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$\bar{\lambda}$-double sequence spaces of fuzzy real numbers defined by Orlicz function

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Abstract. In this paper we define and study two concepts which arise from the notion of de la Vallée-Poussin means, namely: strongly double $\bar{\lambda}$- convergence defined by Orlicz function and $\bar{\lambda}$-statistical convergence and establish a natural characterization for the underline sequence spaces.

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1. Introduction

The concepts of fuzzy sets and fuzzy set operations were first introduced by Zadeh [24]. Subsequently, several authors have discussed various aspects of the theory and applications of fuzzy sets, such as fuzzy topological spaces, similarity relations and fuzzy ordering, fuzzy measures of fuzzy events and fuzzy mathematical programming. In [10], Nanda studied sequences of fuzzy real numbers and showed that the set of all convergent sequences of fuzzy real numbers forms a complete metric space. Nuray [12] proved the inclusion relations between the set of statistically convergent and lacunary statistically convergent sequences of fuzzy real numbers. Savas [15] introduced and discussed double convergent sequences of fuzzy real numbers and showed that the set of all double convergent sequences of fuzzy real numbers is complete. Later on sequence of fuzzy real numbers have been discussed by Savas (see [16, 17, 18, 20, 21]), Mursaleen and Basarir [9] and others.

The study of Orlicz sequence spaces was initiated with a certain specific purpose in Banach space theory. Lindenstrauss and Tzafriri [8] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space l_M contains a subspace isomorphic to $l_p (1 \leq p < \infty)$. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [7]. Orlicz spaces find a number of useful applications in the theory of nonlinear integral equations. While Orlicz sequence spaces are the generalization of l_p-spaces, the l_p-spaces find themselves enveloped in Orlicz spaces [6].

Subsequently, the notion of Orlicz function was used to define sequence spaces by Parashar and B. Choudhary [13] and other authors.

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Recently Savas [19] generalized $c(\Delta)$ and $l_\infty(\Delta)$ by using Orlicz function and also established some inclusion theorems.

In this paper, using an Orlicz function some sequence spaces of fuzzy numbers have been given.

2. Definitions and background

We begin by introducing some preliminary notations and definitions which will be used throughout and we refer readers to ([20, 9] and [23]) for more details.

Recall in [7] that an Orlicz function $M : [0,\infty) \to [0,\infty)$ is a continuous, convex, non-decreasing function defined for $x > 0$ such that $M(0) = 0$ and $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$.

A fuzzy real number X is a fuzzy set on R , i.e., a mapping $X : R \to I(= [0,1])$, associating each real number t with its grade of membership $X(t)$.

The α-cut of a fuzzy real number X is denoted by $[X]_\alpha, 0 < \alpha \le 1$, where $[X]_\alpha = \{t \in R : X(t) \ge \alpha\}$. A fuzzy real number X is said to be upper semi-continuous if for each $\varepsilon > 0$, $X^{-1}([0, a+\varepsilon))$, for all $a \in I$ is open in the usual topology of R. If there exists $t \in R$ such that $X(t) = 1$, then a fuzzy real number X is called normal.

A fuzzy real number X is said to be convex, if $X(t) \ge X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$. The class of all upper semi-continuous, normal, convex fuzzy real numbers is denoted by $R(I)$ and throughout the article by a fuzzy real number we mean that the number belongs to $R(I)$. Let $X, Y \in R(I)$ and the α-level sets be

$$[X]_\alpha = [a_1^\alpha, a_2^\alpha], [Y]_\alpha = [b_1^\alpha, b_2^\alpha], \alpha \in [0,1].$$

Then the arithmetic operations on $R(I)$ are defined as follows:

$$\begin{aligned}
(X \oplus Y)(t) &= \sup\{X(s) \wedge Y(t-s)\}, t \in R,\\
(X \ominus Y)(t) &= \sup\{X(s) \wedge Y(s-t)\}, t \in R,\\
(X \otimes Y)(t) &= \sup\{X(s) \wedge Y(\frac{t}{s})\}, t \in R,\\
(X/Y)(t) &= \sup\{X(st) \wedge Y(s)\}, t \in R.
\end{aligned}$$

