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Soft subalgebras and soft ideals of BCK/BCI-algebras related to fuzzy set theory

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Abstract. The notion of $\in$-soft set and q-soft set based on a fuzzy set is introduced, and characterizations for an $\in$-soft set and a q-soft set to be (idealistic) soft BCK/BCI-algebras are provided. Using the notion of $(\in, \in \vee \mathrm{q})$-fuzzy BCK/BCI subalgebras/ideals, characterizations for an $\in$-soft set and a q-soft set to be (idealistic) soft BCK/BCI-algebras are established.

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1. Introduction

Various problems in system identification involve characteristics which are essentially non-probabilistic in nature [31]. In response to this situation Zadeh [32] introduced fuzzy set theory as an alternative to probability theory. Uncertainty is an attribute of information. In order to suggest a more general framework, the approach to uncertainty is outlined by Zadeh [33]. To solve a complicated problem in economics, engineering, and environment, we cannot successfully use classical methods because of various uncertainties typical for those problems. There are three theories: theory of probability, theory of fuzzy sets, and the interval mathematics which can be considered as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties. Uncertainties cannot be handled using traditional mathematical tools but may be dealt with using a wide range of existing theories such as probability theory, theory of (intuitionistic) fuzzy sets, theory of vague sets, theory of interval mathematics, and theory of rough sets. However, all of these theories have their own difficulties which are pointed out in [28]. Maji et al. [26] and Molodtsov [28] suggested that one reason for these difficulties may be due to the inadequacy of the parametrization tool of the theory. To overcome these difficulties, Molodtsov [28] introduced the concept of a soft set as a new mathematical tool for dealing with uncertainties that is free from difficulties that have troubled usual theoretical approaches. Molodtsov pointed out several directions for the applications of soft sets.

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At present, works on soft set theory are progressing rapidly. Maji et al. [26] described the application of soft set theory to a decision making problem. Maji et al. [25] also studied several operations on the theory of soft sets. Chen et al. [2] presented a new definition of soft set parametrization reduction and compared this definition to the related concept of attributes reduction in rough set theory. The algebraic structure of set theories dealing with uncertainties has been studied by some authors. The study of structures of fuzzy sets in algebraic structures are carried out by several authors (see [1, 3, 4, 5, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23, 24, 30]). The first author discussed (α, β)-type subalgebras/ideals in BCK/BCI-algebras (see [7, 8, 9, 10]). Aktaş and Çağman [1] studied basic concepts of soft set theory and compared soft sets to fuzzy and rough sets, providing examples to clarify their differences. They also discussed the notion of soft groups. Jun [11] applied the notion of soft sets by Molodtsov to the theory of BCK/BCI-algebras and introduced the notion of soft BCK/BCI-algebras and soft subalgebras, and then derived their basic properties. Jun and Park [17] dealt with the algebraic structure of BCK/BCI-algebras by applying soft set theory. They discussed algebraic properties of soft sets in BCK/BCI-algebras and introduced the notion of soft ideals and idealistic soft BCK/BCI-algebras. They investigated relations between soft BCK/BCI-algebras and idealistic soft BCK/BCI-algebras and established the intersection, union, "AND" operation, and "OR" operation of soft ideals and idealistic soft BCK/BCI-algebras.

In this paper, we introduce the notion of an $\in$-soft set and a q-soft set based on a fuzzy set and give characterizations for an $\in$-soft set and a q-soft set to be (idealistic) soft BCK/BCI-algebras. Using the notion of $(\in, \in\vee \mathrm{q})$-fuzzy BCK/BCI subalgebras/ideals, we provide characterizations for an $\in$-soft set and a q-soft set to be (idealistic) soft BCK/BCI-algebras.

2. Preliminaries

By a BCI-algebra we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms:

$$((x*y)*(x*z))*(z*y) = 0, \tag{1}$$

$$(x*(x*y))*y = 0, \tag{2}$$

$$x*x = 0, \tag{3}$$

$$x*y = y*x = 0 \Rightarrow x = y, \tag{4}$$

for all $x, y, z \in X$. We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x*y = 0$. If a BCI-algebra X satisfies $0*x = 0$ for all $x \in X$, then we say that X is a BCK-algebra. A nonempty subset S of a BCK/BCI-algebra X is called a BCK/BCI-subalgebra of X if $x*y \in S$ for all $x, y \in S$. A nonempty subset A of a BCK/BCI-algebra X is called a BCK/BCI-ideal of X if it satisfies:

$$0 \in A, \tag{5}$$

$$x*y \in A,\ y \in A \Rightarrow x \in A, \quad \forall x, y \in X. \tag{6}$$

We refer the reader to the books [6] and [27] for further information regarding BCK/BCI-algebras. A fuzzy set μ in a set X of the form

$$\mu(y) := \begin{cases} t \in (0,1], & \text{if } y = x, \\ 0, & \text{if } y \neq x, \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t.

