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On the convergence (upper boundness) of trigonometric series

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Abstract. In this paper we prove that the condition

$$\sum_{k=\left[\frac{n}{2}\right]}^{2n} \frac{k^r \lambda_k}{|n-k|+1} = o(1)\ (= O(1)),$$

for $r = 0, 1, 2, \dots$, is necessary for the convergence of the $r-th$ derivative of the Fourier series in the L^1-metric. This condition is sufficient under some additional assumptions for Fourier coefficients. In fact, in this paper we generalize some results of A. S. Belov [1].
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1. Introduction

Let

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \right) \tag{1}$$

be a trigonometric series in the complex or real form, respectively, and let us write

$$\begin{aligned}
a_n &= c_n + c_{-n}, \\
b_n &= (c_n - c_{-n})i, \\
\lambda_n &= \sqrt{|a_n|^2 + |b_n|^2} = \sqrt{2(|c_n|^2 + |c_{-n}|^2)}, \\
A_n(x) &= c_n e^{inx} + c_{-n} e^{-inx} = a_n \cos nx + b_n \sin nx, \\
S_n(x) &= c_0 + \sum_{k=1}^{n} A_k(x) \\
\sigma_n(x) &= \frac{1}{n+1} \sum_{k=1}^{n} S_k(x), \\
\widetilde{S}_n(x) &= \sum_{k=1}^{n} (a_k \sin kx - b_k \cos kx) = -i \sum_{k=0}^{n} \left(c_k e^{ikx} - c_{-k} e^{-ikx} \right),\ n \geq 0,
\end{aligned}$$

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for all $n \geq 0$.

It is a well-known fact that for $f \in L_{2\pi}$ the L^1-metric is defined by the equality

$$\|f\|_{L^1} = \|f\| = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|dx.$$

With regard to the series (1) the following theorem is proved [1]:

Theorem 1. *If $n \geq 2$ is an integer, then*

$$\sum_{k=\left[\frac{n}{2}\right]}^{2n} \frac{\lambda_k}{|n-k|+1} \leq 100 \max_{m=[n/2]-1,\ldots,2n} \|\sigma_m - S_m\|.$$

In particular:

1. *If*
$$\|\sigma_m - S_m\| = o(1)\ (= O(1)), \tag{2}$$
then
$$\sum_{k=\left[\frac{n}{2}\right]}^{2n} \frac{\lambda_k}{|n-k|+1} = o(1)\ (= O(1),\ \textit{respectively}). \tag{3}$$

2. *Assume that series (1) converges (possesses bounded partial sums) in the L^1-metric, then condition (3) holds.*

In the same paper the cosine and sine series are considered

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \tag{4}$$

$$\sum_{n=1}^{\infty} a_n \sin nx, \tag{5}$$

where for series (4) or (5) the coefficients a_n are the same as in the trigonometric series (1) except for coefficients of series (5) which are denoted by a_n instead of b_n, and the following corollary is proved.

Corollary 1. *It holds:*

1. *Assume that series (4) or (5) satisfies condition (2), then*
$$\sum_{k=\left[\frac{n}{2}\right]}^{2n} \frac{|a_k|}{|n-k|+1} = o(1)\ \ (O(1),\ \textit{respectively}). \tag{6}$$

2. *Assume that series (4) or (5) converges (possesses bounded partial sums) in the L^1-metric, then condition (6) holds.*

The aim of this paper is to generalize the above results under more general assumptions and to obtain some corollaries.

2. Helpful lemmas

To prove the main results first we need the following lemma.