The above operations can be defined in terms of α-level sets as follows:

$$\begin{aligned}
[X \oplus Y]_\alpha &= [a_1^\alpha + b_1^\alpha, a_2^\alpha + b_2^\alpha],\\
[X \ominus Y]_\alpha &= [a_1^\alpha - b_2^\alpha, a_2^\alpha - b_1^\alpha],\\
[X \otimes Y]_\alpha &= [\min_{i,j\in\{1,2\}} a_i^\alpha.b_j^\alpha, \max_{i,j\in\{1,2\}} a_i^\alpha.b_j^\alpha],\\
[X^{-1}]_\alpha &= [(a_2^\alpha)^{-1}, (a_1^\alpha)^{-1}], 0 \notin X.
\end{aligned}$$

The additive identity and multiplicative identity in $R(I)$ are denoted by $\bar{0}$ and $\bar{1}$, respectively.

Let D be the set of all closed and bounded intervals $X = [X^L, X^R]$. Then we write $X \le Y$, if and only if $X^L \le Y^L$ and $X^R \le Y^R$, and

$$\rho(X,Y) = \max\left\{|X^L - Y^L|, |X^R - Y^R|\right\}.$$

It is obvious that (D, ρ) is a complete metric space. Now we define the metric $d : R(I)\text{x}R(I) \to R$ by

$$d(X, Y) = \sup_{0 \leq \alpha \leq 1} \rho([X]_\alpha, [Y]_\alpha),$$

for $X, Y \in R(I)$.

A fuzzy double sequence is a double infinite array of fuzzy real numbers. We denote a fuzzy real-valued double sequence by (X_{mn}), where X_{mn} are fuzzy real numbers for each $m, n \in N$.
Let $w^{''}$ denote the set of all double sequences of fuzzy real numbers. We give the following definitions (see [20]) for fuzzy real-valued double sequences.

Definition 1. *A fuzzy real-valued double sequence $X = (X_{kl})$ is said to be convergent in the Pringsheim's sense or P-convergent to a fuzzy real number X_0, if for every $\varepsilon > 0$, there exists $N \in \mathcal{N}$ such that*

$$d\left(X_{kl}, X_0\right) < \epsilon \quad for \quad k, l > N,$$

and we denote it by $P - limX = X_0$. The fuzzy real number X_0 is called the Pringsheim limit of X_{kl}. More exactly, we say that a double sequence (X_{kl}) converges to a finite fuzzy real number X_0 if X_{kl} tends to X_0 as both k and l tend to ∞ independently of each another.

Let $c^2(F)$ denote the set of all fuzzy real-valued double convergent sequence of fuzzy real numbers.

Definition 2. *A fuzzy real-valued double sequence $X = (X_{kl})$ is bounded if there exists a positive number M such that $d\left(X_{kl}, \bar{0}\right) \leq M$ for all k and l,*

$$||X||_{(\infty,2)} = \sup_{k,l} d\left(X_{kl}, \bar{0}\right) < \infty.$$

We will denote the set of all bounded fuzzy real-valued double sequences by $l_\infty^{''}(F)$.

Definition 3. *Let $\lambda = (\lambda_i)$ and $\mu = (\mu_j)$ be two non-decreasing sequences of positive real numbers both of which tend to ∞ as i and j approach ∞, respectively. Also let $\lambda_{i+1} \leq \lambda_i + 1, \lambda_1 = 0$ and $\mu_{j+1} \leq \mu_j + 1, \mu_1 = 0$. A fuzzy real-valued double sequence $X = (X_{kl})$ is said to be $\bar{\lambda}$-summable, if there exists a fuzzy real number X_0 such that*

$$P - \lim_{ij} \frac{1}{\bar{\lambda}_{ij}} \sum_{k \in I_i} \sum_{l \in I_j} d\left(X_{kl}, X_0\right) = 0,$$

where $I_i = [i - \lambda_i + 1, i]$, $I_j = [j - \mu_j + 1, j]$ and $\bar{\lambda}_{ij} = \lambda_i \mu_j$.

