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On the approximation properties of two-dimensional q-Bernstein-Chlodowsky polynomials

İbrahim Büyükyazici[1,*]

[1] *Department of Mathematics, Arts and Science Faculty, Gazi University, Ankara-06 500, Turkey*

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Abstract. In the present paper we introduce positive linear operators $q-$Bernstein - Chlodowsky polynomials on a rectangular domain and obtain their Korovkin type approximation properties. The rate of convergence of this generalization is obtained by means of the modulus of continuity, and also by using the K-functional of Peetre. We obtain weighted approximation properties for these positive linear operators and their generalizations in this paper.
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1. Introduction and definitions

The classical Bernstein-Chlodowsky polynomials have the following form

$$\tilde{B}_n(f;x) = \sum_{k=0}^{n} f(\frac{k}{n}\alpha_n)\binom{n}{k}\left(\frac{x}{\alpha_n}\right)^k\left(1-\frac{x}{\alpha_n}\right)^{n-k}$$

where $0 \le x \le \alpha_n$ and $\{\alpha_n\}$ is a sequence of positive numbers such that $\lim\limits_{n\to\infty}\alpha_n = \infty$, $\lim\limits_{n\to\infty}\frac{\alpha_n}{n} = 0$. These polynomials were introduced by Chlodowsky in 1932 as a generalization of Bernstein polynomials (1912) on an unbounded set. There are many investigations devoted to the problem of approximating continuous functions by classical Bernstein polynomials, as well as by two-dimensional Bernstein polynomials and their generalizations. On the other hand, Bernstein-Chlodowsky polynomials have not been studied so extensively and we know only a few papers that are devoted to the two-dimensional case [2,9,14]. Some generalizations of these polynomials in the one-dimensional case may be found in [7,8,16,21].

In recent years, the $q-$Bernstein polynomials, introduced by Phillips [18], have attracted a great deal of interest because of their potential applications in approximation theory and numerical analysis, and many properties of these polynomials have been discovered (see [3,10,12,13,17-20,22,24]). In [16] Karsli and Gupta introduced the q analogue of Bernstein-Chlodowsky polynomials in the one-dimensional case. They investigated approximation properties for these new polynomials.

*Corresponding author. *Email address:* `bibrahim@gazi.edu.tr` (I. Büyükyazici)

In this paper, we define $q-$Bernstein-Chlodowsky polynomials on a rectangular domain $D_{ab} = [0,a] \times [0,b]$ in (2) and investigate their Korovkin type approximation properties. We compute the rate of convergence of these new operators by means of the modulus of continuity. Also by using the K-functional of Peetre the order of approximation is established (in Section 3).

The second purpose of this paper is to obtain weighted approximation properties of these new operators on $IR_+^2 = [0,\infty) \times [0,\infty)$. In order to obtain these results we will use the weighted Korovkin type theorem proved by Gadjiev in [5,6] (in Section 4).

Finally, we give a generalization of (2) and obtain weighted approximation properties of these generalization operators (in Section 5).

Definition 1. *$C^2(D_{ab})$ is the space of functions of f such that $f, \dfrac{\partial^i f}{\partial x^i}, \dfrac{\partial^i f}{\partial y^i}$ $(i = 1, 2)$ belong to $C(D_{ab})$. The norm on the space $C^2(D_{ab})$ can be defined as*

$$\|f\|_{C^2(D_{ab})} = \|f\|_{C(D_{ab})} + \sum_{i=1}^{2} \left(\left\| \frac{\partial^i f}{\partial x^i} \right\|_{C(D_{ab})} + \left\| \frac{\partial^i f}{\partial y^i} \right\|_{C(D_{ab})} \right).$$

Definition 2 (see [1])**.** *For $f \in C(D_{ab})$ and $\delta > 0$, the Peetre-K functional is defined by*

$$K(f;\delta) = \inf_{g \in C^2(D_{ab})} \left\{ \|f-g\|_{C(D_{ab})} + \delta \|g\|_{C^2(D_{ab})} \right\}.$$

It is clear that if $f \in C(D_{ab})$, then we have $\lim\limits_{\delta \to 0} K(f;\delta) = 0$.

Definition 3. *Let $f \in C(D_{ab})$ be a continuous function and δ a positive number. The full continuity modulus of the function $f(x,y)$ is*

$$\omega(f;\delta) = \max_{\substack{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \le \delta \\ x,y \in D_{ab}}} |f(x_1,y_1) - f(x_2,y_2)|$$

and its partial continuity moduli with respect to x and y are

$$\omega^{(1)}(f;\delta) = \max_{0 \le y \le b} \max_{|x_1-x_2| \le \delta} |f(x_1,y) - f(x_2,y)|$$
$$\omega^{(2)}(f;\delta) = \max_{0 \le x \le a} \max_{|y_1-y_2| \le \delta} |f(x,y_1) - f(x,y_2)|.$$

It is also known that $\lim\limits_{\delta \to 0} \omega(f;\delta) = 0$ and for any $\lambda > 0$, $\omega(f;\lambda\delta) \le (\lambda+1)\omega(f;\delta)$. The same properties are satisfied by partial continuity moduli.

Definition 4. *Let $\rho(x,y) = 1 + x^2 + y^2$, $(x,y) \in IR_+^2$. Denote by $B_\rho(IR_+^2)$ and $C_\rho(IR_+^2)$ the following spaces:*

$B_\rho(IR_+^2)$: The space of all functions f satisfying the condition $|f(x,y)| \le M_f \rho(x,y)$, where M_f is a constant depending on the function f only.

