then $t_1^2 = t_2$*
- *if $2 \le i \le n-2$, then $t_i^2 = t_{i-1}t_{i+1}$*
- *if $i = n-1$, then $t_i^2 = t_{i-1}t_{i+1}^2$*
- *if $i = n$, then $t_i^2 = t_{i-1}$*

Let $z = \alpha_n^\vee(-1)$. From [2, Corollary 3.1.3], follows that the center of $Spin(2n+1, F)$ equals $\{1, z\} \simeq \mathbb{Z}_2$. From now on, z stands for the non-trivial element of the center of $Spin(2n+1, F)$, for some $n \geq 1$. We introduce the notion of general spin groups, following Asgari [2]. These groups are defined in the following way:

$$\begin{aligned} GSpin(2n+1, F) &= \frac{GL(1, F) \times Spin(2n+1, F)}{\{(1,1), (-1, z)\}}, n \geq 1, \\ GSpin(1, F) &= GL(1, F). \end{aligned}$$

The derived group of a general spin group is a spin group, so general spin groups are to spin groups as the general linear groups are to special linear groups. An advantage of general spin groups is that their Levi subgroups are isomorphic to a product of general linear groups and a general spin group of a smaller rank. This was proved in [2], using root datum of general spin groups. Another proof can be found in this manuscript.

3. Levi subgroups

Let us fix some notation. Let $\theta \subset \Sigma$, $\theta \neq \emptyset$. Here and subsequently, we will write θ as a union of connected components of its Dyinkin diagram,

$$\theta = \theta_1 \cup \theta_2 \cup \cdots \cup \theta_k$$

where $\theta_i \cap \theta_j = \emptyset$ for $i \neq j$. We choose $\theta_1, \ldots, \theta_k$ in such a way that for $\alpha_{i_1} \in \theta_{j_1}$ and $\alpha_{i_2} \in \theta_{j_2}$, where $j_1 < j_2$, then $i_1 < i_2$. For $1 \leq i \leq k$, let $n_i = |\theta_i|$. For a shorten notation, we write l_i instead of $\sum_{1 \leq j \leq i} n_j$. Now it follows that, if min_i is the minimal index such that $\alpha_{min_i} \in \theta_i$, then $\theta_i = \{\alpha_{min_i}, \alpha_{min_i+1}, \ldots, \alpha_{min_i+n_i-1}\}$. Also, if $\alpha_{i_1} \in \theta_{j_1}$ and $\alpha_{i_2} \in \theta_{j_2}$, where $j_1 < j_2$, then $i_2 - i_1 > 1$.

We write ζ_k for the $k-$th primitive root of identity in F^* and I_n for an $n \times n$ identity matrix.

Now we begin a case-by-case consideration:

(1) Suppose $\alpha_1 \in \theta$, $\alpha_{n-1}, \alpha_n \notin \theta$. Obviously, $\alpha_1 \in \theta_1$, $min_1 = 1$ and $min_k + n_k - 1 < n - 1$.

We obtain M'_θ using [4, Chapter 5., Theorem 1.33, Lemma 1.35 and Example 1.36], where a derived group of M_θ is described. In this case, M'_θ is isomorphic to $SL(n_1+1, F) \times SL(n_2+1, F) \times \cdots \times SL(n_k+1, F)$.

Let $\lambda_1 = t_1$. From Proposition 2 we get $t_2 = \lambda_1^2$, $t_3 = \lambda_1^3$, $\ldots$, $t_{n_1} = \lambda_1^{n_1}$, $t_{n_1+1} = \lambda_1^{n_1+1}$. Next, put $\lambda_2 = t_{n_1+2}$, $\lambda_3 = t_{n_1+3}$, $\ldots$, $\lambda_{min_2-n_1} = t_{min_2}$. If $min_2 = n_1 + 2$, then let $\mu_1 = \lambda_1^{n_1+1}$; let $\mu_1 = \lambda_{min_2-n_1-1}$ otherwise.

From Proposition 2 again, we obtain

$$\begin{aligned} t_{min_2+1} &= t_{min_2}^2 t_{min_2-1}^{-1} = \lambda_{min_2-n_1}^2 \mu_1^{-1}, \\ t_{min_2+2} &= t_{min_2+1}^2 t_{min_2}^{-1} = \lambda_{min_2-n_1}^4 \mu_1^{-2} \lambda_{min_2-n_1}^{-1} = \lambda_{min_2-n_1}^3 \mu_1^{-2}, \end{aligned}$$

$$
\begin{aligned}
t_{min_2+3} &= t^2_{min_2+2}t^{-1}_{min_2+1} = \lambda^4_{min_2-n_1}\mu_1^{-3},\\
&\vdots\\
t_{min_2+n_2-1} &= \lambda^{n_2}_{min_2-n_1}\mu_1^{-n_2+1}\\
t_{min_2+n_2} &= \lambda^{n_2+1}_{min_2-n_1}\mu_1^{-n_2}.
\end{aligned}
$$