Throughout this paper we shall denote $\lambda_i \mu_j$ by $\bar{\lambda}_{i,j}$ and $(k \in I_i, l \in I_j)$ by $(k, l) \in \bar{I}_{i,j}$.

It is quite natural to expect that some new sequence spaces by de la Vallée-poussin mean method can be defined by combining the concept of Orlicz function and $\bar{\lambda}$-method. Such a combination would be a multidimensional analogue of the definition presented by Esi in [3]. We are now ready to present multidimensional sequence spaces.

Definition 4. *Let M be an Orlicz function, $X = (X_{kl})$ a fuzzy real-valued double sequence and $p = (p_{k,l})$ any factorable double sequence of strictly positive real numbers. Let $\lambda = (\lambda_i)$ and $\mu = (\mu_j)$ be the same as above.*

$$[V_{\bar\lambda}^{''}, M, p] = \left\{ X \in w^{''} : P - \lim_{i,j} \frac{1}{\bar\lambda_{i,j}} \sum_{(k,l)\in \bar I_{i,j}} \left[M\left(\frac{d(X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}} = 0, \right.$$
$$\left. \textit{for some} \quad \rho > 0, \textit{and} \quad X_0 \in R(I) \right\},$$
$$[V_{\bar\lambda}^{''}, M, p]_0 = \left\{ X \in w^{''} : P - \lim_{i,j} \frac{1}{\bar\lambda_{i,j}} \sum_{(k,l)\in \bar I_{i,j}} \left[M\left(\frac{d(X_{k,l}, \bar 0)}{\rho} \right) \right]^{p_{k,l}} = 0, \right.$$
$$\left. \textit{for some} \quad \rho > 0 \right\},$$

and

$$[V_{\bar\lambda}^{''}, M, p]_\infty = \left\{ X \in w^{''} : \sup_{i,j} \frac{1}{\bar\lambda_{i,j}} \sum_{(k,l)\in \bar I_{i,j}} \left[M\left(\frac{d(X_{k,l}, \bar 0)}{\rho} \right) \right]^{p_{k,l}} < \infty, \right.$$
$$\left. \textit{for some} \quad \rho > 0 \right\}$$

where

$$\bar 0(t) := \begin{cases} 1, & \textit{if } t = 0, \\ 0, & \textit{otherwise.} \end{cases}$$

If we consider various assignments of M, $\bar\lambda$, and p in the above sequence spaces we obtain the following:

1. If $p_{k,l} = 1$ for all (k,l), then

$$[V_{\bar\lambda}^{''}, M, p](F) = [V_{\bar\lambda}^{''}, M](F),$$
$$[V_{\bar\lambda}^{''}, M, p]_0(F) = [V_{\bar\lambda}^{''}, M]_0(F),$$

and

$$[V_{\bar\lambda}^{''}, M, p]_\infty(F) = [V_{\bar\lambda}^{''}, M]_\infty(F),$$

which were defined as

$$[V_{\bar{\lambda}}^{''}, M](F) = \left\{ X \in w^{''} : P - \lim_{i,j} \frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j}} \left[M\left(\frac{d(X_{k,l}, X_0)}{\rho} \right) \right] = 0, \right.$$

$$\left. \text{for some} \quad \rho > 0, \text{ and } X_0 \in R(I) \right\},$$

$$[V_{\bar{\lambda}}^{''}, M]_0(F) = \left\{ X \in w^{''} : P - \lim_{i,j} \frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j}} \left[M\left(\frac{d(X_{k,l}, \bar{0})}{\rho} \right) \right] = 0, \right.$$