For a fuzzy point x_t and a fuzzy set μ in a set X, Pu and Liu [29] gave meaning to the symbol $x_t \Psi \mu$, where $\Psi \in \{\in, \mathrm{q}, \in\vee\,\mathrm{q}, \in\wedge\,\mathrm{q}\}$. To say that $x_t \in \mu$ (resp. $x_t \mathrm{q} \mu$) means that $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$), and in this case, x_t is said to belong to (resp. be quasi-coincident with) a fuzzy set μ. To say that $x_t \in\vee\,\mathrm{q}\,\mu$ (resp. $x_t \in\wedge\,\mathrm{q}\,\mu$) means that $x_t \in \mu$ or $x_t \mathrm{q} \mu$ (resp. $x_t \in \mu$ and $x_t \mathrm{q} \mu$). For all $t_1, t_2 \in [0,1]$, $\min\{t_1, t_2\}$ and $\max\{t_1, t_2\}$ will be denoted by $m(t_1, t_2)$ and $M(t_1, t_2)$, respectively. To say that $x_t \overline{\Psi} \mu$ means that $x_t \Psi \mu$ does not hold, where $\Psi \in \{\in, \mathrm{q}, \in\vee\,\mathrm{q}, \in\wedge\,\mathrm{q}\}$.

A fuzzy set μ in a BCK/BCI-algebra X is called a fuzzy BCK/BCI-subalgebra of X if it satisfies:

$$\mu(x * y) \geq m(\mu(x), \mu(y)), \quad \forall x, y \in X. \tag{7}$$

Proposition 1 (see [8]). *Let X be a BCK/BCI-algebra. A fuzzy set μ in X is a fuzzy BCK/BCI-subalgebra of X if and only if the following assertion is valid.*

$$x_t \in \mu,\ y_s \in \mu \ \Rightarrow\ (x * y)_{m(t,s)} \in \mu, \quad \forall x, y \in X, \forall t, s \in (0,1]. \tag{8}$$

A fuzzy set μ in a BCK/BCI-algebra X is called a fuzzy BCK/BCI-ideal of X if it satisfies:

$$\mu(0) \geq \mu(x), \quad \forall x \in X \tag{9}$$

$$\mu(x) \geq m(\mu(x * y), \mu(y)), \quad \forall x, y \in X. \tag{10}$$

Proposition 2 (see [7]). *Let X be a BCK/BCI-algebra. A fuzzy set μ in X is a fuzzy BCK/BCI-ideal of X if and only if the following assertions are valid.*

$$x_t \in \mu \Rightarrow 0_t \in \mu, \quad \forall x \in X, \forall t \in (0,1] \tag{11}$$

$$(x * y)_t \in \mu,\ y_s \in \mu \Rightarrow x_{m(t,s)} \in \mu, \quad \forall x, y \in X, \forall t, s \in (0,1]. \tag{12}$$

3. Soft subalgebras and soft ideals

Molodtsov [28] defined the soft set in the following way: Let U be an initial universe set and E a set of parameters. Let $\mathcal{P}(U)$ denote the power set of U and $A \subset E$.

Definition 1 (see [28]). *A pair (ϑ, A) is called a soft set over U, where ϑ is a mapping given by*

$$\vartheta : A \to \mathcal{P}(U).$$

In other words, a soft set over U is a parameterized family of subsets of the universe U. For $\varepsilon \in A$, $\vartheta(\varepsilon)$ may be considered as the set of ε-approximate elements of the soft set (ϑ, A).

Given a fuzzy set μ in X and $A \subseteq [0,1]$, we define two set-valued functions $\vartheta : A \to \mathcal{P}(X)$ and $\vartheta_{\mathrm{q}} : A \to \mathcal{P}(X)$ by

$$\vartheta(t) = \{x \in X \mid x_t \in \mu\} \tag{13}$$

$$\vartheta_{\mathrm{q}}(t) = \{x \in X \mid x_t \mathrm{q} \mu\} \tag{14}$$

for all $t \in A$, respectively. Then (ϑ, A) and $(\vartheta_{\mathrm{q}}, A)$ are soft sets over X, which are called an $\in$-soft set and a q-soft set over X, respectively.

