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The Knopp and statistical α-cores of sequences

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Abstract. In this paper, we give some non-trivial generalizations of the Knopp core and statistical core theorems introduced by Knopp [Math. Z. 31 (1930) 97-127] and by Fridy and Orhan [J. Math. Anal. Appl. 208 (1997) 520-527], respectively.

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1. Introduction

Let m and c be the linear spaces of complex bounded and convergent sequences $x = \{x_n\}$, respectively, endowed with the normed by $\|x\| = \sup |x_n|$. Let $A = (a_{nk})$ be an infinite matrix and we write $(Ax)_n := \sum_k a_{nk} x_k$ provided that the series converges for each $n \in \mathbb{N}$. By Ax we denote the sequence $\{(Ax)_n\}$. If $\lim Ax = \lim x$ for each $x \in c$, then we say that A is regular [1, 4, 18] and write $A \in (c, c; p)$. The Silverman-Toeplitz theorem gives the necessary and sufficient conditions for the regularity of the matrix A (see, e.g., [1]). A matrix $A = (a_{nk})$ is called normal if it is a lower semi triangular matrix with non-zero diagonal entries [4].

The concept of the core of a complex number sequence was introduced by Knopp [11]. For brevity we shall denote the Knopp core of x by $K - core\,\{x\}$. Recall that it is defined by

$$K - core\,\{x\} := \bigcap_{n=1}^{\infty} C_n(x),$$

where $C_n(x)$ is the least closed convex hull of $\{x_k\}_{k \geq n}$. The famous Knopp's core theorem (see e.g., [4, 11, 5, 14, 17, 20]) gives necessary and sufficient conditions on a matrix A so that the Knopp core of Ax is contained in the Knopp core of x; that is,

$$K - core\,\{Ax\} \subseteq K - core\,\{x\}$$

holds. Let $\mathbb{C}$ denote the set of complex numbers. Shcherbakoff [19] proved for every bounded x that

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$$K - core\,\{x\} := \bigcap_{z\in\mathbb{C}} B_x(z),$$

where

$$B_x(z) := \left\{ w \in \mathbb{C} : |w - z| \leq \limsup_k |x_k - z| \right\}.$$

He also generalized the notion of the core of a bounded complex sequence by introducing the idea of the generalized $\alpha - core$ of a bounded complex sequence x as follows:

$$K^{(\alpha)} - core\,\{x\} := \bigcap_{z\in\mathbb{C}} B_x^{\alpha}(z),$$

where

$$B_x^{\alpha}(z) := \left\{ w \in \mathbb{C} : |w - z| \leq \alpha \;\limsup_k |x_k - z|\,,\; \alpha \geq 1 \right\}.$$

Observe that the case of $\alpha = 1$ in the above definition reduces the usual Knopp core. In [16] Natarajan has proved the following theorem.

Theorem 1 (see [16])**.** *When $K = \mathbb{R}$ or $\mathbb{C}$, an infinite matrix $A = (a_{nk})$, $a_{nk} \in K$ $n, k = 0, 1, 2, ...$ is such that*

$$K - core\,\{Ax\} \subseteq K^{(\alpha)} - core\,\{x\}\,,\; \alpha \geq 1, \tag{1}$$

for any bounded sequence x if and only if A is regular and satisfies

$$\limsup_{n\to\infty} \left(\sum_{k=0}^{\infty} |a_{nk}| \right) \leq \alpha\ .$$