These equations cover kernels of all the roots in θ_2, so for each root between θ_2 and θ_3 we put

$$
\lambda_{min_2-n_1+1} = t_{min_2+n_2+1}, \lambda_{min_2-n_1+2} = t_{min_2+n_2+2}, \dots, \lambda_{min_3-l_2} = t_{min_3}.
$$

If $min_3 = min_2+n_2+1$, then let $\mu_2 = \lambda^{n_2+1}_{min_2-n_1}\mu_1^{-n_2}$; let $\mu_2 = \lambda_{min_3-l_2-1}$ otherwise. Repeating the procedure similar to that in the previous paragraph, we get

$$
\begin{aligned}
t_{min_3+1} &= t^2_{min_3}t^{-1}_{min_3-1} = \lambda^2_{min_3-l_2}\mu_2^{-1},\\
&\vdots\\
t_{min_3+n_3-1} &= \lambda^{n_3}_{min_3-l_2}\mu_2^{-n_3+1},\\
t_{min_3+n_3} &= \lambda^{n_3+1}_{min_3-l_2}\mu_2^{-n_3}.
\end{aligned}
$$

We continue by repeating this process for all the remaining subsets $\theta_4, \dots, \theta_k$ of θ. At the end we get $t_{min_k+n_k-1} = \lambda^{n_k}_{min_k-l_{k-1}}\mu_{k-1}^{-n_k+1}$ and $t_{min_k+n_k} = \lambda^{n_k+1}_{min_k-l_{k-1}}\mu_{k-1}^{-n_k}$.

Since in this case $min_k + n_k < n$, we also have to put

$$
\lambda_{min_k-l_{k-1}+1} = t_{min_k+n_k+1}, \dots, \lambda_{n-l_k} = t_n.
$$

Finally, we have:

$$
\begin{aligned}
A_\theta = \{&\alpha_1^\vee(\lambda_1)\alpha_2^\vee(\lambda_1^2)\cdots\alpha^\vee_{n_1+1}(\lambda_1^{n_1+1})\alpha^\vee_{n_1+2}(\lambda_2)\cdots\alpha^\vee_{min_2}(\lambda_{min_2-n_1})\\
&\cdot\alpha^\vee_{min_2+1}(\lambda^2_{min_2-n_1}\mu_1^{-1})\alpha^\vee_{min_2+2}(\lambda^3_{min_2-n_1}\mu_1^{-2})\cdots\\
&\cdot\alpha^\vee_{min_2+n_2}(\lambda^{n_2+1}_{min_2-n_1}\mu_1^{-n_2})\alpha^\vee_{min_2+n_2+1}(\lambda_{min_2-n_1+1})\cdots\alpha^\vee_{min_3}(\lambda_{min_3-l_2})\\
&\cdot\alpha^\vee_{min_3+1}(\lambda^2_{min_3-l_2}\mu_2^{-1})\cdots\alpha^\vee_{min_3+n_3}(\lambda^{n_3+1}_{min_3-l_2}\mu_2^{-n_3})\cdots\\
&\cdot\alpha^\vee_{min_k+n_k}(\lambda^{n_k+1}_{min_k-l_{k-1}}\mu_{k-1}^{-n_k})\alpha^\vee_{min_k+n_k+1}(\lambda_{min_k-l_{k-1}+1})\cdots\alpha^\vee_n(\lambda_{n-l_k})\\
&: \lambda_1, \cdots, \lambda_{n-l_k} \in F^*\}\\
\simeq{}& (F^*)^{n-l_k}
\end{aligned}
$$

After identifying A_θ with $GL(1,F)^{n-l_k} \simeq (F^*)^{n-l_k}$, we fix (as in [4, Example 1.36]) an identification of M'_θ with $SL(n_1+1,F)\times SL(n_2+1,F)\times\cdots\times SL(n_k+1,F)$ under which the element $\alpha_1^\vee(\lambda_1)\alpha_2^\vee(\lambda_1^2)\cdots\alpha^\vee_{n_1}(\lambda_1^{n_1})$ goes to the diagonal element $diag(\lambda_1, \lambda_1, \dots, \lambda_1, \lambda_1^{-n_1})$ of $SL(n_1+1,F)$,

$$
\alpha^\vee_{min_2}(\lambda_{min_2-n_1})\alpha^\vee_{min_2+1}(\lambda^2_{min_2-n_1}\mu_1^{-1})\cdots\alpha^\vee_{min_2+n_2-1}(\lambda^{n_2}_{min_2-n_1}\mu_1^{-n_2+1})
$$