Definition 2 (see [11]). *Let (ϑ, A) be a soft set over a BCK/BCI-algebra X. Then (ϑ, A) is called a soft BCK/BCI-algebra over X if $\vartheta(x)$ is a BCK/BCI-subalgebra of X for all $x \in A$; for our convenience, the empty set $\emptyset$ is regarded as a BCK/BCI-subalgebra of X.*

Theorem 1. *Let μ be a fuzzy set in a BCK/BCI-algebra X and let (ϑ, A) be an $\in$-soft set over X with $A = (0,1]$. Then (ϑ, A) is a soft BCK/BCI-algebra over X if and only if μ is a fuzzy BCK/BCI-subalgebra of X.*

Proof. Assume that (ϑ, A) is a soft BCK/BCI-algebra over X. If μ is not a fuzzy BCK/BCI-subalgebra of X, then there exist $a, b \in X$ such that $\mu(a*b) < m(\mu(a), \mu(b))$. Take $t \in A$ such that $\mu(a*b) < t \leq m(\mu(a), \mu(b))$. Then $a_t \in \mu$ and $b_t \in \mu$ but $(a*b)_{m(t,t)} = (a*b)_t \notin \mu$. Hence $a, b \in \vartheta(t)$, but $a*b \notin \vartheta(t)$. This is a contradiction. Therefore $\mu(x*y) \geq m(\mu(x), \mu(y))$ for all $x, y \in X$. Conversely, suppose that μ is a fuzzy BCK/BCI-subalgebra of X. Let $t \in A$ and $x, y \in \vartheta(t)$. Then $x_t \in \mu$ and $y_t \in \mu$. It follows from Proposition 1 that $(x*y)_t = (x*y)_{m(t,t)} \in \mu$ so that $x*y \in \vartheta(t)$. Thus $\vartheta(t)$ is a BCK/BCI-subalgebra of X, i.e., (ϑ, A) is a soft BCK/BCI-algebra over X. $\square$

Theorem 2. *Let μ be a fuzzy set in a BCK/BCI-algebra X and let $(\vartheta_{\mathrm{q}}, A)$ be a* q*-soft set over X with $A = (0,1]$. Then $(\vartheta_{\mathrm{q}}, A)$ is a soft BCK/BCI-algebra over X if and only if μ is a fuzzy BCK/BCI-subalgebra of X.*

Proof. Suppose that μ is a fuzzy BCK/BCI-subalgebra of X. Let $t \in A$ and $x, y \in \vartheta_{\mathrm{q}}(t)$. Then $x_t \mathrm{q} \mu$ and $y_t \mathrm{q} \mu$, i.e., $\mu(x) + t > 1$ and $\mu(y) + t > 1$. It follows from (7) that

$$\mu(x*y) + t \geq m(\mu(x), \mu(y)) + t = m(\mu(x) + t, \mu(y) + t) > 1$$

so that $(x*y)_t \mathrm{q} \mu$, i.e., $x*y \in \vartheta_{\mathrm{q}}(t)$. Hence $\vartheta_{\mathrm{q}}(t)$ is a BCK/BCI-subalgebra of X for all $t \in A$, and so $(\vartheta_{\mathrm{q}}, A)$ is a soft BCK/BCI-algebra over X. Conversely, assume that $(\vartheta_{\mathrm{q}}, A)$ is a soft BCK/BCI-algebra over X. If $\mu(a*b) < m(\mu(a), \mu(b))$ for some $a, b \in X$, then we can select $t \in A$ such that

$$\mu(a*b) + t \leq 1 < m(\mu(a), \mu(b)) + t.$$

Hence $a_t \mathrm{q} \mu$ and $b_t \mathrm{q} \mu$, but $(a*b)_t \overline{\mathrm{q}} \mu$, i.e., $a \in \vartheta_{\mathrm{q}}(t)$ and $b \in \vartheta_{\mathrm{q}}(t)$, but $a*b \notin \vartheta_{\mathrm{q}}(t)$. This is a contradiction. Therefore μ is a fuzzy BCK/BCI-subalgebra of X. $\square$

Theorem 3. *Let μ be a fuzzy set in a BCK/BCI-algebra X and let (ϑ, A) be an $\in$-soft set over X with $A = (0.5, 1]$. Then the following assertions are equivalent:*

(i) (ϑ, A) *is a soft BCK/BCI-algebra over* X.