If $K \subseteq \mathbb{N}$, then let $K_n := \{k \in K : k \leq n\}$; and $|K_n|$ will denote the cardinality of K_n. The natural density of K is given by $\delta(K) := \lim_n n^{-1}\,|K_n|$ provided that the limit exists. In [8] a statistical cluster point of a sequence x is defined as a number γ such that for every $\varepsilon > 0$ the set $\{k \in N : |x_k - \gamma| < \varepsilon\}$ does not have density zero. In [9] the sequence x is defined to be statistically bounded if x has a bounded subsequence of density one; and the statistical core of such an x of real values is the closed interval $[st-\liminf x, st-\limsup x]$, where $st-\liminf x$ and $st-\limsup x$ are the least and greatest statistical cluster points of x, respectively (see [9, 10, 6]). It is known [9] that for a sequence x the number β is the $st - \limsup x$ if and only if for every $\varepsilon > 0$, $\delta\{k : x_k > \beta - \varepsilon\} \neq 0$ and $\delta\{k : x_k > \beta + \varepsilon\} = 0$. The dual statement for $st - \liminf x$ is as follows: The number η is the $st - \liminf x$ if and only if for every $\varepsilon > 0$, $\delta\{k : x_k < \eta + \varepsilon\} \neq 0$ and $\delta\{k : x_k < \eta - \varepsilon\} = 0$. A statistically bounded sequence x is statistically convergent if and only if $st - \limsup x = st - \liminf x$ (see [9]). We denote the all statistical convergent sequences by st. Some results on statistical convergence may be found in the papers [8, 9, 3, 7, 15].

In [10] Fridy and Orhan defined a statistical core of a complex sequence x as follows:

Definition 1 (see [10]). *Let x be a statistically bounded sequence, and let, for each $z \in C$,*

$$B_x^*(z) := \left\{ w \in \mathbb{C} : |w - z| \leq st - \limsup_k |x_k - z| \right\}.$$

Then the statistical core of x is defined by

$$st - core\,\{x\} := \bigcap_{z \in \mathbb{C}} B_x^*(z).$$

In [10] the statistical core analogs of the Knopp core theorem was obtained.

Theorem 2 (see [10]). *If the matrix A satisfies $\sup_n \sum_k |a_{nk}| < \infty$, then $K - core\,\{Ax\} \subseteq st - core\,\{x\}$ for every $x \in m$ if and only if the following conditions hold:*

(i) *A is regular and* $\lim_n \sum_{k \in E} |a_{nk}| = 0$ *whenever $\delta(E) = 0$, $E \subseteq N$,*

(ii) $\lim_n \sum_{k=1}^{\infty} |a_{nk}| = 1.$

We can generalize the notion of the statistical core of a bounded complex sequence by introducing the idea of the generalized statistical α-core of a bounded complex sequence x as

$$st^{(\alpha)} - core\,\{x\} := \bigcap_{z \in \mathbb{C}} C_x^{\alpha}(z),$$

where $C_x^{\alpha}(z) := \left\{ w \in \mathbb{C} : |w - z| \leq \alpha \; st - \limsup_k |x_k - z|\,,\, \alpha \geq 1 \right\}$. When $\alpha = 1$, $st^{(\alpha)} - core\,\{x\}$ coincides with the usual statistical core.

2. The main results

In this paper, with the help of the method used in Natarajan [16], we improve the results introduced by Fridy and Orhan [10].

Theorem 3. *If A satisfies $\|A\| := \sup_n \sum_k |a_{nk}| < \infty$, then*

$$K - core\,\{Ax\} \subseteq st^{(\alpha)} - core\,\{x\} \tag{2}$$

for every $x \in m$ if and only if the following conditions hold:

(i) *A is regular,*

(ii) $\lim_n \sum_{k \in E} |a_{nk}| = 0$ *whenever $\delta(E) = 0$, $E \subseteq \mathbb{N}$,*

(iii) $\lim_n \sum_{k=1}^{\infty} |a_{nk}| \leq \alpha$, $(\alpha \geqslant 1)$.

Proof. Necessity: Let $\|A\| < \infty$. Assume that, for $\alpha \geqslant 1$, $x \in m$, (2) holds. Then, for all $x \in m$,

$$K - core\,\{Ax\} \subseteq st^{(\alpha)} - core\,\{x\} \subseteq K^{(\alpha)} - core\,\{x\}\,.$$

Then by [19, 16], A is regular and (iii) holds. If x is statistically convergent to L, then using the idea of Fridy and Orhan [10], we show that A maps $st \cap m$ into c. Hence, by Theorem 2 of [3] and Theorem 1 of [13], we conclude that A satisfies (ii) (see also [12]).