to $diag(\lambda_{min_2-n_1}, \dots, \lambda_{min_2-n_1}, \lambda^{-n_2}_{min_2-n_1})$ of $SL(n_2+1,F)$ and proceed in the same way for all connected components $\theta_3, \dots, \theta_k$ (similar identifications are used in all cases). Using these identifications, we conclude that in $A_\theta \bigcap M'_\theta$ we have:

$$\begin{aligned}\lambda_1^{n_1+1} &= 1, \lambda_2 = \lambda_3 = \cdots = \mu_1 = 1,\\ \lambda_{min_2-n_1}^{n_2+1} &= 1, \lambda_{min_2-n_1+1} = \lambda_{min_2-n_1+2} = \cdots = \mu_2 = 1,\\ \lambda_{min_3-l_2}^{n_3+1} &= 1, \ldots, \mu_{k-1} = 1, \lambda_{min_k-l_{k-1}}^{n_k+1} = 1,\\ \lambda_{min_k-l_{k-1}+1} &= \cdots = \lambda_{n-l_k} = 1,\end{aligned}$$

therefore

$$\begin{aligned}A_\theta \cap M'_\theta &= \{\alpha_1^\vee(\lambda_1)\alpha_2^\vee(\lambda_1^2)\cdots\alpha_{n_1}^\vee(\lambda_1^{n_1})\alpha_{min_2}^\vee(\lambda_{min_2-n_1})\cdots\\ &\quad \cdot \alpha_{min_2+n_2-1}^\vee(\lambda_{min_2-n_1}^{n_2})\cdots\alpha_{min_k+n_k}^\vee(\lambda_{min_k-l_{k-1}}^{n_k})\\ &\quad : \lambda_1^{n_1+1} = 1, \lambda_{min_2-n_1}^{n_2+1} = 1, \ldots, \lambda_{min_k-l_{k-1}}^{n_k+1} = 1\}\\ &\simeq \langle\zeta_{n_1+1}\rangle \times \langle\zeta_{n_2+1}\rangle \times \cdots \times \langle\zeta_{n_k+1}\rangle\end{aligned}$$

It follows immediately that

$$\begin{aligned}M_\theta &\simeq \frac{(F^*)^{n-l_k} \times SL(n_1+1,F) \times \cdots \times SL(n_k+1,F)}{\langle\zeta_{n_1+1}\rangle \times \cdots \times \langle\zeta_{n_k+1}\rangle}\\ &\simeq \frac{F^* \times SL(n_1+1,F)}{\langle\zeta_{n_1+1}\rangle} \times \cdots \times \frac{F^* \times SL(n_k+1,F)}{\langle\zeta_{n_k+1}\rangle} \times (F^*)^{n-l_k-k}\\ &\simeq GL(n_1+1,F) \times \cdots \times GL(n_k+1,F) \times GL(1,F)^{n-l_k-k}\end{aligned}$$

because the mapping $F^* \times SL(n,F) \to GL(n,F)$, $(x,S) \mapsto xI_n \cdot S$, is a surjective homomorphism whose kernel is isomorphic to $\langle\zeta_n\rangle$.

(2) Suppose $\alpha_1, \alpha_{n-1}, \alpha_n \notin \theta$. Of course, $min_k + n_k - 1 < n - 1$. M'_θ is again isomorphic to $SL(n_1+1,F) \times SL(n_2+1,F) \times \cdots \times SL(n_k+1,F)$. We start with

$$\lambda_1 = t_1, \lambda_2 = t_2, \ldots, \lambda_{min_1} = t_{min_1}.$$

It follows

$$t_{min_1+1} = \lambda_{min_1}^2\lambda_{min_1-1}^{-1}, \ldots, t_{min_1+n_1-1} = \lambda_{min_1}^{n_1}\lambda_{min_1-1}^{-n_1+1}$$

and

$$t_{min_1+n_1} = \lambda_{min_1}^{n_1+1}\lambda_{min_1-1}^{-n_1}.$$

We can now proceed analogously to case (1):

$$\begin{aligned}A_\theta &= \{\alpha_1^\vee(\lambda_1)\cdots\alpha_{min_1}^\vee(\lambda_{min_1})\alpha_{min_1+1}^\vee(\lambda_{min_1}^2\lambda_{min_1-1}^{-1})\cdots\\ &\quad \cdot \alpha_{min_1+n_1}^\vee(\lambda_{min_1}^{n_1+1}\lambda_{min_1-1}^{-n_1})\cdots\alpha_{min_k}^\vee(\lambda_{min_k-l_{k-1}})\cdots\\ &\quad \cdot \alpha_{min_k+n_k}^\vee(\lambda_{min_k-l_{k-1}}^{n_k+1}\mu_{k-1}^{-n_k})\alpha_{min_k+n_k+1}^\vee(\lambda_{min_k-l_{k-1}+1})\cdots\\ &\quad \cdot \alpha_n^\vee(\lambda_{n-l_k}) : \lambda_1, \cdots, \lambda_{n-l_k} \in F^*\}\\ &\simeq (F^*)^{n-l_k}\end{aligned}$$