$C_\rho(IR_+^2)$: The subspace of all continuous functions in the space $B_\rho(IR_+^2)$.

C_ρ is obviously a linear normed space with the $\rho-$norm:

$$\|f\|_\rho = \sup_{x,y\geq 0} \frac{|f(x,y)|}{\rho(x,y)}.$$

Lemma 1. *In order to have the sequence of positive linear operators* $\{L_{n,m}\}_{n,m\geq 1}$ *act from* $C_\rho(IR_+^2)$ *to* $B_\rho(IR_+^2)$, *it is necessary and sufficient that the inequality*

$$L_{n,m}(\rho;x,y) \leq K\rho(x,y)$$

is fulfilled with some positive constant K (see [5,6]).

2. Construction of operators

Let $q > 0$. For each nonnegative integer n, we define the q-integer $[n]_q$ as

$$[n]_q = \begin{cases} (1-q^n)/(1-q), & \text{if } q \neq 1 \\ n, & \text{if } q = 1 \end{cases}$$

and the q-factorial $[n]_q!$ as

$$[n]_q! = \begin{cases} [n]_q[n-1]_q \cdots [1]_q, & \text{if } n \geq 1 \\ 1, & \text{if } n = 0. \end{cases}$$

For integers n and k, with $0 \leq k \leq n$, q-binomial coefficients are then defined as follows (see [15,19]):

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Let $\{\alpha_n\}$ and $\{\beta_m\}$ be increasing sequences of positive real numbers and let them satisfy the properties: $\lim\limits_{n\to\infty} \alpha_n = \lim\limits_{m\to\infty} \beta_m = \infty$ that the sequences

$$\left\{\frac{\alpha_n}{[n]_{q_n}}\right\} \text{ and } \left\{\frac{\beta_m}{[m]_{q_m}}\right\}$$

decrease to zero as $n, m \to \infty$, where $\{q_n\}$ is a sequence of real numbers such that $0 < q_n \leq 1$ for all n and $\lim\limits_{n\to\infty} q_n = 1$.

For any $\alpha_n > 0, \beta_m > 0$ we denote by $D_{\alpha_n\beta_m}$:

$$D_{\alpha_n\beta_m} = \{(x,y)\colon\ 0 \leq x \leq \alpha_n,\ 0 \leq y \leq \beta_m\}. \tag{1}$$

We can introduce the Bernstein-Chlodowsky type polynomials for a function f of two variables as follows:

$$\tilde{B}_{n,m}^{q_n,q_m}(f;x,y) = \sum_{k=0}^{n}\sum_{j=0}^{m} f(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n, \frac{[j]_{q_m}}{[m]_{q_m}}\beta_m)\Omega_{k,n,q_n}(\frac{x}{\alpha_n})\Omega_{j,m,q_m}(\frac{y}{\beta_m}), \tag{2}$$

where $(x,y) \in D_{\alpha_n\beta_m}$, $\Omega_{k,n,q_n}(u) = \begin{bmatrix} n \\ k \end{bmatrix}_{q_n} u^k \prod\limits_{s=0}^{n-k-1}(1-q_n^s u)$ and $\sum\limits_{k=0}^{n} \Omega_{k,n,q_n}(u) = 1$.

Remark 1. *It should be mentioned here that the q analogue of the Durrmeyer operators in the one-dimensional case has been studied in [4] and [11]. Also, we can define the two-dimensional q analogue of the Bernstein-Chlodowsky-Durrmeyer operators as follows:*

$$\tilde{D}_{n,m}^{q_n,q_m}(f;x,y) = \frac{[n+1]_{q_n}}{\alpha_n}\frac{[m+1]_{q_m}}{\beta_m}\sum_{k=0}^{n}\sum_{j=0}^{m} q_n^{-k}q_m^{-j}\Psi_{n,m}^{q_n,q_m}(\frac{x}{\alpha_n},\frac{y}{\beta_m})$$
$$\times\left(\int_{t=0}^{\alpha_n}\int_{s=0}^{\beta_m}\Psi_{n,m}^{q_n,q_m}(q_n\frac{t}{\alpha_n},q_m\frac{s}{\beta_m})f(t,s)d_{q_n}td_{q_m}s\right),$$

where $\Psi_{n,m}^{q_n,q_m}(u,v)=\Omega_{k,n,q_n}(u)\Omega_{j,m,q_m}(v)$ *and* $0\le x\le\alpha_n,\ 0\le y\le\beta_m$.

3. Convergence and rate of approximation

Theorem 1. *If* $f\in C(D_{ab})$, *then for any sufficiently large fixed positive real numbers* a *and* b $(a\le\alpha_n, b\le\beta_m)$ *the equality*

$$\lim_{n,m\to\infty}\max_{(x,y)\in D_{ab}}|\tilde{B}_{n,m}^{q_n,q_m}(f;x,y)-f(x,y)|=0$$

holds (see Example 1).