Sufficiency: Assume that (i), (ii) and (iii) hold. Let $w \in K - core\,\{Ax\}$ and $\alpha \geq 1$. For any $z \in \mathbb{C}$, we have

$$|w - z| \leq \limsup_n \left| \sum_{k=1}^{\infty} a_{nk}(z - x_k) \right|.$$

Let $r = st - \limsup_k |x_k - z|$. Then we have $\alpha r = st - \limsup_k (\alpha\,|x_k - z|)$, $\alpha \geq 1$. Now, for given $\varepsilon > 0$, setting $E = \{k : \alpha\,|z - x_k| > \alpha r + \varepsilon\}$, $\alpha \geq 1$, we see that $\delta(E) = 0$. Then, we obtain

$$\left| \sum_{k=1}^{\infty} a_{nk}(z - x_k) \right| \leq \sup_k |z - x_k| \sum_{k \in E} |a_{nk}| + (\alpha r + \varepsilon) \sum_{k \notin E} |a_{nk}| \tag{3}$$

Now (i), (ii) and (iii) yield that

$$\limsup_n \left| \sum_{k=1}^{\infty} a_{nk}(z - x_k) \right| \leq \alpha r + \varepsilon. \tag{4}$$

By (4), we have $|w - z| \leq \alpha r + \varepsilon$. Since ε was arbitrary, we may write that $|w - z| \leq \alpha r$. Hence, we get $w \in C_x^{\alpha}(z)$, i.e.,

$$w \in st^{\alpha} - core\,\{x\}\,, \quad \alpha \geq 1.$$

The theorem is proved. □

Since $st - core$ of any sequence is a subset of the $K - core$, therefore the preceding theorem gives the following result immediately.

Corollary 1. *If the matrix A satisfies $\sup_n \sum_{k=1}^{\infty} |a_{nk}| < \infty$ and properties (ii) and (iii) of Theorem 3, then*

$$st - core\,\{Ax\} \subseteq st^{(\alpha)} - core\,\{x\}\,.$$

Before giving some further results, we ... restate a lemma due to Choudhary [2] that we need for our purposes.

Lemma 1 (see [2]). *Let n be fixed. In order to define* $(Ax)_n$*, whenever* Bx *bounded, it is necessary and sufficient that*

(*i*) $c_{nk} = \sum_{v=k}^{\infty} a_{nv} b_{vk}^{-1}$ *exists for all* k,

(*ii*) $\sum_{k=0}^{\infty} |c_{nk}| < \infty$ *for all* n,

(*iii*) $\sum_{k=0}^{j} \left| \sum_{v=j+1}^{\infty} a_{nv} b_{vk}^{-1} \right| \to 0$ *as* $j \to \infty$

should hold for the n considered. If these conditions are satisfied, then for bounded Bx,

$$(Ax)_n = (Cy)_n, \tag{5}$$

where $y := Bx$.

Whenever B is normal, B has a reciprocal. Denote its reciprocal by $B^{-1} = (b_{nk}^{-1})$. Note that if B is a normal matrix, then the space $m_B := \{x : Bx \in m\}$ is isometrically isomorphic to m. Hence given a sequence $y \in m_B$, then there exists a unique sequence $x \in m_B$ so that $y := Bx$.

Our next result extends Theorem 2 of Fridy and Orhan [10] to $st^{(\alpha)} - core$.