$$\left. \text{for some} \quad \rho > 0 \right\}$$

and

$$[V_{\bar{\lambda}}^{''}, M]_\infty(F) = \left\{ X \in w^{''} : \sup_{i,j} \frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j}} \left[M\left(\frac{d(X_{k,l}, \bar{0})}{\rho} \right) \right] < \infty, \right.$$

$$\left. \text{for some} \quad \rho > 0 \right\}.$$

2. If $\bar{\lambda}_{i,j} = ij$, then $[V_{\bar{\lambda}}^{''}, M, p](F), [V_{\bar{\lambda}}^{''}, M, p]_0(F)$ and $[V_{\bar{\lambda}}^{''}, M, p]_\infty(F)$ reduce to the following sequence spaces:

$$[C^{''}, M, p](F) = \left\{ X \in w^{''} : P - \lim_{i,j} \frac{1}{ij} \sum_{k,l=1,1}^{i,j} \left[M\left(\frac{d(X_{k,l}, X_0)}{\rho} \right) \right]^{p_{k,l}} = 0, \right.$$

$$\left. \text{for some} \quad \rho > 0, \text{ and } X_0 \in R(I) \right\},$$

$$[C^{''}, M, p]_0(F) = \left\{ X \in w^{''} : P - \lim_{i,j} \frac{1}{ij} \sum_{k,l=1,1}^{i,j} \left[M\left(\frac{d(X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} = 0, \right.$$

$$\left. \text{for some} \quad \rho > 0 \right\}$$

and

$$[C^{''}, M, p]_\infty(F) = \left\{ X \in w^{''} : \sup_{i,j} \frac{1}{ij} \sum_{k,l=1,1}^{i,j} \left[M\left(\frac{d(X_{k,l}, \bar{0})}{\rho} \right) \right]^{p_{k,l}} < \infty, \right.$$

$$\left. \text{for some} \quad \rho > 0 \right\}.$$

3. If $M(X) = X$, $\bar{\lambda}_{i,j} = ij$, and $p_{k,l} = 1$ for all (k, l), then

$$[V_{\bar{\lambda}}^{''}, M, p](F) = [C^{''}](F),$$
$$[V_{\bar{\lambda}}^{''}, M, p]_0(F) = [C^{''}]_0(F),$$

and

$$[V_{\bar{\lambda}}^{''}, M, p]_\infty(F) = [C^{''}]_\infty(F),$$

which were defined as follows:

$$[C^{''}](F)=\left\{X \in w^{''} : P - \lim_{i,j} \frac{1}{ij} \sum_{k,l=1,1}^{i,j} d(X_{k,l}, X_0) = 0, \text{ for some } X_0 \in R(I)\right\}$$

$$[C^{''}]_0(F)=\left\{X \in w^{''} : P - \lim_{i,j} \frac{1}{ij} \sum_{k,l=1,1}^{i,j} d(X_{k,l}, \bar{0}) = 0\right\}$$

and

$$[C^{''}]_\infty(F) = \left\{X \in w^{''} : \sup_{i,j} \frac{1}{ij} \sum_{k,l=1,1}^{i,j} d(X_{k,l}, \bar{0}) < \infty\right\}.$$

3. Main results

We begin the characterization of the above sequence spaces by presenting the following theorem:

Theorem 1. *Let the sequence $p_{k,l}$ be bounded, then*

$$[V_{\bar{\lambda}}^{''}, M, p]_0(F) \subset [V_{\bar{\lambda}}^{''}, M, p](F) \subset [V_{\bar{\lambda}}^{''}, M, p]_\infty(F)$$