Proof. Simple calculations show that

$$\tilde{B}_{n,m}^{q_n,q_m}(1;x,y) = 1\,, \tag{3}$$
$$\tilde{B}_{n,m}^{q_n,q_m}(t_1;x,y) = x\,, \tag{4}$$
$$\tilde{B}_{n,m}^{q_n,q_m}(t_2;x,y) = y\,, \tag{5}$$
$$\tilde{B}_{n,m}^{q_n,q_m}(t_1^2;x,y) = x^2+\frac{x(\alpha_n-x)}{[n]_{q_n}}\,, \tag{6}$$
$$\tilde{B}_{n,m}^{q_n,q_m}(t_2^2;x,y) = y^2+\frac{y(\beta_m-y)}{[m]_{q_m}}. \tag{7}$$

If we use the above equalities, we can see that,

$$\left\|\tilde{B}_{n,m}^{q_n,q_m}(1;x,y)-1\right\|_{C(D_{ab})} = 0,$$
$$\left\|\tilde{B}_{n,m}^{q_n,q_m}(t_1;x,y)-x\right\|_{C(D_{ab})} = 0,$$
$$\left\|\tilde{B}_{n,m}^{q_n,q_m}(t_2;x,y)-y\right\|_{C(D_{ab})} = 0,$$
$$\left\|\tilde{B}_{n,m}^{q_n,q_m}(t_1^2+t_2^2;x,y)-(x^2+y^2)\right\|_{C(D_{ab})} \le a\frac{\alpha_n}{[n]_{q_n}}+b\frac{\beta_m}{[m]_{q_m}}.$$

The proof of uniform convergence is then completed by applying a Korovkin-type theorem [23]. □

Theorem 2. *For any $f \in C(D_{ab})$, the following inequalities*

$$\left|\tilde{B}_{n,m}^{q_n,q_m}(f;x,y) - f(x,y)\right| \leq 2\left[\omega^{(1)}(f;\sqrt{a\frac{\alpha_n}{[n]_{q_n}}}) + \omega^{(2)}(f;\sqrt{b\frac{\beta_m}{[m]_{q_m}}})\right], \quad (8)$$

$$\left|\tilde{B}_{n,m}^{q_n,q_m}(f;x,y) - f(x,y)\right| \leq 2\omega(f;\sqrt{a\frac{\alpha_n}{[n]_{q_n}} + b\frac{\beta_m}{[m]_{q_m}}}) \quad (9)$$

hold.

Proof. Using the relation $\sum_{k=0}^{n} \Omega_{k,n,q_n}(\frac{x}{\alpha_n}) = \sum_{j=0}^{m} \Omega_{j,m,q_m}(\frac{y}{\beta_m}) = 1$, express the difference between $\tilde{B}_{n,m}^{q_n,q_m}(f;x,y)$ and $f(x,y)$ as

$$\begin{aligned}\tilde{B}_{n,m}^{q_n,q_m}(f;x,y) - f(x,y) &= \sum_{k=0}^{n}\sum_{j=0}^{m}\left\{f(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n, \frac{[j]_{q_m}}{[m]_{q_m}}\beta_m) - f(x,y)\right\} \\ &\quad \times\Omega_{k,n,q_n}(\frac{x}{\alpha_n})\Omega_{j,m,q_m}(\frac{y}{\beta_m}) \\ &= \sum_{k=0}^{n}\sum_{j=0}^{m}\left\{f(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n, \frac{[j]_{q_m}}{[m]_{q_m}}\beta_m) - f(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n, y)\right. \\ &\quad \left. + f(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n, y) - f(x,y)\right\}\Omega_{k,n,q_n}(\frac{x}{\alpha_n})\Omega_{j,m,q_m}(\frac{y}{\beta_m})\end{aligned} \quad (10)$$

and so

$$\begin{aligned}\left|\tilde{B}_{n,m}^{q_n,q_m}(f;x,y) - f(x,y)\right| &\leq \sum_{k=0}^{n}\sum_{j=0}^{m}\left|f(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n, \frac{[j]_{q_m}}{[m]_{q_m}}\beta_m) - f(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n, y)\right| \\ &\quad \times\Omega_{k,n,q_n}(\frac{x}{\alpha_n})\Omega_{j,m,q_m}(\frac{y}{\beta_m}) \\ &\quad + \sum_{k=0}^{n}\sum_{j=0}^{m}\left|f(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n, y) - f(x,y)\right| \\ &\quad \times\Omega_{k,n,q_n}(\frac{x}{\alpha_n})\Omega_{j,m,q_m}(\frac{y}{\beta_m}) \\ &\leq \sum_{k=0}^{n}\sum_{j=0}^{m}\omega^{(2)}(f;\left|\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m - y\right|)\Omega_{k,n,q_n}(\frac{x}{\alpha_n})\Omega_{j,m,q_m}(\frac{y}{\beta_m}) \\ &\quad + \sum_{k=0}^{n}\sum_{j=0}^{m}\omega^{(1)}(f;\left|\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n - x\right|)\Omega_{k,n,q_n}(\frac{x}{\alpha_n})\Omega_{j,m,q_m}(\frac{y}{\beta_m}) \\ &= \Phi_1(x,y) + \Phi_2(x,y).\end{aligned}$$