Lemma 1. *Given an arbitrary trigonometric series (1) and arbitrary natural numbers n and N such that $N \le 2n+1$, the following estimates hold:*

$$\max_{k=n,\dots,N} \|\widetilde{S}_k^{(r)} - \widetilde{S}_{n-1}^{(r)}\| \le 2 \max_{k=n,\dots,N} \|S_k^{(r)} - S_{n-1}^{(r)}\| \tag{7}$$

$$\max_{m=n,\dots,N} \left\| \left(\sum_{j=n}^{m} c_j e^{ijx} \right)^{(r)} \right\| \le \frac{3}{2} \max_{m=n,\dots,N} \|S_m^{(r)} - S_{n-1}^{(r)}\|; \tag{8}$$

$$\max_{m=n,\dots,N} \left\| \left(\sum_{j=n}^{m} c_{-j} e^{-ijx} \right)^{(r)} \right\| \le \frac{3}{2} \max_{m=n,\dots,N} \|S_m^{(r)} - S_{n-1}^{(r)}\|; \tag{9}$$

$$\max_{k=n,\dots,N} \|S_k^{(r)} - S_{n-1}^{(r)}\| \le 4 \max_{k=n-1,\dots,N} \|S_k^{(r)} - \sigma_k^{(r)}\|; \tag{10}$$

$$\sum_{k=n}^{N} \frac{k^r \lambda_k}{k+1-n} \le 15 \max_{k=n,\dots,N} \|S_k^{(r)} - S_{n-1}^{(r)}\|; \tag{11}$$

$$\sum_{k=n}^{N} \frac{k^r \lambda_k}{N+1-k} \le 10 \|S_N^{(r)} - S_{n-1}^{(r)}\|, \quad (r = 0, 1, \dots). \tag{12}$$

Proof. (7): Let m, n be two natural numbers such that $m \ge n$. The $r-$th derivative of the equality

$$\widetilde{S}_{n-1}(x) - \widetilde{S}_m(x) = \frac{1}{m} \left(S'_m(x) - S'_{n-1}(x) \right) + \sum_{k=n}^{m-1} \frac{1}{k(k+1)} \left(S'_k(x) - S'_{n-1}(x) \right)$$

is

$$\widetilde{S}_{n-1}^{(r)}(x) - \widetilde{S}_m^{(r)}(x) = \frac{1}{m} \left(S_m^{(r+1)}(x) - S_{n-1}^{(r+1)}(x) \right) + \sum_{k=n}^{m-1} \frac{1}{k(k+1)} \left(S_k^{(r+1)}(x) - S_{n-1}^{(r+1)}(x) \right).$$

Using the well-known Bernstein's inequality (see [3, Chapter 10, Theorems 3.13 and 3.16]) we have

$$\|S_k^{(r+1)} - S_{n-1}^{(r+1)}\| \le k \|S_k^{(r)} - S_{n-1}^{(r)}\|,$$

and

$$\begin{aligned} \|\widetilde{S}_{n-1}^{(r)} - \widetilde{S}_m^{(r)}\| &\le \|S_m^{(r)} - S_{n-1}^{(r)}\| + \sum_{k=n}^{m-1} \frac{1}{k+1} \|S_k^{(r)} - S_{n-1}^{(r)}\| \\ &\le \left(1 + \sum_{k=n}^{m-1} \frac{1}{k+1} \right) \max_{k=n,\dots,m} \|S_k^{(r)} - S_{n-1}^{(r)}\|. \end{aligned}$$

Therefore since for $n \le N \le 2n+1$

$$1+\sum_{k=n}^{N-1}\frac{1}{k+1}\le 1+\frac{1}{n+1}(N-n)\le 2,$$

then we obtain

$$\max_{k=n,\dots,N}\|\widetilde{S}_{n-1}^{(r)}-\widetilde{S}_{m}^{(r)}\|\le 2\max_{k=n,\dots,N}\|S_{k}^{(r)}-S_{n-1}^{(r)}\|.$$

(8): From the equality

$$2\sum_{j=n}^{m}c_je^{ijx}=(S_m(x)-S_{n-1}(x))-i\left(\widetilde{S}_m(x)-\widetilde{S}_{n-1}(x)\right)$$

we find

$$2\left(\sum_{j=n}^{m}c_je^{ijx}\right)^{(r)}=\left(S_m^{(r)}(x)-S_{n-1}^{(r)}(x)\right)-i\left(\widetilde{S}_m^{(r)}(x)-\widetilde{S}_{n-1}^{(r)}(x)\right),$$

therefore using estimate (7) we get

$$\begin{aligned}2\max_{m=n,\dots,N}\left\|\left(\sum_{j=n}^{m}c_je^{ijx}\right)^{(r)}\right\| &\le \max_{m=n,\dots,N}\|S_m^{(r)}-S_{n-1}^{(r)}\|+\max_{m=n,\dots,N}\|\widetilde{S}_m^{(r)}-\widetilde{S}_{n-1}^{(r)}\|\\ &\le 3\max_{m=n,\dots,N}\|S_m^{(r)}-S_{n-1}^{(r)}\|,\end{aligned}$$

which is the required estimate.