(ii) $M(\mu(x * y), 0.5) \geq m(\mu(x), \mu(y)), \quad \forall x, y \in X.$

Proof. Assume that (ϑ, A) is a soft BCK/BCI-algebra over X. Then $\vartheta(t)$ is a BCK/BCI-subalgebra of X for all $t \in A$. If there exist $a, b \in X$ such that

$$M(\mu(a * b), 0.5) < t = m(\mu(a), \mu(b)),$$

then $t \in A$, $a_t \in \mu$ and $b_t \in \mu$ but $(a*b)_t \overline{\in} \mu$. It follows that $a, b \in \vartheta(t)$ and $a*b \notin \vartheta(t)$. This is a contradiction, and so

$$M(\mu(x * y), 0.5) \geq m(\mu(x), \mu(y))$$

for all $x, y \in X$. Conversely, suppose that (ii) is valid. Let $t \in A$ and $x, y \in \vartheta(t)$. Then $x_t \in \mu$ and $y_t \in \mu$, or equivalently, $\mu(x) \geq t$ and $\mu(y) \geq t$. Hence

$$M(\mu(x * y), 0.5) \geq m(\mu(x), \mu(y)) \geq t > 0.5,$$

and thus $\mu(x * y) \geq t$, i.e., $(x * y)_t \in \mu$. Therefore $x * y \in \vartheta(t)$ which shows that (ϑ, A) is a soft BCK/BCI-algebra over X. □

A fuzzy set μ in a BCK/BCI-algebra X is called an $(\in, \in\vee\mathrm{q})$-fuzzy BCK/BCI-subalgebra of X (see [8]) if it satisfies the following condition:

$$x_t \in \mu,\ y_s \in \mu \Rightarrow (x * y)_{m(t,s)} \in\vee\,\mathrm{q}\ \mu, \quad \forall x, y \in X\ \forall t, s \in (0, 1]. \tag{15}$$

Lemma 1 (see [8])**.** *A fuzzy set* μ *in a BCK/BCI-algebra* X *is an* $(\in, \in\vee\mathrm{q})$-*fuzzy BCK/BCI-subalgebra of* X *if and only if it satisfies:*

$$\mu(x * y) \geq m(\mu(x), \mu(y), 0.5), \quad \forall x, y \in X. \tag{16}$$

Theorem 4. *Let* μ *be a fuzzy set in a BCK/BCI-algebra* X *and let* (ϑ, A) *be an* $\in$-*soft set over* X *with* $A = (0, 0.5]$. *Then the following assertions are equivalent:*

(i) μ *is an* $(\in, \in\vee\mathrm{q})$-*fuzzy BCK/BCI-subalgebra of* X.

(ii) (ϑ, A) *is a soft BCK/BCI-algebra over* X.

Proof. Assume that μ is an $(\in, \in\vee\mathrm{q})$-fuzzy BCK/BCI-subalgebra of X. Let $t \in A$ and $x, y \in \vartheta(t)$. Then $x_t \in \mu$ and $y_t \in \mu$ or equivalently $\mu(x) \geq t$ and $\mu(y) \geq t$. It follows from Lemma 1 that

$$\mu(x * y) \geq m(\mu(x), \mu(y), 0.5) \geq m(t, 0.5) = t$$

so that $(x * y)_t \in \mu$ or equivalently $x * y \in \vartheta(t)$. Hence (ϑ, A) is a soft BCK/BCI-algebra over X. Conversely, suppose that (ii) is valid. If there exist $a, b \in X$ such that $\mu(a * b) < m(\mu(a), \mu(b), 0.5)$, then we can select $t \in (0, 1)$ such that

$$\mu(a * b) < t \leq m(\mu(a), \mu(b), 0.5).$$

Thus $t \leq 0.5$, $a_t \in \mu$ and $b_t \in \mu$, that is, $a \in \vartheta(t)$ and $b \in \vartheta(t)$. Since $\vartheta(t)$ is a BCK/BCI-subalgebra of X, it follows that $a * b \in \vartheta(t)$ for all $t \leq 0.5$ so that $(a * b)_t \in \mu$ or equivalently $\mu(a * b) \geq t$ for all $t \leq 0.5$. This is a contradiction. Hence $\mu(x * y) \geq m(\mu(x), \mu(y), 0.5)$ for all $x, y \in X$. It follows from Lemma 1 that μ is an $(\in, \in\vee\mathrm{q})$-fuzzy BCK/BCI-subalgebra of X. □