Proof. Let X be an element of $[V_{\bar{\lambda}}^{''}, M, p](F)$. Then we have

$$\begin{aligned}\frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j}} \left[M\left(\frac{d(X_{k,l}, \bar{0})}{2\rho}\right)\right]^{p_{k,l}} &\leq \frac{C}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j}} \frac{1}{2^{p_{kl}}} \left[M\left(\frac{d(X_{k,l}, X_0)}{\rho}\right)\right]^{p_{k,l}} \\ &\quad + \frac{C}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j}} \frac{1}{2^{p_{kl}}} \left[M\left(\frac{d(X_0, \bar{0})}{\rho}\right)\right]^{p_{k,l}} \\ &\leq \frac{C}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j}} \left[M\left(\frac{d(X_{k,l}, X_0)}{\rho}\right)\right]^{p_{k,l}} \\ &\quad + C \max\left(1, \sup\left[M\left(\frac{d(X_0, \bar{0})}{\rho}\right)\right]^{H}\right),\end{aligned}$$

where sup $p_{kl} = H$ and $C = \max(1, 2^{H-1})$. Thus we have $X \in [V_{\bar{\lambda}}^{''}, M, p]_\infty(F)$. The inclusion $[V_{\bar{\lambda}}^{''}, M, p]_0(F) \subset [V_{\bar{\lambda}}^{''}, M, p](F)$ is obvious. □

Theorem 2. *If* $0 < p_{k,l} < q_{k,l}$ *and* $\frac{q_{k,l}}{p_{k,l}}$ *are bounded, then* $[V_{\bar{\lambda}}^{''}, M, p](F) \subset [V_{\bar{\lambda}}^{''}, M, q](F)$.

Proof. This can be proved by using the techniques similar to those used in Theorem 3.3. of Mursaleen and Basarir [9]. □

A real number sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} |\{k < n : |x_k - L| \geq \epsilon\}| = 0,$$

where by $k < n$ we mean that $k = 0, 1, 2, ..., n$ and the vertical bars indicate the number of elements in the enclosed set. In this case we write $st_1 - \lim x = L$ or $x_k \to L(st_1)$. Statistical convergence is a generalization of the usual notion of convergence for real valued sequences that parallels the usual theory of convergence. The idea of statistical convergence was first introduced by Fast [4]. Today, statistical convergence has become one of the most active areas of research in the field of summability theory.

Before we present new definitions and the main theorems, we shall state a few known results. The following definition was presented by Savas [16] for a single sequence of fuzzy real numbers. A sequence X is said to be λ-statistically convergent or S_λ-convergent to X_0, if for every $\epsilon > 0$

$$\lim_n \frac{1}{\lambda_n} \Big| \{k \in I_n : d(X_k, X_0) \geq \epsilon\} \Big| = 0,$$

where the vertical bars indicate the numbers of elements in the enclosed set. In this case we write $S_\lambda - \lim X = X_0$ or $X_k \to X_0(S_\lambda)$.

Let $K \subseteq \mathcal{N} \times \mathcal{N}$ be a two-dimensional set of positive integers and let $K_{m,n}$ be the numbers of (i, j) in K such that $i \leq n$ and $j \leq m$. Then the lower asymptotic density of K is defined as

$$P - \liminf_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K).$$

In the case when the sequence $\left(\frac{K_{m,n}}{mn}\right)_{m,n=1}^{\infty}$ has a limit, then we say that K has a natural density and it is defined as

$$P - \lim_{m,n} \frac{K_{m,n}}{mn} = \delta_2^*(K).$$

For example, let $K = \{(i^2, j^2) : i, j \in \mathcal{N}\}$, where $\mathcal{N}$ is the set of natural numbers. Then

$$\delta_2(K) = P - \lim_{m,n} \frac{K_{m,n}}{mn} \leq P - \lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0$$

(i.e. the set K has double natural density zero).

Recently, Savas and Mursaleen [22] introduced statistical convergence for a fuzzy real-valued double sequence of fuzzy real numbers as follows:

Definition 5. *A fuzzy real-valued double sequence $X = (X_{kl})$ of fuzzy real numbers is said to be statistically convergent to X_0 provided that for each $\epsilon > 0$*

$$P - \lim_{m,n} \frac{1}{nm} |\{(j,k); j \leq m \text{ and } k \leq n : d(X_{kl}, X_0) \geq \epsilon\}| = 0.$$

In this case we write $st_2 - \lim_{k,l} X_{k,l} = X_0$ and we denote the set of all double statistically convergent sequences of fuzzy real numbers by $st^2(F)$.