Estimate (9) can be proved in the same line as estimate (8). In fact, it is sufficient to use the $r-$th derivative of the equality

$$2\sum_{j=n}^{m}c_{-j}e^{-ijx}=(S_m(x)-S_{n-1}(x))+i\left(\widetilde{S}_m(x)-\widetilde{S}_{n-1}(x)\right),$$

therefore by reason of its simplicity we omit it.
(10): Since the $r-$th derivative of the equality

$$\begin{aligned}S_m(x)-S_{n-1}(x)=&\frac{m+1}{m}\left(S_m(x)-\sigma_m(x)\right)\\&+\sum_{k=n}^{m-1}\frac{1}{k}\left(S_k(x)-\sigma_k(x)\right)-\left(S_{n-1}(x)-\sigma_{n-1}(x)\right)\end{aligned}$$

is

$$\begin{aligned}S_m^{(r)}(x)-S_{n-1}^{(r)}(x)=&\frac{m+1}{m}\left(S_m^{(r)}(x)-\sigma_m^{(r)}(x)\right)\\&+\sum_{k=n}^{m-1}\frac{1}{k}\left(S_k^{(r)}(x)-\sigma_k^{(r)}(x)\right)-\left(S_{n-1}^{(r)}(x)-\sigma_{n-1}^{(r)}(x)\right),\end{aligned}$$

then

$$\begin{aligned}
\|S_m^{(r)} - S_{n-1}^{(r)}\| &\leq \frac{m+1}{m}\|S_m^{(r)} - \sigma_m^{(r)}\| + \sum_{k=n}^{m-1} \frac{1}{k}\|S_k^{(r)} - \sigma_k^{(r)}\| + \|S_{n-1}^{(r)} - \sigma_{n-1}^{(r)}\| \\
&= \|S_m^{(r)} - \sigma_m^{(r)}\| + \sum_{k=n}^{m} \frac{1}{k}\|S_k^{(r)} - \sigma_k^{(r)}\| + \|S_{n-1}^{(r)} - \sigma_{n-1}^{(r)}\| \\
&\leq \left(2 + \sum_{k=n}^{m} \frac{1}{k}\right) \max_{k=n-1,\dots,m} \|S_k^{(r)} - \sigma_k^{(r)}\| \\
&< 4 \max_{k=n-1,\dots,N} \|S_k^{(r)} - \sigma_k^{(r)}\|, \quad \text{for} \quad n = 1.
\end{aligned}$$

Let us consider now the case when $n \geq 2$. Indeed, since for $n \leq N \leq 2n+1$, we have

$$2 + \sum_{k=n}^{m} \frac{1}{k} \leq 2 + \frac{N-n+1}{n} \leq 3 + \frac{2}{n} \leq 4,$$

then estimate (10) holds for all $n \geq 1$.
(11): By estimate (8) we have

$$H := \pi \left\| \left(\sum_{j=n}^{N} c_j e^{ijx} \right)^{(r)} \right\| \leq \frac{3\pi}{2} \max_{k=n,\dots,N} \|S_k^{(r)} - S_{n-1}^{(r)}\|. \tag{13}$$

But, by the Hardy's inequality (see [2, Chapter 7, Theorem 8.7]) we have

$$H := \pi \left\| \sum_{j=n}^{N} (ij)^r c_j e^{ijx} \right\| \geq \sum_{k=n}^{N} \frac{k^r |c_k|}{k+1-n}. \tag{14}$$