Theorem 5. *Let S be a BCK/BCI-subalgebra of a BCK/BCI-algebra X and let (ϑ, A) be a soft set over X. If $A = (0, 0.5]$, then there exists an $(\in, \in \vee \mathrm{q})$-fuzzy BCK/BCI-subalgebra μ of X such that*

$$\vartheta(t) := \{x \in X \mid x_t \in \mu\} = S, \quad \forall t \in A.$$

Proof. Let μ be a fuzzy set in X defined by

$$\mu(x) := \begin{cases} t, \text{ if } \ x \in S, \\ 0, \text{ otherwise,} \end{cases}$$

for all $x \in X$ where $t \in A$. Obviously, $\vartheta(t) = S$. Assume that

$$\mu(a * b) < m(\mu(a), \mu(b), 0.5)$$

for some $a, b \in X$. Since $|\mathrm{Im}(\mu)| = 2$, we have

$$\mu(a * b) = 0 \text{ and } m(\mu(a), \mu(b), 0.5) = t.$$

It follows that $\mu(a) = t = \mu(b)$ so that $a, b \in S$. But $\mu(a * b) = 0$, whence $a * b \notin S$. This is a contradiction, and so $\mu(x * y) \geq m(\mu(x), \mu(y), 0.5)$ for all $x, y \in X$. Using Lemma 1, we know that μ is an $(\in, \in \vee \mathrm{q})$-fuzzy BCK/BCI-subalgebra of X. □

Definition 3 (see [17])**.** *Let (ϑ, A) be a soft set over a BCK/BCI-algebra X. Then (ϑ, A) is called an idealistic soft BCK/BCI-algebra over X if $\vartheta(x)$ is a BCK/BCI-ideal of X for all $x \in A$.*

Theorem 6. *Let μ be a fuzzy set in a BCK/BCI-algebra X and let (ϑ, A) be an $\in$-soft set over X with $A = (0, 1]$. Then (ϑ, A) is an idealistic soft BCK/BCI-algebra over X if and only if μ is a fuzzy BCK/BCI-ideal of X.*

Proof. Suppose that μ is a fuzzy BCK/BCI-ideal of X and let $t \in A$. If $x \in \vartheta(t)$, then $x_t \in \mu$. It follows from (11) that $0_t \in \mu$, i.e., $0 \in \vartheta(t)$. Let $x, y \in X$ be such that $x * y \in \vartheta(t)$ and $y \in \vartheta(t)$. Then $(x * y)_t \in \mu$ and $y_t \in \mu$, which imply from (12) that $x_t = x_{m(t,t)} \in \mu$. Hence $x \in \vartheta(t)$, and thus (ϑ, A) is an idealistic soft BCK/BCI-algebra over X. Conversely, assume that (ϑ, A) is an idealistic soft BCK/BCI-algebra over X. If there exists $a \in X$ such that $\mu(0) < \mu(a)$, then we can select $t \in A$ such that $\mu(0) < t \leq \mu(a)$. Thus $0_t \notin \mu$, i.e., $0 \notin \vartheta(t)$. This is a contradiction. Thus $\mu(0) \geq \mu(x)$ for all $x \in X$. Suppose there exist $a, b \in X$ such that $\mu(a) < m(\mu(a * b), \mu(b))$. Take $s \in A$ such that $\mu(a) < s \leq m(\mu(a * b), \mu(b))$. Then $(a * b)_s \in \mu$ and $b_s \in \mu$, but $a_s \notin \mu$, that is, $a * b \in \vartheta(s)$ and $b \in \vartheta(s)$ but $a \notin \vartheta(s)$. This is a contradiction, and so

$$\mu(x) \geq m(\mu(x * y), \mu(y))$$

for all $x, y \in X$. Therefore μ is a fuzzy BCK/BCI-ideal of X. □

Theorem 7. *Let μ be a fuzzy set in a BCK/BCI-algebra X and let $(\vartheta_{\mathrm{q}}, A)$ be a* q*-soft set over X with $A = (0, 1]$. Then the following assertions are equivalent:*

(i) *μ is a fuzzy BCK/BCI-ideal of X.*

(ii) $(\vartheta_{\mathrm{q}}(t) \neq \emptyset \Rightarrow \vartheta_{\mathrm{q}}(t)$ is a BCK/BCI-ideal of $X)$, $\quad \forall t \in A$