In $A_\theta \cap M'_\theta$ we have:

$$\begin{aligned}
\lambda_1 &= \cdots = \lambda_{min_1-1} = 1, \lambda_{min_1}^{n_1+1} = 1,\\
\lambda_{min_1+1} &= \cdots = \lambda_{min_2-n_1-1} = \mu_1 = 1, \lambda_{min_2-n_1}^{n_2+1} = 1,\\
&\vdots\\
\lambda_{min_{k-1}-l_{k-2}} &= \cdots = \lambda_{min_k-l_{k-1}-1} = \mu_{k-1} = 1,\\
\lambda_{min_k-l_{k-1}}^{n_k+1} &= 1, \lambda_{min_k-l_{k-1}+1} = \cdots = \lambda_{n-l_k} = 1.
\end{aligned}$$

Therefore, $A_\theta \cap M'_\theta \simeq \langle \zeta_{n_1+1} \rangle \times \langle \zeta_{n_2+1} \rangle \times \cdots \times \langle \zeta_{n_k+1} \rangle$ and again

$$M_\theta \simeq GL(n_1+1, F) \times \cdots \times GL(n_k+1, F) \times GL(1, F)^{n-l_k-k}$$

(3) Suppose $\alpha_1, \alpha_{n-1}, \alpha_n \in \theta$. Obviously, $min_1 = 1$ and $min_k + n_k = n+1$. M'_θ is isomorphic to $SL(n_1+1, F) \times SL(n_2+1, F) \times \cdots \times SL(n_{k-1}+1, F) \times Spin(2n_k+1, F)$.

On the set $\theta \setminus \theta_k = \theta_1 \cup \theta_2 \cup \cdots \cup \theta_{k-1}$ we apply the same analysis as in case (1) and get

$$\begin{aligned}
\lambda_1 = t_1, \ldots, \lambda_1^{n_1+1} &= t_{n_1+1}, \lambda_2 = t_{n_1+2},\\
&\vdots\\
\lambda_{min_{k-1}-l_{k-2}} &= t_{min_{k-1}},\\
&\vdots\\
t_{min_{k-1}+n_{k-1}-1} &= \lambda_{min_{k-1}-l_{k-2}}^{n_{k-1}} \mu_{k-2}^{-n_{k-1}+1},\\
t_{min_{k-1}+n_{k-1}} &= \lambda_{min_{k-1}-l_{k-2}}^{n_{k-1}+1} \mu_{k-2}^{-n_{k-1}}.
\end{aligned}$$

Next, put $\lambda_{min_{k-1}-l_{k-2}+1} = t_n$. From Proposition 2 applied to the set θ_k we obtain: $t_{n-1} = t_{n-2} = \cdots = t_{n-n_k} = \lambda_{min_{k-1}-l_{k-2}+1}^2$. We have two possibilities which are considered separately:

- $min_{k-1} + n_{k-1} = n - n_k$
  It follows directly that $min_{k-1} - l_{k-2} = n - l_k$ and $\lambda_{n-l_k}^{n_{k-1}+1} \mu_{k-2}^{-n_{k-1}} = \lambda_{n-l_k+1}^2$. So, $A_\theta \simeq (F^*)^{n-l_k}$. In $A_\theta \cap M'_\theta$ we have:

$$\begin{aligned}
\lambda_1^{n_1+1} &= 1, \lambda_2 = \lambda_3 = \cdots = \mu_1 = 1,\\
\lambda_{min_2-n_1}^{n_2+1} &= 1, \lambda_{min_2-n_1+1} = \lambda_{min_2-n_1+2} = \cdots = \mu_2 = 1,\\
&\vdots\\
\lambda_{n-l_k}^{n_{k-1}+1} &= 1 = \lambda_{n-l_k+1}^2.
\end{aligned}$$

  That implies $A_\theta \cap M'_\theta \simeq \langle \zeta_{n_1+1} \rangle \times \langle \zeta_{n_2+1} \rangle \times \cdots \times \langle \zeta_{n_{k-2}+1} \rangle \times \langle \zeta_{2(n_{k-1}+1)} \rangle$ (this $2(n_{k-1}+1)$-th root of identity comes from the last equation). This gives

$$\begin{aligned}
M_\theta &\simeq GL(n_1+1, F) \times \cdots \times GL(n_{k-2}+1, F) \times GL(1, F)^{n-l_k-k}\\
&\times \frac{GL(1, F) \times SL(n_{k-1}+1, F) \times Spin(2n_k+1, F)}{B},
\end{aligned}$$

where $B = \{(\zeta, \zeta^2 \cdot I_{n_{k-1}+1}, \zeta^{n_{k-1}+1}) : \zeta^{2(n_{k-1}+1)} = 1\}$. Observe that the set $\{\zeta^{n_{k-1}+1} : \zeta^{2(n_{k-1}+1)} = 1\}$ can be identified with $\{1, z\}$, the center of $Spin(2n_k + 1, F)$.