From (13) and (14) we obtain

$$\sum_{k=n}^{N} \frac{k^r |c_k|}{k+1-n} \leq \frac{3\pi}{2} \max_{k=n,\dots,N} \|S_k^{(r)} - S_{n-1}^{(r)}\|. \tag{15}$$

In a very similiar way we can find the following estimate

$$\sum_{k=n}^{N} \frac{k^r |c_{-k}|}{k+1-n} \leq \frac{3\pi}{2} \max_{k=n,\dots,N} \|S_k^{(r)} - S_{n-1}^{(r)}\|. \tag{16}$$

Since

$$\lambda_k = \sqrt{2\left(|c_k|^2 + |c_{-k}|^2\right)} \leq \sqrt{2}\left(|c_k| + |c_{-k}|\right),$$

then by (15) and (16) we have

$$\begin{aligned}
\sum_{k=n}^{N} \frac{k^r \lambda_k}{k+1-n} &\leq \sqrt{2} \sum_{k=n}^{N} \frac{k^r \left(|c_k| + |c_{-k}|\right)}{k+1-n} \\
&\leq 15 \max_{k=n,\dots,N} \|S_k^{(r)} - S_{n-1}^{(r)}\|,
\end{aligned}$$

which proves estimate (11).
(12): The $r-$th derivative of the equality

$$S_N(x)-S_{n-1}(x)=\sum_{j=n}^{N}c_je^{ijx}+\sum_{j=n}^{N}c_{-j}e^{-ijx}$$

is

$$S_N^{(r)}(x)-S_{n-1}^{(r)}(x)=\sum_{j=n}^{N}(ij)^{(r)}c_je^{ijx}+\sum_{j=n}^{N}(-ij)^{(r)}c_{-j}e^{-ijx},$$

therefore using the Hardy's inequality we get

$$\sum_{k=n}^{N}\frac{k^r|c_k|}{N+1-k}\leq \pi\|S_N^{(r)}-S_{n-1}^{(r)}\|,$$

and similarly

$$\sum_{k=n}^{N}\frac{k^r|c_{-k}|}{N+1-k}\leq \pi\|S_N^{(r)}-S_{n-1}^{(r)}\|.$$

Using the last two estimates we obtain

$$\begin{aligned}\sum_{k=n}^{N}\frac{k^r\lambda_k}{N+1-k}&\leq \sum_{k=n}^{N}\frac{k^r\sqrt{2\left(|c_k|+|c_{-k}|\right)^2}}{N+1-k}\\&\leq \sqrt{2}\sum_{k=n}^{N}\frac{k^r\left(|c_k|+|c_{-k}|\right)}{N+1-k}\\&\leq 2\pi\sqrt{2}\|S_N^{(r)}-S_{n-1}^{(r)}\|\\&\leq 10\|S_N^{(r)}-S_{n-1}^{(r)}\|.\end{aligned}$$

This completes the proof of Lemma 1. □

We shall prove now another lemma which is not needed in this paper. Its only importance is that it generalizes Lemma 2 in [1].

Lemma 2. *For any trigonometric series (1) and an arbitrary natural number n, the following estimate holds ($r=0,1,\dots$):*

$$\|\sigma_n^{(r)}-S_n^{(r)}\|\leq \frac{(n-1)^r}{n+1}\sum_{j=1}^{n-1}\|S_j-S_{[j/2]}\|+2n^r\max_{k=[n/2],\dots,n}\|S_k-S_{[n/2]}\|. \tag{17}$$

If

$$n^r\max_{k=[n/2],\dots,n}\|S_k-S_{[n/2]}\|=o(1)\ (=O(1))\,, \tag{18}$$

then condition (21) (see section 3 below in this paper) is satisfied.