Proof. Assume that μ is a fuzzy BCK/BCI-ideal of X. Let $t \in A$ be such that $\vartheta_{\mathrm{q}}(t) \neq \emptyset$. If $0 \notin \vartheta_{\mathrm{q}}(t)$, then $0_t \overline{\mathrm{q}} \mu$ and so $\mu(0) + t < 1$. It follows from (9) that $\mu(x) + t \leq \mu(0) + t < 1$ for all $x \in X$ so that $\vartheta_{\mathrm{q}}(t) = \emptyset$. This is a contradiction, and hence $0 \in \vartheta_{\mathrm{q}}(t)$. Let $x, y \in X$ be such that $x * y \in \vartheta_{\mathrm{q}}(t)$ and $y \in \vartheta_{\mathrm{q}}(t)$. Then $(x * y)_t \mathrm{q} \mu$ and $y_t \mathrm{q} \mu$, or equivalently, $\mu(x * y) + t > 1$ and $\mu(y) + t > 1$. Using (10), we have

$$\mu(x) + t \geq m(\mu(x * y), \mu(y)) + t = m(\mu(x * y) + t, \mu(y) + t) > 1,$$

and so $x_t \mathrm{q} \mu$, i.e., $x \in \vartheta_{\mathrm{q}}(t)$. Thus $\vartheta_{\mathrm{q}}(t)$ is a BCK/BCI-ideal of X.

Conversely, assume that (ii) is valid. If $\mu(0) < \mu(a)$ for some $a \in X$, then $\mu(0) + t \leq 1 < \mu(a) + t$ for some $t \in A$. Thus $a_t \mathrm{q} \mu$, and so $\vartheta_{\mathrm{q}}(t) \neq \emptyset$. Hence $0 \in \vartheta_{\mathrm{q}}(t)$, and thus $0_t \mathrm{q} \mu$, i.e., $\mu(0)+t > 1$. This is impossible, and hence $\mu(0) \geq \mu(x)$ for all $x \in X$. Suppose there exist $a, b \in X$ such that $\mu(a) < m(\mu(a * b), \mu(b))$. Then $\mu(a) + s \leq 1 < m(\mu(a * b), \mu(b)) + s$ for some $s \in A$. It follows that $(a * b)_s \mathrm{q} \mu$ and $b_s \mathrm{q} \mu$, i.e., $a * b \in \vartheta_{\mathrm{q}}(s)$ and $b \in \vartheta_{\mathrm{q}}(s)$. Since $\vartheta_{\mathrm{q}}(s)$ is a BCK/BCI-ideal of X, we get $a \in \vartheta_{\mathrm{q}}(s)$, and so $a_s \mathrm{q} \mu$ or equivalently $\mu(a) + s > 1$. This is a contradiction. Therefore $\mu(x) \geq m(\mu(x * y), \mu(y))$ for all $x, y \in X$. Hence μ is a fuzzy BCK/BCI-ideal of X. $\square$

A fuzzy set μ in a BCK/BCI-algebra X is called an $(\in, \in\vee\,\mathrm{q})$-fuzzy BCK/BCI-ideal of X (see [7]) if it satisfies the following condition:

$$x_t \in \mu \Rightarrow 0_t \in\vee\,\mathrm{q}\;\mu, \quad \forall x \in X, \forall t \in (0,1] \tag{17}$$

$$(x * y)_t \in \mu,\; y_s \in \mu \Rightarrow x_{m(t,s)} \in\vee\,\mathrm{q}\;\mu, \quad \forall x, y \in X,\; \forall t, s \in (0,1]. \tag{18}$$

Lemma 2 (see [7]). *A fuzzy set μ in a BCK/BCI-algebra X is an $(\in, \in\vee\,\mathrm{q})$-fuzzy BCK/BCI-ideal of X if and only if it satisfies the following assertions:*

(i) $\mu(0) \geq m(\mu(x), 0.5), \quad \forall x \in X$

(ii) $\mu(x) \geq m(\mu(x * y), \mu(y), 0.5), \quad \forall x, y \in X.$

Theorem 8. *Let μ be a fuzzy set in a BCK/BCI-algebra X and let (ϑ, A) be an $\in$-soft set over X with $A = (0, 0.5]$. Then the following assertions are equivalent:*

(i) *μ is an $(\in, \in\vee\,\mathrm{q})$-fuzzy BCK/BCI-ideal of X.*

(ii) *(ϑ, A) is an idealistic soft BCK/BCI-algebra over X.*

Proof. Assume that μ is an $(\in, \in\vee\,\mathrm{q})$-fuzzy BCK/BCI-ideal of X. Let $t \in A$. Using Lemma 2(i), we get $\mu(0) \geq m(\mu(x), 0.5)$ for all $x \in \vartheta(t)$. It follows that

$$\mu(0) \geq m(\mu(x), 0.5) \geq m(t, 0.5) = t, \text{ i.e., } 0_t \in \mu.$$

Hence $0 \in \vartheta(t)$. Let $x, y \in X$ be such that $x * y \in \vartheta(t)$ and $y \in \vartheta(t)$. Then $(x * y)_t \in \mu$ and $y_t \in \mu$, or equivalently, $\mu(x * y) \geq t$ and $\mu(y) \geq t$. Using Lemma 2(ii), we have

$$\mu(x) \geq m(\mu(x * y), \mu(y), 0.5) \geq m(t, 0.5) = t, \text{ i.e., } x_t \in \mu.$$