Consider $\Phi_1(x,y)$. By using (3) and well-known properties of the modulus of continuity, we get

$$\Phi_1(x,y) = \sum_{k=0}^{n}\sum_{j=0}^{m}\omega^{(2)}(f;\left|\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m - y\right|)\Omega_{k,n,q_n}(\frac{x}{\alpha_n})\Omega_{j,m,q_m}(\frac{y}{\beta_m})$$
$$= \sum_{j=0}^{m}\omega^{(2)}(f;\left|\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m - y\right|)\Omega_{j,m,q_m}(\frac{y}{\beta_m})$$
$$\le \omega^{(2)}(f;\delta_m)\left\{1+\frac{1}{\delta_m}\left[\sum_{j=0}^{m}\left(\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m - y\right)^2\Omega_{j,m,q_m}(\frac{y}{\beta_m})\right]^{1/2}\right\},$$

where we have invoked the Cauchy-Schwartz inequality. Expanding the squared term and making use of (3), (5) and (7), we obtain

$$\Phi_1(x,y) \le \omega^{(2)}(f;\delta_m)\left\{1+\frac{1}{\delta_m}\sqrt{\frac{y(\beta_m-y)}{[m]_{q_m}}}\right\} \le \omega^{(2)}(f;\delta_m)\left\{1+\frac{1}{\delta_m}\sqrt{b\frac{\beta_m}{[m]_{q_m}}}\right\},$$

and by choosing $\delta_m = \sqrt{b\frac{\beta_m}{[m]_{q_m}}}$ we obtain

$$\Phi_1(x,y) \le 2\omega^{(2)}(f;\sqrt{b\frac{\beta_m}{[m]_{q_m}}}).$$

In the same way we obtain

$$\Phi_2(x,y) \le 2\omega^{(1)}(f;\sqrt{a\frac{\alpha_n}{[n]_{q_n}}})$$

which proves (8). From (10) we have

$$\left|\tilde{B}_{n,m}^{q_n,q_m}(f;x,y) - f(x,y)\right| \le \sum_{k=0}^{n}\sum_{j=0}^{m}\left|f(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n,\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m) - f(x,y)\right| \times\Omega_{k,n,q_n}(\frac{x}{\alpha_n})\Omega_{j,m,q_m}(\frac{y}{\beta_m}). \quad (11)$$

Using properties of the modulus of continuity and applying the Cauchy–Schwartz inequality directly in (12), we obtain (9). □

Corollary 1. *Let f be Hölder continuous on D_{ab}, which is denoted by $f \in Lip_M(\gamma, D_{ab})$. Then*

$$\left|\tilde{B}_{n,m}^{q_n,q_m}(f;x,y) - f(x,y)\right| \le M'\left(a\frac{\alpha_n}{[n]_{q_n}} + b\frac{\beta_m}{[m]_{q_m}}\right)^{\gamma/2},$$

where $M' = 2M$ and $0<\gamma\le 1$.

Corollary 2. *If f satisfies the Lipschitz conditions*

$$|f(x_1,y)-f(x_2,y)| \leq M_1 |x_1-x_2|^{\gamma_1}$$

and

$$|f(x,y_1)-f(x,y_2)| \leq M_2 |y_1-y_2|^{\gamma_2},$$

then

$$\left|\tilde{B}_{n,m}^{q_n,q_m}(f;x,y)-f(x,y)\right| \leq M_1^{'}\left(a\frac{\alpha_n}{[n]_{q_n}}\right)^{\gamma_1/2}+M_2^{'}\left(b\frac{\beta_m}{[m]_{q_m}}\right)^{\gamma_2/2},$$

where $M_1^{'}=2M_1$, $M_2^{'}=2M_2$ and $0<\gamma_1,\gamma_2\leq 1$.

Theorem 3. *Let $f(x,y)$ have continuous partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$, let $\varpi^{(1)}(f,.)$ and $\varpi^{(2)}(f,.)$ denote the partial moduli of continuity of $\partial f/\partial x$ and $\partial f/\partial y$, respectively. Then the inequality*

$$\begin{aligned}\left|\tilde{B}_{n,m}^{q_n,q_m}(f;x,y)-f(x,y)\right| \leq 2\Bigg(&\sqrt{a\frac{\alpha_n}{[n]_{q_n}}}\varpi^{(1)}(f;\sqrt{a\frac{\alpha_n}{[n]_{q_n}}})\\ &+\sqrt{b\frac{\beta_m}{[m]_{q_m}}}\varpi^{(2)}(f;\sqrt{b\frac{\beta_m}{[m]_{q_m}}})\Bigg)\end{aligned}$$

holds.

Proof. By the mean value theorem, we can write

$$\begin{aligned} f\left(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n,\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m\right)-f(x,y) &= f\left(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n,y\right)-f(x,y)\\ &\quad+f\left(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n,\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m\right)-f\left(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n,y\right)\\ &=(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n-x)\frac{\partial f(x,y)}{\partial x}\\ &\quad+(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n-x)\left[\frac{\partial f(\zeta_1,y)}{\partial x}-\frac{\partial f(x,y)}{\partial x}\right]\\ &\quad+(\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m-y)\frac{\partial f(x,y)}{\partial y} \qquad (12)\\ &\quad+(\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m-y)\left[\frac{\partial f(x,\zeta_2)}{\partial y}-\frac{\partial f(x,y)}{\partial y}\right]\end{aligned}$$

for any fixed $y\in[0,b]$, where ζ_1 is some point between x and $\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n$ and any fixed $x\in[0,a]$, where ζ_2 is some point between y and $\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m$. Let us apply the operator