Proof. Applying the Bernstein's inequality to the $r-$th derivative of the equality

$$(n+1)\left(S_n(x)-\sigma_n(x)\right)=\sum_{j=1}^{n-1}\left(S_j(x)-S_{[j/2]}(x)\right)+n\left(S_n(x)-S_{[n/2]}(x)\right)$$
$$-2\sum_{j=[n/2]+1}^{n-1}\left(S_j(x)-S_{[n/2]}(x)\right),$$

we obtain

$$\begin{aligned}(n+1)\|S_n^{(r)}-\sigma_n^{(r)}\| &\le \sum_{j=1}^{n-1}\|S_j^{(r)}-S_{[j/2]}^{(r)}\|+n\|S_n^{(r)}-S_{[n/2]}^{(r)}\| \\ &\quad +2\sum_{j=[n/2]+1}^{n-1}\|S_j^{(r)}-S_{[n/2]}^{(r)}\| \\ &\le \sum_{j=1}^{n-1}\|S_j^{(r)}-S_{[j/2]}^{(r)}\|+(2n-1)\max_{k=[n/2],\dots,n}\|S_k^{(r)}-S_{[n/2]}^{(r)}\| \\ &\le (n-1)^r\sum_{j=1}^{n-1}\|S_j-S_{[j/2]}\| \\ &\quad +2(n+1)n^r\max_{k=[n/2],\dots,n}\|S_k-S_{[n/2]}\|.\end{aligned}$$

Supposing that (18) holds, then obviously from (17) the estimate (21) holds. □

Lemma 3. *Given an arbitrary trigonometric series (1) and arbitrary natural numbers n and N such that $N\le 2n+1$, the following estimates hold:*

$$\max_{k=n,\dots,N}\|\widetilde{S}_k^{(r)}-\widetilde{S}_{n-1}^{(r)}\|\le 2N^r\max_{k=n,\dots,N}\|S_k-S_{n-1}\|;$$
$$\max_{m=n,\dots,N}\left\|\left(\sum_{j=n}^{m}c_je^{ijx}\right)^{(r)}\right\|\le\frac{3}{2}N^r\max_{m=n,\dots,N}\|S_m-S_{n-1}\|;$$
$$\max_{m=n,\dots,N}\left\|\left(\sum_{j=n}^{m}c_{-j}e^{-ijx}\right)^{(r)}\right\|\le\frac{3}{2}N^r\max_{m=n,\dots,N}\|S_m-S_{n-1}\|;$$
$$\max_{k=n,\dots,N}\|S_k^{(r)}-S_{n-1}^{(r)}\|\le 4N^r\max_{k=n-1,\dots,N}\|S_k-\sigma_k\|;$$
$$\sum_{k=n}^{N}\frac{k^r\lambda_k}{k+1-n}\le 15N^r\max_{k=n,\dots,N}\|S_k-S_{n-1}\|;$$
$$\sum_{k=n}^{N}\frac{k^r\lambda_k}{N+1-k}\le 10N^r\|S_N-S_{n-1}\|,\quad (r=0,1,\dots).$$

Proof. This lemma can be proved in a very same manner as Lemma 1. In this case it is sufficient to use the well-known Bernstein's inequality, therefore we shall omit it. □

Remark 1. *Putting $r = 0$ to Lemma 1 and Lemma 2 we obtain Lemma 1 and Lemma 2, respectively ,proved in [1]. Lemma 1 in [1] is a consequence of Lemma 3 as well.*

3. Main results

Let

$$\sum_{n=-\infty}^{\infty}(in)^r c_n e^{inx}\left(\sum_{n=1}^{\infty} n^r\left[a_n\cos\left(nx+\frac{r\pi}{2}\right)+b_n\sin\left(nx+\frac{r\pi}{2}\right)\right]\right) \qquad (19)$$

be the $r-$th derivative of a trigonometric series (1) in the complex or real form, respectively.

In this section we shall prove the following theorems which generalize Theorem 1 and Corollary 1.