Hence $x \in \vartheta(t)$, and so (ϑ, A) is an idealistic soft BCK/BCI-algebra over X. Conversely, suppose that (ii) is valid. If there is $a \in X$ such that $\mu(0) < m(\mu(a), 0.5)$, then $\mu(0) < t \leq m(\mu(a), 0.5)$ for some $t \in A$. It follows that $0_t \overline{\in} \mu$, i.e., $0 \notin \vartheta(t)$, a contradiction. Hence $\mu(0) \geq m(\mu(x), 0.5)$ for all $x \in X$. Assume that there exist $a, b \in X$ such that

$$\mu(a) < m(\mu(a * b), \mu(b), 0.5).$$

Taking $t := \frac{1}{2}(\mu(a) + m(\mu(a * b), \mu(b), 0.5))$, we have $t \in A$ and

$$\mu(a) < t < m(\mu(a * b), \mu(b), 0.5).$$

Hence $(a * b)_t \in \mu$ and $b_t \in \mu$, but $a_t \overline{\in} \mu$. These imply that $a * b \in \vartheta(t)$ and $b \in \vartheta(t)$ but $a \notin \vartheta(t)$. This is a contradiction. Therefore

$$\mu(x) \geq m(\mu(x * y), \mu(y), 0.5)$$

for all $x, y \in X$. It follows from Lemma 2 that μ is an $(\in, \in \vee\, \mathrm{q})$-fuzzy BCK/BCI-ideal of X. □

Theorem 9. *Let μ be a fuzzy set in a BCK/BCI-algebra X and let (ϑ, A) be an $\in$-soft set over X with $A = (0.5, 1]$. Then (ϑ, A) is an idealistic soft BCK/BCI-algebra over X if and only if μ satisfies the following assertions:*

(i) $M(\mu(0), 0.5) \geq \mu(x)$, $\forall x \in X$

(ii) $M(\mu(x), 0.5) \geq m(\mu(x * y), \mu(y))$, $\forall x, y \in X$.

Proof. Assume that (ϑ, A) is an idealistic soft BCK/BCI-algebra over X. If there is an element $a \in X$ such that condition (i) is not valid, then $\mu(a) \in A$ and $a \in \vartheta(\mu(a))$. But $\mu(0) < \mu(a)$ implies $0 \notin \vartheta(\mu(a))$, a contradiction. Hence (i) is valid. Suppose that

$$M(\mu(a), 0.5) < m(\mu(a * b), \mu(b)) = t$$

for some $a, b \in X$. Then $t \in A$ and $a * b, b \in \vartheta(t)$. But $a \notin \vartheta(t)$ since $\mu(a) < t$. This is a contradiction, and so (ii) is valid. Conversely, assume that μ satisfies conditions (i) and (ii). Let $t \in A$. For any $x \in \vartheta(t)$, we have

$$M(\mu(0), 0.5) \geq \mu(x) \geq t > 0.5$$

and so $\mu(0) \geq t$, i.e., $0_t \in \mu$. Hence $0 \in \vartheta(t)$. Let $x, y \in X$ be such that $x * y \in \vartheta(t)$ and $y \in \vartheta(t)$. Then $(x * y)_t \in \mu$ and $y_t \in \mu$, or equivalently, $\mu(x * y) \geq t$ and $\mu(y) \geq t$. Hence

$$M(\mu(x), 0.5) \geq m(\mu(x * y), \mu(y)) \geq t > 0.5,$$

which implies that $\mu(x) \geq t$, i.e., $x_t \in \mu$. Thus $x \in \vartheta(t)$, and therefore (ϑ, A) is an idealistic soft BCK/BCI-algebra over X. □

4. Examples

Example 1. *Consider a BCI-algebra $X = \{0, a, b, c\}$ with the following Cayley table:*

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let μ be a fuzzy set in X defined by $\mu(0) = 0.6$, $\mu(a) = 0.7$ and $\mu(b) = \mu(c) = 0.3$. Then μ is an $(\in, \in\vee\mathrm{q})$-fuzzy BCI-subalgebra of X (see [9]). Take $A = (0, 0.5]$ and let (ϑ, A) be an $\in$-soft set over X. Then

$$\vartheta(t) = \begin{cases} X, & \text{if } t \in (0, 0.3], \\ \{0, a\}, & \text{if } t \in (0.3, 0.5], \end{cases}$$

which are BCI-subalgebras of X. Hence (ϑ, A) is a soft BCI-algebra over X.