- $min_{k-1} + n_{k-1} < n - n_k$
  We put $\lambda_{min_{k-1}-l_{k-2}+2} = t_{min_{k-1}+n_{k-1}+1}$, $\lambda_{min_{k-1}-l_{k-2}+3} = t_{min_{k-1}+n_{k-1}+2}$, $\ldots, \lambda_{n-l_k} = t_{n-n_k-1}$.
  Again, $A_\theta \simeq (F^*)^{n-l_k}$, while in $A_\theta \cap M'_\theta$ we have

$$\lambda_1^{n_1+1} = 1, \lambda_2 = \lambda_3 = \cdots = \mu_1 = 1,$$
$$\vdots$$
$$\lambda_{min_{k-1}-l_{k-2}}^{n_{k-1}+1} = 1, \mu_{k-2} = 1,$$
$$\lambda^2_{min_{k-1}-l_{k-2}+1} = 1, \lambda_{min_{k-1}-l_{k-2}+2} = \cdots = \lambda_{n-l_k} = 1,$$

  that implies $A_\theta \cap M'_\theta \simeq \langle\zeta_{n_1+1}\rangle \times \langle\zeta_{n_2+1}\rangle \times \cdots \times \langle\zeta_{n_{k-1}+1}\rangle \times \langle\zeta_2\rangle$.
  Observe that $\langle\zeta_2\rangle \simeq \{(1,1), (-1, z)\}$. We thus get

$$\begin{aligned} M_\theta &\simeq GL(n_1+1,F) \times \cdots \times GL(n_{k-1}+1,F) \times GL(1,F)^{n-l_k-k} \\ &\quad \times \frac{GL(1,F) \times Spin(2n_k+1,F)}{\langle\zeta_2\rangle} \\ &\simeq GL(n_1+1,F) \times \cdots \times GL(n_{k-1}+1,F) \times GL(1,F)^{n-l_k-k} \\ &\quad \times GSpin(2n_k+1,F). \end{aligned}$$

(4) Suppose $\alpha_1, \alpha_n \in \theta, \alpha_{n-1} \notin \theta$. Clearly, $min_1 = 1$, $\theta_k = \{\alpha_n\}$ and $n_k = 1$. M'_θ is isomorphic to $SL(n_1+1,F) \times SL(n_2+1,F) \times \cdots \times SL(n_{k-1}+1,F) \times Spin(3,F)$. This case can be handled in pretty much the same way as case (3), so we only state final results.

- if $min_{k-1} + n_{k-1} = n - 1$, then

$$\begin{aligned} M_\theta &\simeq GL(n_1+1,F) \times \cdots \times GL(n_{k-2}+1,F) \times GL(1,F)^{n-l_k-k} \\ &\quad \times \frac{GL(1,F) \times SL(n_{k-1}+1,F) \times Spin(3,F)}{B}, \end{aligned}$$

  where $B = \{(\zeta, \zeta^2 \cdot I_{n_{k-1}+1}, \zeta^{n_{k-1}+1}) : \zeta^{2(n_{k-1}+1)} = 1\}$

- if $min_{k-1} + n_{k-1} < n - 1$, then

$$\begin{aligned} M_\theta &\simeq GL(n_1+1,F) \times \cdots \times GL(n_{k-2}+1,F) \times GL(1,F)^{n-l_k-k} \\ &\quad \times GSpin(3,F). \end{aligned}$$

(5) Suppose $\alpha_1 \notin \theta, \alpha_{n-1}, \alpha_n \in \theta$. Obviously, $min_1 > 1$ and $min_k + n_k = n + 1$. M'_θ is isomorphic to

$$SL(n_1+1,F) \times SL(n_2+1,F) \times \cdots \times SL(n_{k-1}+1,F) \times Spin(2n_k+1,F).$$

Let $\lambda_1 = t_n$. From Proposition 2 we conclude that

$$t_{n-1} = \cdots = t_{min_k} = t_{min_k - 1} = \lambda_1^2.$$

Next, let

$$\lambda_2 = t_{min_k - 2}, \ldots, \lambda_{min_k - min_{k-1} - n_{k-1} + 1} = t_{min_{k-1} + n_{k-1} - 1}.$$