Theorem 2. *If $n \geq 2$ is an integer and $r = 0, 1, \dots,$ then*

$$\sum_{k=\left[\frac{n}{2}\right]}^{2n}\frac{k^r\lambda_k}{|n-k|+1} \leq 100 \max_{m=[n/2]-1,\dots,2n}\|\sigma_m^{(r)} - S_m^{(r)}\|. \qquad (20)$$

In particular:

1. *If*

$$\|\sigma_m^{(r)} - S_m^{(r)}\| = o(1)\ (= O(1))\,, \qquad (21)$$

then

$$\sum_{k=\left[\frac{n}{2}\right]}^{2n}\frac{k^r\lambda_k}{|n-k|+1} = o(1)\ (= O(1),\ \textit{respectively})\,. \qquad (22)$$

2. *Assume that series (19) converges (possesses bounded partial sums) in the L^1-metric; then condition (22) holds.*

Proof. From Lemma 1, according to estimates (11) and (10)

$$\sum_{k=n}^{2n}\frac{k^r\lambda_k}{k+1-n} \leq 15 \max_{k=n,\dots,2n}\left\|S_k^{(r)} - S_{n-1}^{(r)}\right\| \leq 60 \max_{k=n,\dots,2n}\left\|S_k^{(r)} - \sigma_k^{(r)}\right\|. \qquad (23)$$

On the other hand, according to estimates (12) and (10), for $2[n/2]+1 \geq n$ we have

$$\sum_{k=\left[\frac{n}{2}\right]}^{n}\frac{k^r\lambda_k}{n+1-k} \leq 10\left\|S_n^{(r)} - S_{\left[\frac{n}{2}\right]-1}^{(r)}\right\| \leq 40 \max_{k=\left[\frac{n}{2}\right]-1,\dots,n}\left\|S_k^{(r)} - \sigma_k^{(r)}\right\|. \qquad (24)$$

Adding (23) and (24) we obtain (20). In addition, (21) and (20) imply (22).

Let series (19) converge (possess bounded partial sums) in the L^1−metric, then

$$\left\|\sigma_m^{(r)} - S_m^{(r)}\right\| \leq \left\|f^{(r)} - S_m^{(r)}\right\| + \left\|\sigma_m^{(r)} - f^{(r)}\right\| = o(1) \ (= O(1)) \,.$$

Therefore (21) implies (22). This completes the proof of the theorem. □

The following corollaries are direct consequeces of Theorem 2.

Corollary 2. *It holds:*

1. *Assume that series (4) or (5) satisfies condition (2), then*

$$\sum_{k=\left[\frac{n}{2}\right]}^{2n} \frac{k^r |a_k|}{|n-k|+1} = o(1) \ \ (O(1), \ \textit{respectively}) \,.$$

2. *Assume that series (4) or (5) converges (possesses bounded partial sums) in the L^1−metric, then condition (6) holds.*

Remark 2. *If we put $r = 0$ to Theorem 2, we obtain the Theorem 1. In other words, Theorem 2 is a generalization of Theorem 1. Likewise Corollary 1 is a direct consequence of Corollary 2 (the case $r = 0$).*

Finally, let us formulate a statement that generalizes only part (1) of Theorem 1.

Corollary 3. *If $n \geq 2$ is an integer and $r = 0, 1, \ldots,$ then*

$$\sum_{k=\left[\frac{n}{2}\right]}^{2n} \frac{k^r \lambda_k}{|n-k|+1} \leq 100 \max_{m=[n/2]-1,\ldots,2n} \left\{ m^r \|\sigma_m - S_m\| \right\}.$$

If

$$m^r \|\sigma_m - S_m\| = o(1) \ (= O(1)) \,,$$

then

$$\sum_{k=\left[\frac{n}{2}\right]}^{2n} \frac{k^r \lambda_k}{|n-k|+1} = o(1) \ (= O(1), \ \textit{respectively}) \,.$$

Proof. The proof of this corollary is obvious, therefore we shall omit it. □

Remark 3. *For $L^p_{2\pi}$ we write*

$$\|f\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p \, dx \right)^{1/p} \quad \textit{for} \quad 1 \leq p < \infty,$$

$$\|f\|_\infty = ess \sup_x |f(x)| \quad \textit{for} \quad p = \infty.$$

We observe that estimates (7)-(10) in Lemma 1 and estimate (17) in Lemma 2 with all the corresponding proofs hold true when the norm $\|\cdot\|$ is replaced by the norm $\|\cdot\|_p$ for $1 \leq p \leq \infty$.

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