Example 2. *Let $\mathbb{Z}$ be the set of all integers. Then $(\mathbb{Z}; *, 0)$ is a BCI-algebra where the operation $*$ is the minus operation, i.e., $x * y = x - y$ for all $x, y \in \mathbb{Z}$. Let μ be a fuzzy set in $\mathbb{Z}$ defined by*

$$\mu(x) := \begin{cases} 0, & \text{if } x \in \{2k+1 \mid k \in \mathbb{Z},\, k < 0\}, \\ 0.3, & \text{if } x \in \{2k-1 \mid k \in \mathbb{Z},\, k > 0\}, \\ 0.5, & \text{if } x \in \{2k \mid k \in \mathbb{Z}\} \setminus \{4k \mid k \in \mathbb{Z}\}, \\ 0.8, & \text{if } x \in \{4k \mid k \in \mathbb{Z}\} \setminus \{8k \mid k \in \mathbb{Z}\}, \\ 0.9, & \text{if } x \in \{8k \mid k \in \mathbb{Z},\, k < 0\}, \\ 1, & \text{if } x \in \{8k \mid k \in \mathbb{Z},\, k \geq 0\}. \end{cases}$$

Let (ϑ, A) be an $\in$-soft set over $\mathbb{Z}$ where $A = (0.3, 0.9]$. Then

$$\vartheta(t) = \begin{cases} 2\mathbb{Z}, & \text{if } t \in (0.3, 0.5], \\ 4\mathbb{Z}, & \text{if } t \in (0.5, 0.8], \\ 8\mathbb{Z}, & \text{if } t \in (0.8, 0.9], \end{cases}$$

which are BCI-subalgebras of $\mathbb{Z}$. Hence (ϑ, A) is a soft BCI-algebra over X. But μ is neither a fuzzy BCI-subalgebra nor an $(\in, \in\vee\mathrm{q})$-fuzzy BCI-subalgebra of X since

$$\mu(4 * 7) = \mu(-3) = 0 \ngeq 0.3 = m(\mu(4), \mu(7))$$

and

$$4_{0.7} \in \mu \text{ and } 7_{0.2} \in \mu, \text{ but } (4 * 7)_{m(0.7, 0.2)} \overline{\in\vee\mathrm{q}}\, \mu,$$

respectively.

Example 2 shows that there exist a set of parameters A and a fuzzy set μ in X such that

1. μ is neither a fuzzy BCI-subalgebra nor an $(\in, \in\vee\mathrm{q})$-fuzzy BCI-subalgebra of X,

2. An $\in$-soft set (ϑ, A) over X is a soft BCI-algebra over X.

Example 3. *Consider a BCK-algebra $X = \{0, a, b, c, d\}$ with the following Cayley table:*

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	b	0
c	c	a	c	0	c
d	d	d	b	d	0

Then a fuzzy set μ in X given by $\mu(0) = 0.6$, $\mu(a)=\mu(c)=0.7$ and $\mu(b)= \mu(d)= 0.2$ is an $(\in, \in\vee\mathrm{q})$-fuzzy BCI-ideal of X (see [7]). Let (ϑ, A) be an $\in$-soft set over X with $A = (0, 0.5]$. Then

$$\vartheta(t) = \begin{cases} X, & \text{if } t \in (0, 0.2], \\ \{0, a, c\}, & \text{if } t \in (0.2, 0.5], \end{cases}$$

which are BCK-ideals of X. Hence (ϑ, A) is an idealistic soft BCK-algebra over X.

5. Conclusions

Soft sets are deeply related to fuzzy sets and rough sets. Soft set theory is applied to BCK/BCI-algebras by the first author (see [11]), and applications of soft sets in ideal theory of BCK/BCI-algebras are carried out by Jun and Park [17]. In this paper, we introduced the notion of an $\in$-soft set and a q-soft set based on a fuzzy set, and gave characterizations for an $\in$-soft set and a q-soft set to be (idealistic) soft BCK/BCI-algebras. Using the notion of $(\in, \in\vee\mathrm{q})$-fuzzy BCK/BCI subalgebras/ideals, we provided characterizations for an $\in$-soft set and a q-soft set to be (idealistic) soft BCK/BCI-algebras. Based on these results, we will consider another types of fuzzy BCK/BCI subalgebras/ideals to obtain characterizations of (idealistic) soft BCK/BCI-algebras.

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