Theorem 4. *Let* $B = (b_{nk})$ *be a normal matrix and* A *any matrix. In order to have that, whenever* Bx *is bounded,* Ax *should exist and be bounded and*

$$K - core\,\{Ax\} \subseteq st^{(\alpha)} - core\,\{Bx\}\,,\ \alpha \geq 1 \tag{6}$$

it is necessary and sufficient that

(*i*) $C = AB^{-1}$ *exists,*

(*ii*) C *is regular and* $\lim_n \sum_{k \in E} |c_{nk}| = 0$*, for every* $E \subseteq \mathbb{N}$ *with* $\delta(E) = 0$,

(*iii*) $\limsup_{n\to\infty} \left(\sum_{k=0}^{\infty} |c_{nk}| \right) \leq \alpha$,

(*iv*) *for any fixed* n, $\sum_{k=0}^{j} \left| \sum_{v=j+1}^{\infty} a_{nv} b_{vk}^{-1} \right| \to 0 \quad (j \to \infty)$.

Proof. Necessity: Assume that (6) holds. Write $y := Bx$. Let $(Ax)_n$ exist for each n whenever y is bounded. Then, by Lemma 1, (*i*) and (*iv*) hold. Moreover, for every bounded y, we have (5). Hence, by (6) we get $K - core\,\{Cy\} \subseteq st^{(\alpha)} - core\,\{y\}$, $\alpha \geq 1$, for every bounded y. Now it follows from Theorem 3 that (*ii*) and (*iii*) hold.

Sufficiency: Observe that conditions (*i*) - (*iv*) imply the conditions of Lemma 1. So (5) holds and Cy is bounded whenever $y \in m$. Now from Theorem 3, (*ii*) and (*iii*) imply that $K - core\,\{Cy\} \subseteq st^{(\alpha)} - core\,\{y\}$, $\alpha \geq 1$, provided that y is bounded. Writing $y = Bx$ we immediately get (6), hence the result. □

Since $st - core\,\{Ax\} \subseteq K - core\,\{Ax\}$, we get the following result at once.

Corollary 2. *If A and B satisfy conditions (i)-(iv) of Theorem 4, then*

$$st-core\,\{Ax\} \subseteq st^{(\alpha)}-core\,\{Bx\}\,,\ \alpha \geq 1$$

for every x such that $Bx \in m$.

Recall that the matrix A is called row-finite if every row contains only a finite number of non-zero elements. In this case (iii) of Theorem 4 is zero for sufficiently large j. Hence, (iii) is evidently satisfied. In this case Theorem 4 reduces to the following

Theorem 5. *Let $B = (b_{nk})$ be a normal matrix. Then, for a row-finite matrix A such that $\|A\| < \infty$,*

$$K-core\,\{Ax\} \subseteq st^{(\alpha)}-core\,\{Bx\}\,,\ \alpha \geq 1,\ \text{for all } x \in m_B$$

if and only if (i) and (iii) of Theorem 4 hold.

If we interchange the roles of the matrices A and B in Theorem 4, we immediately get the following

Theorem 6. *Let $B = (b_{nk})$ and $A = (a_{nk})$ be normal matrices. Then we have, for all $x \in m_B \cap m_A$, that*

$$K-core\,\{Ax\} = st^{(\alpha)}-core\,\{Bx\}$$

if and only if

(i) *$C = AB^{-1}$ and $D = BA^{-1}$ exist,*

(ii) *C and D are regular, i.e.*

$$\lim_n \sum_{k\in E} |c_{nk}| = 0 \ \text{ and } \ \lim_n \sum_{k\in E} |d_{nk}| = 0$$

for every $E \subseteq \mathbb{N}$ with $\delta(E) = 0$,

(iii) $\limsup\limits_{n\to\infty} \left(\sum\limits_{k=0}^{\infty} |c_{nk}| \right) \leq a$ *and* $\limsup\limits_{n\to\infty} \left(\sum\limits_{k=0}^{\infty} |d_{nk}| \right) \leq \alpha$,

(iv) *for any fixed n*

$$\sum_{k=0}^{j} \left| \sum_{v=j+1}^{\infty} a_{nv} b_{vk}^{-1} \right| \to 0 \ \text{ as } \ j \to \infty,$$

and

$$\sum_{k=0}^{j} \left| \sum_{v=j+1}^{\infty} b_{nv} a_{vk}^{-1} \right| \to 0 \ \text{ as } \ j \to \infty.$$

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