If $min_{k-1} + n_{k-1} = min_k - 1$, then put $\mu_1 = \lambda_1^2$, otherwise put $\mu_1 = \lambda_{min_k - min_{k-1} - n_{k-1}}$. Using standard calculations, it easily follows:

$$\begin{aligned} t_{min_{k-1}+n_{k-1}-2} &= \lambda^2_{min_k - min_{k-1} - n_{k-1}+1}\mu_1^{-1}, \\ t_{min_{k-1}+n_{k-1}-3} &= \lambda^3_{min_k - min_{k-1} - n_{k-1}+1}\mu_1^{-2}, \\ &\vdots \\ t_{min_{k-1}-1} &= \lambda^{n_{k-1}+1}_{min_k - min_{k-1} - n_{k-1}+1}\mu_1^{-n_k}. \end{aligned}$$

In the next step, let

$$\begin{aligned} \lambda_{min_k - min_{k-1} - n_{k-1}+2} &= t_{min_{k-1}-2}, \\ \lambda_{min_k - min_{k-1} - n_{k-1}+3} &= t_{min_{k-1}-3}, \\ &\vdots \\ \lambda_{min_k - min_{k-2} - n_{k-1} - n_{k-2}+1} &= t_{min_{k-2}+n_{k-2}-1}. \end{aligned}$$

If $min_{k-2} + n_{k-2} = min_{k-1} - 1$, then put $\mu_2 = \lambda^{n_{k-1}+1}_{min_k - min_{k-1} - n_{k-1}+1}\mu_1^{-n_k}$, otherwise put $\mu_2 = \lambda_{min_k - min_{k-2} - n_{k-1} - n_{k-2}}$. The rest of this construction runs as before:

$$\begin{aligned} t_{min_{k-2}+n_{k-2}-2} &= \lambda^2_{min_k - min_{k-2} - n_{k-1} - n_{k-2}+1}\mu_2^{-1}, \\ &\vdots \\ t_{min_{k-2}-1} &= \lambda^{n_{k-2}+1}_{min_k - min_{k-2} - n_{k-1} - n_{k-2}+1}\mu_2^{-n_{k-1}}, \\ &\vdots \\ t_{min_1 - 1} &= \lambda^{n_1+1}_{min_k - min_1 - l_{k-1}+1}\mu_{k-1}^{-n_1}. \end{aligned}$$

Also, we have to add $\lambda_{min_k - min_1 - l_{k-1}+2} = t_{min_1 - 2}$, $\ldots$, $\lambda_{min_k - l_{k-1} - 1} = t_1$. From $min_k + n_k = n + 1$ we easily get that $min_k - l_{k-1} - 1 = n - l_k$.

$$\begin{aligned} A_\theta = \{ & \alpha_1^\vee(\lambda_{n-l_k})\alpha_2^\vee(\lambda_{n-l_k-1})\cdots\alpha^\vee_{min_1-2}(\lambda_{min_k - min_1 - l_{k-1}+2}) \\ & \cdot \alpha^\vee_{min_1-1}(\lambda^{n_1+1}_{min_k - min_1 - l_k + n_k + 1}\mu_{k-1}^{-n_1})\cdots\alpha^\vee_{min_k-1}(\lambda_1^2)\cdots\alpha_n^\vee(\lambda_1) \\ & : \lambda_1, \ldots, \lambda_{n-l_k} \in F^*\} \\ \simeq & \ (F^*)^{n-l_k}. \end{aligned}$$

In $A_\theta \cap M'_\theta$ we have:

$$\begin{aligned}
&\lambda_1^2 = 1,\\
&\lambda_2 = \cdots = \lambda_{min_k - min_{k-1} - n_{k-1}} = \mu_1 = 1,\\
&\lambda_{min_k - min_{k-1} - n_{k-1}+1}^{n_{k-1}+1} = 1,\\
&\qquad\qquad\vdots\\
&\mu_{k-1} = 1, \lambda_{min_k - min_1 - l_{k-1}+1}^{n_1+1} = 1,\\
&\lambda_{min_k - min_1 - l_{k-1}+2} = \cdots = \lambda_{n-l_k} = 1,
\end{aligned}$$

that implies
$A_\theta \cap M'_\theta \simeq \langle \zeta_{n_1+1}\rangle \times \langle \zeta_{n_2+1}\rangle \times \cdots \times \langle \zeta_{n_{k-2}+1}\rangle \times \langle \zeta_2\rangle$.
Finally,

$$\begin{aligned}
M_\theta &\simeq GL(n_1+1,F) \times \cdots \times GL(n_{k-2}+1,F) \times GL(1,F)^{n-l_k-k}\\
&\quad \times \frac{GL(1,F) \times Spin(2n_k+1,F)}{\langle \zeta_2\rangle}\\
&\simeq GL(n_1+1,F) \times \cdots \times GL(n_{k-2}+1,F) \times GL(1,F)^{n-l_k-k}\\
&\quad \times GSpin(2n_k+1,F).
\end{aligned}$$

Observe that, for $\theta = \Sigma \setminus \{\alpha_1\}$ we have $\theta = \theta_1$, $k = 1$, $n_1 = n-1$ and

$$M_{\Sigma\setminus\{\alpha_1\}} \simeq M_\theta = GSpin(2(n-1)+1,F),$$

which implies that $GSpin(2n-1,F)$ is the maximal Levi subgroup of $Spin(2n+1,F)$.