$\tilde{B}^{q_n,q_m}_{n,m}$ to (12) to obtain

$$\tilde{B}^{q_n,q_m}_{n,m}(f;x,y)-f(x,y)=\frac{\partial f(x,y)}{\partial x}\sum_{k=0}^{n}\sum_{j=0}^{m}\left(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n-x\right)$$
$$\times\Omega_{k,n,q_n}(\frac{x}{\alpha_n})\Omega_{j,m,q_m}(\frac{y}{\beta_m})$$
$$+\sum_{k=0}^{n}\sum_{j=0}^{m}\left(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n-x\right)\left(\frac{\partial f(\zeta_1,y)}{\partial x}-\frac{\partial f(x,y)}{\partial x}\right)$$
$$\times\Omega_{k,n,q_n}(\frac{x}{\alpha_n})\Omega_{j,m,q_m}(\frac{y}{\beta_m})$$
$$+\frac{\partial f(x,y)}{\partial y}\sum_{k=0}^{n}\sum_{j=0}^{m}\left(\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m-y\right)\Omega_{k,n,q_n}(\frac{x}{\alpha_n})\Omega_{j,m,q_m}(\frac{y}{\beta_m})$$
$$+\sum_{k=0}^{n}\sum_{j=0}^{m}\left(\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m-y\right)\left(\frac{\partial f(x,\zeta_2)}{\partial y}-\frac{\partial f(x,y)}{\partial y}\right)$$
$$\times\Omega_{k,n,q_n}(\frac{x}{\alpha_n})\Omega_{j,m,q_m}(\frac{y}{\beta_m}).$$

Since

$$\sum_{k=0}^{n}\left(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n-x\right)\Omega_{k,n,q_n}(\frac{x}{\alpha_n})=\sum_{j=0}^{m}\left(\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m-y\right)\Omega_{j,m,q_m}(\frac{y}{\beta_m})=0$$

and

$$|\zeta_1-x|\le\left|\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n-x\right|,\ |\zeta_2-y|\le\left|\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m-y\right|,$$

we obtain the inequality

$$\left|\tilde{B}^{q_n,q_m}_{n,m}(f;x,y)-f(x,y)\right|\le\varpi^{(1)}(f;\delta_n)\sum_{k=0}^{n}\left|\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n-x\right|\Omega_{k,n,q_n}(\frac{x}{\alpha_n})$$
$$+\frac{\varpi^{(1)}(f;\delta_n)}{\delta_n}\sum_{k=0}^{n}\left(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n-x\right)^2\Omega_{k,n,q_n}(\frac{x}{\alpha_n})$$
$$+\varpi^{(2)}(f;\delta_m)\sum_{j=0}^{m}\left|\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m-y\right|\Omega_{j,m,q_m}(\frac{y}{\beta_m})$$
$$+\frac{\varpi^{(2)}(f;\delta_m)}{\delta_m}\sum_{j=0}^{m}\left(\frac{[j]_{q_m}}{[m]_{q_m}}\beta_m-y\right)^2\Omega_{j,m,q_m}(\frac{y}{\beta_m}).$$

From (3), (4), (5), (6) and (7), we have

$$\left|\tilde{B}_{n,m}^{q_n,q_m}(f;x,y)-f(x,y)\right| \leq \varpi^{(1)}(f;\delta_n)\left[\sqrt{a\frac{\alpha_n}{[n]_{q_n}}}+\frac{1}{\delta_n}a\frac{\alpha_n}{[n]_{q_n}}\right] +\varpi^{(2)}(f;\delta_m)\left[\sqrt{b\frac{\beta_m}{[m]_{q_m}}}+\frac{1}{\delta_m}b\frac{\beta_m}{[m]_{q_m}}\right]$$

and choosing $\delta_n=\sqrt{a\frac{\alpha_n}{[n]_{q_n}}}$ and $\delta_m=\sqrt{b\frac{\beta_m}{[m]_{q_m}}}$ we obtain

$$\left|\tilde{B}_{n,m}^{q_n,q_m}(f;x,y)-f(x,y)\right| \leq 2\left(\sqrt{a\frac{\alpha_n}{[n]_{q_n}}}\varpi^{(1)}(f;\sqrt{a\frac{\alpha_n}{[n]_{q_n}}}) +\sqrt{b\frac{\beta_m}{[m]_{q_m}}}\varpi^{(2)}(f;\sqrt{b\frac{\beta_m}{[m]_{q_m}}})\right).$$

Hence Theorem 3 is proved. □

Theorem 4. *If* $f\in C(D_{ab})$, *then* $\left\|\tilde{B}_{n,m}^{q_n,q_m}(f;x,y)-f(x,y)\right\|_{C(D_{ab})}\leq 2K(f;\frac{\delta_{n,m}}{2})$ *where*

$$\delta_{n,m}=\frac{1}{2}\max\left(a\frac{\alpha_n}{[n]_{q_n}},b\frac{\beta_m}{[m]_{q_m}}\right).$$

Proof. Let $g\in C^2(D_{ab})$. Using Taylor's theorem, we can write

$$\begin{aligned} g(t_1,t_2)-g(x,y) &= g(t_1,y)-g(x,y)+g(t_1,t_2)-g(t_1,y) \\ &= \frac{\partial g(x,y)}{\partial x}(t_1-x)+\int_x^{t_1}(t_1-u)\frac{\partial^2 g(u,y)}{\partial u^2}du \\ &\quad +\frac{\partial g(x,y)}{\partial y}(t_2-y)+\int_y^{t_2}(t_2-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv \\ &= \frac{\partial g(x,y)}{\partial x}(t_1-x)+\int_0^{t_1-x}(t_1-x-u)\frac{\partial^2 g(u+x,y)}{\partial u^2}du \\ &\quad +\frac{\partial g(x,y)}{\partial y}(t_2-y)+\int_0^{t_2-y}(t_2-y-v)\frac{\partial^2 g(x,v+y)}{\partial v^2}dv. \end{aligned}$$