Quite recently Savas [21] defined $\bar{\lambda}$-statistical analogues of convergence for fuzzy real-valued double sequences. We now write $\bar{\lambda}$-statistical analogues for a fuzzy real-valued double sequence as follows:

Definition 6. *A fuzzy real-valued double sequence $X = (X_{kl})$ is said to be $S^{''}_{\bar{\lambda}}(F)$-convergent or $\bar{\lambda}$-statistical convergent to X_0, provided that for every $\epsilon > 0$*

$$P - \lim_{i,j} \frac{1}{\bar{\lambda}_{i,j}} \left|\{(k,l) \in \bar{I}_{i,j} : d(X_{k,l}, X_0) > \epsilon\}\right| = 0.$$

In this case we write $S^{''}_{\bar{\lambda}} - \lim X = X_0$ or $X_{k,l} \to X_0(S^{''}_{\bar{\lambda}})$ and $S^{''}_{\bar{\lambda}}(F) = \{X : \exists X_0 \in R(I), S^{''}_{\bar{\lambda}} - \lim X = X_0\}$. We now have the following theorem:

Theorem 3. *Let $\bar{\lambda} = (\lambda_{i,j})$ be the same as above, and let $0 < p < \infty$, then*

1. $X_{k,l} \to L[V^{''}_{\bar{\lambda}}]^p(F)$ *implies* $X_{k,l} \to X_0(S^{''}_{\bar{\lambda}}(F))$,
2. *if* $X \in l^{''}_{\infty}(F)$ *and* $X_{k,l} \to X_0(S^{''}_{\bar{\lambda}})(F)$, *then* $X_{k,l} \to X_0[V^{''}_{\bar{\lambda}}]^p(F)$,
3. $S^{''}_{(\bar{\lambda})}(F) \cap l^{''}_{\infty}(F) = [V^{''}_{\bar{\lambda}}]^p(F) \cap l^{''}_{\infty}(F)$,

where

$$[V^{''}_{\bar{\lambda}}](F) = \left\{ X \in w^{''} : P - \lim_{i,j} \frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l) \in \bar{I}_{i,j}} d(X_{k,l}, X_0)^p = 0, \right.$$
$$\left. \text{for some} \quad X_0 \in R(I) \right\}.$$

Proof. Omitted. □

If we let $\bar{\lambda}_{i,j} = ij$ and $p = 1$ in Theorem 3, then we have the following corollary which was proved in [22] :

Corollary 1. *It holds:*

1. $X_{k,l} \to X_0[C^{''}](F)$ *implies* $X_{k,l} \to X_0(S^{''})(F)$,
2. *If* $X \in l^{''}_{\infty}(F)$ *and* $X_{k,l} \to X_0(S^{''})(F)$, *then* $X_{k,l} \to L[C^{''}](F)$,
3. $S^{''}(F) \cap l^{''}_{\infty}(F) = [C^{''}](F) \cap l^{''}_{\infty}(F)$.

Now we have

Theorem 4. *If M is an Orlicz function and $0 < h = \inf p_k \leq p_k \leq \sup_k p_k = H < \infty$, then $[V_{\bar{\lambda}}^{''}, M, p](F) \subset S_{\bar{\lambda}}^{''}(F)$.*

Proof. Let $X \in [V_{\bar{\lambda}}^{''}, M, p]$. Then there exists $\rho > 0$ such that

$$\frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j}} \left[M\left(\frac{d(X_{k,l}, X_0)}{\rho}\right)\right]^{p_{k,l}} \to 0$$

as $(i, j) \to \infty$ in the Pringsheim sense . If $\epsilon > 0$ and let $\epsilon_1 = \frac{\epsilon}{\rho}$, then we obtain the following:

$$\begin{aligned}
\frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j}} \left[M\left(\frac{d(X_{k,l}, X_0)}{\rho}\right)\right]^{p_{k,l}} &= \frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j} d(X_{k,l},X_0)\geq\epsilon} \left[M\left(\frac{d(X_{k,l}, X_0)}{\rho}\right)\right]^{p_{k,l}} \\
&\quad + \frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j} d(X_{k,l},X_0)<\epsilon} \left[M\left(\frac{d(X_{k,l}, X_0)}{\rho}\right)\right]^{p_{k,l}} \\
&\geq \frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j} d(X_{k,l},X_0)\geq\epsilon} \left[M\left(\frac{d(X_{k,l}, X_0)}{\rho}\right)\right]^{p_{k,l}} \\
&\geq \frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j} d(X_{k,l},X_0)\geq\epsilon} [M(\epsilon_1)]^{p_{k,l}} \\
&\geq \frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j}, d(X_{k,l},X_0)\geq\epsilon} \min\{[M(\varepsilon_1)]^h, [M(\varepsilon_1)]^H\} \\
&\geq \frac{1}{\bar{\lambda}_{i,j}} \left|\{(k,l) \in \bar{I}_{i,j} : d(X_{k,l}, X_0) \geq \epsilon\}\right| \\
&\quad \times \min\{[M(\epsilon_1)]^h, [M(\epsilon_1)]^H\}.
\end{aligned}$$

Hence $X \in S_{\bar{\lambda}}^{''}(F)$. This completes the proof. □

Theorem 5. *Let M be an Orlicz function, $X = (X_k)$ a bounded sequence of fuzzy real numbers and $0 < h = \inf p_k \leq p_k \leq \sup_k p_k = H < \infty$. Then $S_{\bar{\lambda}}^{''}(F) \subset [V_{\bar{\lambda}}^{''}, M, p](F)$.*

Proof. Suppose that $X \in l_{\infty}^{''}(F)$ and $X_{k,l} \to X_0(S_{\bar{\lambda}}^{''})(F)$. Since $X \in l_{\infty}^{''}(F)$, there is a constant $K > 0$ such that $d(X_{k,l}, X_0) \leq K$ for all k, l . Given $\varepsilon > 0$ we have

$$\begin{aligned}
&\frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j}} \left[M\left(\frac{d(X_{k,l},X_0)}{\rho}\right)\right]^{p_{k,l}} \\
&\qquad = \frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j} \, d(X_{k,l},X_0)\geq \epsilon} \left[M\left(\frac{d(X_{k,l},X_0)}{\rho}\right)\right]^{p_{k,l}} \\
&\qquad\quad + \frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j} \, d(X_{k,l},X_0)< \epsilon} \left[M\left(\frac{d(X_{k,l},X_0)}{\rho}\right)\right]^{p_{k,l}} \\
&\qquad \leq \frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j} \, d(X_{k,l},X_0)\geq \epsilon} \max\left\{\left[M\left(\frac{K}{\rho}\right)\right]^{h}, \left[M\left(\frac{K}{\rho}\right)\right]^{H}\right\} \\
&\qquad\quad + \frac{1}{\bar{\lambda}_{i,j}} \sum_{(k,l)\in \bar{I}_{i,j} \, d(X_{k,l},X_0)< \epsilon} \left[M\left(\frac{\epsilon}{\rho}\right)\right]^{p_{k,l}} \\
&\qquad \leq \max\left\{[M(T)]^{h}, [M(T)]^{H}\right\} \frac{1}{\bar{\lambda}_{i,j}} \left|\{(k,l)\in \bar{I}_{i,j} : d(X_{k,l},X_0) \geq \epsilon\}\right| \\
&\qquad\quad + \max\left\{\left[M\left(\frac{\epsilon}{\rho}\right)\right]^{h}, \left[M\left(\frac{\epsilon}{\rho}\right)\right]^{H}\right\}, \frac{K}{\rho} = T.
\end{aligned}$$

Hence $X \in [V_{\bar{\lambda}}^{''}, M, p](F)$. This completes the proof. □

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