(6) Suppose $\alpha_1, \alpha_{n-1} \notin \theta, \alpha_n \in \theta$. Of course, $min_1 > 1$ and $n_k = 1$. M'_θ is isomorphic to $SL(n_1+1,F) \times SL(n_2+1,F) \times \cdots \times SL(n_{k-1}+1,F) \times Spin(3,F)$. Analysis similar to that in the (5) shows that:

$$\begin{aligned}
M_\theta &\simeq GL(n_1+1,F) \times \cdots \times GL(n_{k-2}+1,F) \times GL(1,F)^{n-l_k-k}\\
&\quad \times \frac{GL(1,F) \times Spin(3,F)}{\{1,z\}}\\
&\simeq GL(n_1+1,F) \times \cdots \times GL(n_{k-2}+1,F) \times GL(1,F)^{n-l_k-k}\\
&\quad \times GSpin(3,F).
\end{aligned}$$

(7) Suppose $\alpha_1, \alpha_{n-1} \in \theta, \alpha_n \notin \theta$. Clearly, $min_1 = 1$ and $min_k + n_k = n$. M'_θ is isomorphic to $SL(n_1+1,F) \times SL(n_2+1,F) \times \cdots \times SL(n_k+1,F)$.

Proceeding analogously to case (1) we obtain:

$$\begin{aligned}
\lambda_1 &= t_1, t_2 = \lambda_1^2, t_3 = \lambda_1^3, \ldots, t_{n_1} = \lambda_1^{n_1}, t_{n_1+1} = \lambda_1^{n_1+1},\\
\lambda_2 &= t_{n_1+2}, \lambda_3 = t_{n_1+3}, \ldots, \lambda_{min_2-n_1} = t_{min_2},\\
t_{min_2+1} &= \lambda_{min_2-n_1}^2 \mu_1^{-1}, \ldots, t_{min_2+n_2} = \lambda_{min_2-n_1}^{n_2+1}\mu_1^{-n_2},\\
&\vdots\\
t_{min_k+n_k-1} &= \lambda_{min_k-l_{k-1}}^{n_k}\mu_{k-1}^{-n_k+1}, t_n^2 = t_{min_k+n_k}^2 = \lambda_{min_k-l_{k-1}}^{n_k+1}\mu_{k-1}^{-n_k}.
\end{aligned}$$

Suppose $\theta = \Sigma\backslash\{\alpha_n\}$. Then $k=1$, $n_1 = n-1$, $M'_\theta = SL(n,F)$ and $t_n^2 = \lambda_1^n = t_1^n$. If n is even, say $n = 2m$, then
$A_\theta = \{\alpha_1^\vee(\lambda_1)\alpha_2^\vee(\lambda_1^2)\cdots\alpha_{n-1}^\vee(\lambda_1^{n-1})\alpha_n^\vee(\lambda_1^m) : \lambda_1 \in F^*\} \simeq F^*$
Observe that t_k could not be equal to $-\lambda_1^m$ in A_θ, because A_θ is a connected component of the center. In $A_\theta \cap M'_\theta$ we have $\lambda_1^m = 1$, so $A_\theta \cap M'_\theta \simeq \langle\zeta^m\rangle$, therefore

$$M_\theta \simeq \frac{GL(1,F)\times SL(n,F)}{\langle\zeta^m\rangle}$$

If n is odd, then $M_\theta \simeq GL(n,F)$, as Shahidi asserts in [5, Remark 2.2].

If θ has more than one component, then $t_n^2 = \lambda_{min_k-l_{k-1}}^{n_k+1}\mu_{k-1}^{-n_k}$. Since n_k+1 and $-n_k$ are of different parities, if n_k is even or μ_{k-1} is not equal to λ^m for some $\lambda \in F^*$ and m even, we can proceed in the same way as above and get

$$M_\theta \simeq GL(n_1+1,F)\times\cdots\times GL(n_k+1,F)\times GL(1,F)^{n-l_k-k}$$