Applying the operator $\tilde{B}_{n,m}^{q_n,q_m}$ to both sides, we deduce that

$$\begin{aligned}\left|\tilde{B}_{n,m}^{q_n,q_m}(g;x,y)-g(x,y)\right| &\leq \left|\frac{\partial g(x,y)}{\partial x}\right|\left|\tilde{B}_{n,m}^{q_n,q_m}(t_1-x;x,y)\right| \\ &+\left|\tilde{B}_{n,m}^{q_n,q_m}(\int_0^{t_1-x}(t_1-x-u)\frac{\partial^2 g(u+x,y)}{\partial u^2}du;x,y)\right| \\ &+\left|\frac{\partial g(x,y)}{\partial y}\right|\left|\tilde{B}_{n,m}^{q_n,q_m}(t_2-y;x,y)\right| \\ &+\left|\tilde{B}_{n,m}^{q_n,q_m}(\int_0^{t_2-y}(t_2-y-v)\frac{\partial^2 g(x,v+y)}{\partial v^2}dv;x,y)\right|.\end{aligned}$$

Since $\tilde{B}_{n,m}^{q_n,q_m}(t_1-x;x,y)=0$ and $\tilde{B}_{n,m}^{q_n,q_m}(t_2-y;x,y)=0$, we obtain

$$\begin{aligned}\left\|\tilde{B}_{n,m}^{q_n,q_m}(g;x,y)-g(x,y)\right\|_{C(D_{ab})} &\leq \frac{1}{2}\left\|\frac{\partial^2 g}{\partial x^2}\right\|_{C(D_{ab})}\left|\tilde{B}_{n,m}^{q_n,q_m}((t_1-x)^2;x,y)\right| \\ &+\frac{1}{2}\left\|\frac{\partial^2 g}{\partial y^2}\right\|_{C(D_{ab})}\left|\tilde{B}_{n,m}^{q_n,q_m}((t_2-y)^2;x,y)\right|.\end{aligned}$$

From (6) and (7), we get

$$\begin{aligned}\left\|\tilde{B}_{n,m}^{q_n,q_m}(g;x,y)-g(x,y)\right\|_{C(D_{ab})} &\leq \frac{1}{2}a\frac{\alpha_n}{[n]_{q_n}}\left\|\frac{\partial^2 g}{\partial x^2}\right\|_{C(D_{ab})}+\frac{1}{2}b\frac{\beta_m}{[m]_{q_m}}\left\|\frac{\partial^2 g}{\partial y^2}\right\|_{C(D_{ab})} \\ &\leq \frac{1}{2}\max\left(a\frac{\alpha_n}{[n]_{q_n}},b\frac{\beta_m}{[m]_{q_m}}\right) \\ &\quad\times\left[\left\|\frac{\partial^2 g}{\partial x^2}\right\|_{C(D_{ab})}+\left\|\frac{\partial^2 g}{\partial y^2}\right\|_{C(D_{ab})}\right] \\ &\leq \delta_{n,m}\|g\|_{C^2(D_{ab})} \qquad (13)\end{aligned}$$

where $\delta_{n,m}=\frac{1}{2}\max\left(a\frac{\alpha_n}{[n]_{q_n}},b\frac{\beta_m}{[m]_{q_m}}\right)$. By the linearity property of $\tilde{B}_{n,m}^{q_n,q_m}$, we have

$$\begin{aligned}\left\|\tilde{B}_{n,m}^{q_n,q_m}(f;x,y)-f(x,y)\right\|_{C(D_{ab})} &\leq \left\|\tilde{B}_{n,m}^{q_n,q_m}f-\tilde{B}_{n,m}^{q_n,q_m}g\right\|_{C(D_{ab})}+\left\|\tilde{B}_{n,m}^{q_n,q_m}g-g\right\|_{C(D_{ab})} \\ &\quad+\|f-g\|_{C(D_{ab})} \\ &\leq \|f-g\|_{C(D_{ab})}\left|\tilde{B}_{n,m}^{q_n,q_m}(1;x,y)\right|+\|f-g\|_{C(D_{ab})} \\ &\quad+\left\|\tilde{B}_{n,m}^{q_n,q_m}g-g\right\|_{C(D_{ab})} \qquad (14)\end{aligned}$$

and from (13) and (14), we obtain

$$\left\|\tilde{B}_{n,m}^{q_n,q_m}(f;x,y)-f(x,y)\right\|_{C(D_{ab})}\leq 2\left(\|f-g\|_{C(D_{ab})}+\frac{\delta_{n,m}}{2}\|g\|_{C^2(D_{ab})}\right). \qquad (15)$$

We complete the proof by taking the infimum over $g \in C^2(D_{ab})$. □

Example 1. *For $q_n = 1 - \frac{1}{\sqrt{n}}$ and $\alpha_n = \ln(n), \beta_m = \ln(m)$, the convergence of $\tilde{B}_{n,m}^{q_n,q_m}(f;x,y)$ (green) to $f(x,y) = \sin(x-y)$ (red) is illustrated in Figure 1.*