Now we have to consider the situation when n_k is odd and $\mu_{k-1} = \lambda^m$, for $\lambda \in F^*$ and m even. If this is the case, then $\mu_{k-1} = \lambda_{min_{k-1}-l_{k-2}}^{n_{k-1}+1}\mu_{k-2}^{-n_{k-1}}$. Again, this implies that n_{k-1} is odd and $\mu_{k-2} = \lambda_{min_{k-2}-l_{k-3}}^{n_{k-2}+1}\mu_{k-3}^{-n_{k-2}}$. We continue in this fashion to obtain $\mu_2 = \lambda_{min_2-n_1}^{n_2+1}\mu_1^{-n_2}$, n_2 is odd, $\mu_1 = \lambda_1^{n_1+1}$ and n_1 is odd. We conclude that n_k is odd and $\mu_{k-1} = \lambda^m$, for $\lambda \in F^*$ and m even, only if n_i is odd for each $1 \le i \le k$ and $min_i + n_i = min_{i+1} - 1$ for each $1 \le i \le k-1$. Observe that this implies $min_k - l_{k-1} = k = n - l_k$. If this is the case, then

$$\begin{aligned}
A_\theta &= \{\alpha_1^\vee(\lambda_1)\alpha_2^\vee(\lambda_1^2)\cdots\alpha_{n_1+1}^\vee(\lambda_1^{n_1+1})\alpha_{min_2}^\vee(\lambda_2)\\
&\quad\cdot\alpha_{min_2+1}^\vee(\lambda_2^2\mu_1^{-1})\alpha_{min_2+2}^\vee(\lambda_2^3\mu_1^{-2})\cdots\\
&\quad\cdot\alpha_{min_k}^\vee(\lambda_{n-l_k})\cdots\alpha_{n-1}^\vee(\lambda_{n-l_k}^{n_k}\mu_{k-1}^{-n_k+1})\alpha_n^\vee(\lambda_{n-l_k}^{\frac{n_k+1}{2}}\mu)\\
&\quad : \lambda_1,\cdots,\lambda_{n-l_k} \in F^*, \mu^2 = \mu_{k-1}^{-n_k}\}\\
&\simeq (F^*)^{n-l_k}.
\end{aligned}$$

In $A_\theta \cap M'_\theta$ we have:

$$\lambda_1^{n_1+1} = \lambda_2^{n_2+1} = \cdots = \lambda_{k-1}^{n_{k-1}+1} = \lambda_{n-l_k}^{\frac{n_k+1}{2}} = \mu_1 = \mu_2 = \cdots = \mu_{k-1} = 1,$$

we easily get that $\lambda_{n-l_k}^{n_k+1} = 1$, so $A_\theta \cap M'_\theta \simeq \langle\zeta_{n_1+1}\rangle \times \langle\zeta_{n_2+1}\rangle \times \cdots \times \langle\zeta_{n_k+1}\rangle$ and $M_\theta \simeq GL(n_1+1, F) \times \cdots \times GL(n_k+1, F)$.

(8) Suppose $\alpha_1, \alpha_n \notin \theta, \alpha_{n-1} \in \theta$. Clearly, $min_1 > 1$, $\theta \neq \Sigma \setminus \{\alpha_n\}$ and $min_k + n_k = n$. M'_θ is isomorphic to $SL(n_1+1, F) \times SL(n_2+1, F) \times \cdots \times SL(n_k+1, F)$. By the same method as in case (7), we obtain

$$M_\theta \simeq GL(n_1+1, F) \times \cdots \times GL(n_k+1, F) \times GL(1, F)^{n-l_k-k}$$

Remark 1. *Cases (2), (5), (6) and (8) together imply that Levi subgroups of the general spin group* $GSpin(2n+1, F)$ *are isomorphic to* $GL(n_1, F) \times GL(n_2, F) \times \cdots \times GL(n_k, F) \times GSpin(2m+1, F)$, $m \leq n$.

Remark 2. *Observe that* $\frac{F^* \times SL(n,F)}{\langle\zeta_n\rangle}$ *is not isomorphic to* $GL(n, F)$ *over p-adic field* F*, because the image of the given mapping consists of matrices whose determinants are n-th powers.*

Let F_1 *be a p-adic field. We will denote an algebraic closure of* F_1 *by* $\overline{F}_1$*. Since spin groups are double coverings of special orthogonal groups, we have the next exact sequence*

$1 \to \{\pm 1\} \hookrightarrow Spin(2n+1, \overline{F}_1) \xrightarrow{f} SO(2n+1, \overline{F}_1) \to 1$, *where* f *is a central isogeny.* F_1-*rational points of* $Spin(2n+1)$ *may be obtained by using the following exact sequence:*

$1 \to \{\pm 1\} \hookrightarrow Spin(2n+1, F_1) \xrightarrow{f} SO(2n+1, F_1) \xrightarrow{\delta} F_1^*/(F_1^*)^2$
(homomorphism δ *is called the spinor norm)*

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