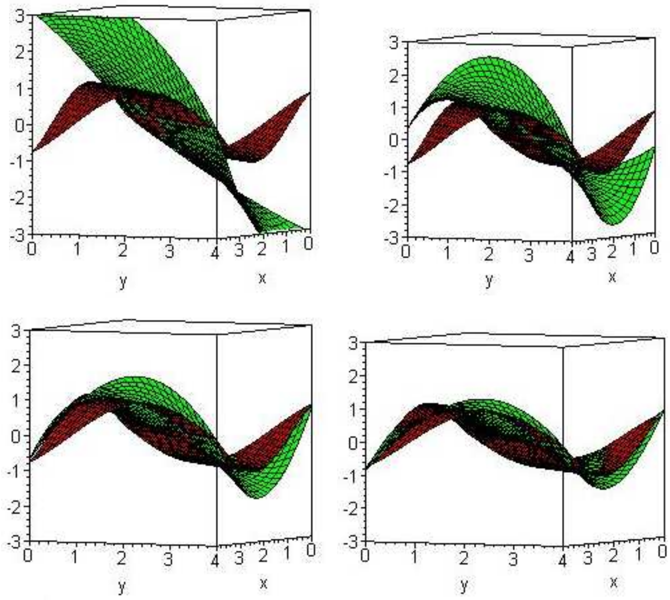

Figure 1. *Convergence of two-dimensional q-Bernstein-Chlodowsky polynomials*

4. Weighted approximation properties

A Korovkin type theorem (in weighted spaces) for linear positive operators $L_{n,m}$, acting from C_ρ to B_ρ, has been proved by Gadjiev in [5,6].

Theorem 5 (see [5, 6]). *If there exists a sequence of positive linear operators $L_{n,m}$, acting from $C_\rho(IR_+^2)$ to $B_\rho(IR_+^2)$, satisfying the conditions*

$$\lim_{n,m\to\infty} \|L_{n,m}(1;x,y) - 1\|_\rho = 0, \tag{16}$$

$$\lim_{n,m\to\infty} \|L_{n,m}(t_1;x,y) - x\|_\rho = 0, \tag{17}$$

$$\lim_{n,m\to\infty} \|L_{n,m}(t_2;x,y) - y\|_\rho = 0, \tag{18}$$

$$\lim_{n,m\to\infty} \left\|L_{n,m}(t_1^2 + t_2^2;x,y) - (x^2+y^2)\right\|_\rho = 0, \tag{19}$$

then there exists a function $f^* \in C_\rho(IR_+^2)$ for which

$$\lim_{n,m\to\infty} \|L_{n,m}f^* - f^*\|_\rho \geq 1.$$

Theorem 6 (see [5, 6]). *Let $L_{n,m}$ be a sequence of positive linear operators acting from $C_\rho(IR_+^2)$ to $B_\rho(IR_+^2)$ and let $\rho_1(x,y) \geq 1$ be a continuous function for which*

$$\lim_{|\upsilon|\to\infty} \frac{\rho(\upsilon)}{\rho_1(\upsilon)} = 0, (\text{where } \upsilon = (x,y)). \tag{20}$$

Conditions (16), (17), (18), (19) *imply*

$$\lim_{n,m\to\infty} \|L_{n,m}f - f\|_{\rho_1} = 0$$

for all $f \in C_\rho(IR_+^2)$.

We consider positive linear operators $L^*_{n,m}$, defined by

$$L^*_{n,m}(f;x,y) = \begin{cases} \tilde{B}^{q_n,q_m}_{n,m}(f;x,y), & \text{if } (x,y) \in D_{\alpha_n\beta_m} \\ f(x,y), & \text{if } (x,y) \in IR_+^2 \backslash D_{\alpha_n\beta_m} \end{cases} \tag{21}$$

where $D_{\alpha_n\beta_m}$ is defined by (1).

Theorem 7. *Let $L^*_{n,m}$ be the sequence of linear positive operators defined by* (21). *Then for all $f \in C_\rho(IR_+^2)$ we have*

$$\lim_{n,m\to\infty} \left\|L^*_{n,m}f - f\right\|_{\rho_1} = 0$$

where $\rho(x,y) = 1 + x^2 + y^2$ and $\rho_1(x,y)$ is the continuous function satisfying conditions (20) *and $\{\alpha_n\}$ and $\{\beta_m\}$ are increasing sequences of positive real numbers that satisfy the properties:*

$$\lim_{n\to\infty} \alpha_n = \lim_{m\to\infty} \beta_m = \infty \text{ and } \lim_{n\to\infty} \frac{\alpha_n}{[n]_{q_n}} = \lim_{m\to\infty} \frac{\beta_m}{[m]_{q_m}} = 0. \tag{22}$$

Proof. Firstly, let us show that L^*_n is acting from $C_\rho(IR_+^2)$ to $B_\rho(IR_+^2)$. Using (3), (6) and (7), one can write

$$\begin{aligned} \left\|L^*_{n,m}(\rho;x,y)\right\|_\rho &\leq 1 + \frac{1}{[n]_{q_n}} \sup_{0\leq x\leq\alpha_n} \frac{x(\alpha_n - x)}{1+x^2+y^2} + \frac{1}{[m]_{q_m}} \sup_{0\leq y\leq\beta_m} \frac{y(\beta_m - y)}{1+x^2+y^2} \\ &\leq 1 + \frac{\alpha_n}{[n]_{q_n}} + \frac{\beta_m}{[m]_{q_m}} \\ &\leq 1 + \sigma_{n,m} \end{aligned}$$

where $\sigma_{n,m} = \dfrac{\alpha_n}{[n]_{q_n}} + \dfrac{\beta_m}{[m]_{q_m}}$. Since $\sigma_{n,m} \to 0$ as $n,m\to\infty$, there is a positive constant M such that $\sigma_{n,m} < M$ for all natural numbers n and m. Hence we have

$$\left\|L^*_{n,m}(\rho;x,y)\right\|_\rho \leq 1 + M.$$

From Lemma 1 we have $L^*_{n,m} : C_\rho(IR_+^2) \to B_\rho(IR_+^2)$. If we can show that the conditions (16), (17), (18) and (19) are satisfied, then the proof is completed by Theorem 6. By using (3), (4), (5), we have (16), (17) and (18). Lastly, using (6) and (7), we get

$$\left\|L^*_{n,m}(t_1^2 + t_2^2;x,y) - (x^2+y^2)\right\|_\rho \leq \sigma_{n,m}$$

and since $\sigma_{n,m} \to 0$ as $n,m\to\infty$, we obtain the desired result. □

5. A generalization of $\tilde{B}_{n,m}^{q_n,q_m}$

We now give a generalization of the q-Bernstein-Chlodowsky polynomials, which can be used to approximate continuous functions on more general weighted spaces.

Let $\psi(x,y) \geq 1$ be any continuous function for $x, y \geq 0$. Also let

$$F_f(u,v) = f(u,v)\frac{\psi(u,v)}{1+u^2+v^2}$$

and consider the following generalization of polynomials (2)

$$\tilde{C}_{n,m}^{q_n,q_m}(f;x,y) = \frac{1+x^2+y^2}{\psi(x,y)}\sum_{k=0}^{n}\sum_{j=0}^{m} F_f(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n, \frac{[j]_{q_m}}{[m]_{q_m}}\beta_m) \times\Omega_{k,n,q_n}(\frac{x}{\alpha_n})\Omega_{j,m,q_m}(\frac{y}{\beta_m}), \quad (23)$$

where $(x,y) \in D_{\alpha_n\beta_m}$ and $\{\alpha_n\}, \{\beta_m\}$ have the same property as in (22). In the case of $\psi(x,y) = 1+x^2+y^2$ operators (23) coincide with (2).

Theorem 8. *If* $\lim\limits_{|v|\to\infty}\frac{1+|v|^2}{\psi(v)} = 0$*(where* $v=(x,y)$*), then for a continuous function* f *satisfying the inequality* $|f(x,y)|\,\psi(x,y) \leq M_f$, $x,y \geq 0$,

$$\lim_{n,m\to\infty}\left\|\tilde{C}_{n,m}^{q_n,q_m}(f;x,y) - f(x,y)\right\|_\rho = 0$$

holds, where $\rho(x,y) = 1+x^2+y^2$ *(see Example 2).*

Proof. Obviously, we can write

$$\left|\tilde{C}_{n,m}^{q_n,q_m}(f;x,y) - f(x,y)\right| = \frac{1+x^2+y^2}{\psi(x,y)}\left|\sum_{k=0}^{n}\sum_{j=0}^{m} F_f(\frac{[k]_{q_n}}{[n]_{q_n}}\alpha_n, \frac{[j]_{q_m}}{[m]_{q_m}}\beta_m) \times\Omega_{k,n,q_n}(\frac{x}{\alpha_n})\Omega_{j,m,q_m}(\frac{y}{\beta_m}) - F_f(x,y)\right|$$

and therefore we get

$$\sup_{x,y\in D_{\alpha_n\beta_m}}\frac{\left|\tilde{C}_{n,m}^{q_n,q_m}(f;x,y) - f(x,y)\right|}{1+x^2+y^2} = \sup_{x,y\in D_{\alpha_n\beta_m}}\frac{\left|\tilde{B}_{n,m}^{q_n,q_m}(F_f;x,y) - F_f(x,y)\right|}{\psi(x,y)}.$$

Also, $F_f(x,y)$ is a continuous function on IR_+^2 satisfying $|F_f(x,y)| \leq M_f\rho(x,y)$, $x,y \geq 0$, since we have the inequality $|f(x,y)|\,\psi(x,y) \leq M_f$ for f. Therefore, by Theorem 7 we obtain the desired result. □

Example 2. *For* $q_n = 1-\frac{1}{\sqrt{n}}$, $\alpha_n = \ln(n), \beta_m = \ln(m)$ *and* $\psi(x,y) = (1+xy)^3$ *the convergence of* $\tilde{C}_{n,m}^{q_n,q_m}(f;x,y)$*(green) to* $f(x,y) = \frac{\sin\pi(x+y)}{(1+xy)^3}$*(red) is illustrated in Figure 2.*

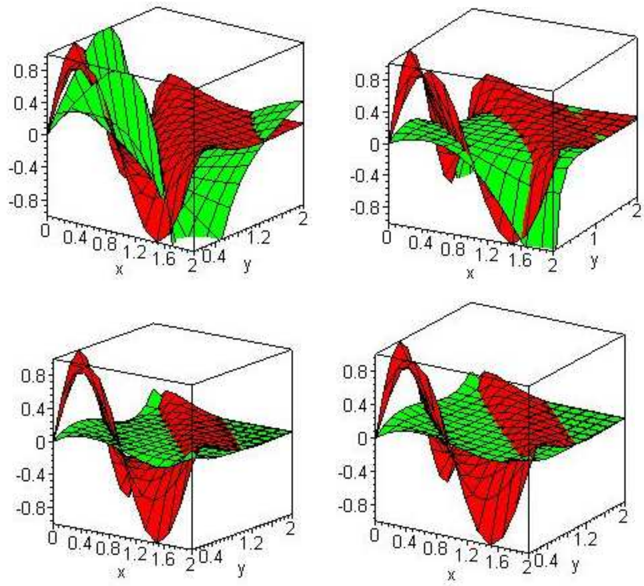

Figure 2. *Convergence of the generalized two-dimensional q-Bernstein-Chlodowsky polynomials*

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