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V_n-slant helices in Euclidean n-space E^n

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Abstract. In this paper, we give a definition of harmonic curvature functions in terms of V_n and we define a new kind of a slant helix. We call this new slant helix a V_n-slant helix in n-dimensional Euclidean space E^n and define it by using new harmonic curvature functions. We also define a vector field D which we call a Darboux vector field of a V_n-slant helix in n-dimensional Euclidean space E^n and we give a new characterization as:

$$\text{“}\alpha : I \subset \mathbb{R} \longrightarrow E^n \text{ is a } V_n\text{-slant helix } \Leftrightarrow H^{*'}_{n-2} - k_1 H^*_{n-3} = 0\text{”},$$

where H^*_{n-2}, H^*_{n-3} are harmonic curvature functions and k_1 shows the first curvature function of the curve α.
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1. Introduction

Harmonic curvature functions were defined earlier by Özdamar and Hacısalihoğlu [9]. In [9], the authors generalize inclined curves in E^3 to E^n and then give a characterization for the inclined curves in E^n : “If a curve α is an inclined curve, then $\sum_{i=1}^{n-2} H_i^2 = constant$”. Recently, many studies have been reported on generalized helices and inclined curves [1], [4], [9].

The definition given in [4] by Hayden is restrictive: The fixed direction makes a constant angle with all the vectors of the Frenet frame. This definition only works in the odd dimensional case. Moreover, in the same reference it is proved that the definition is equivalent to the fact that the ratios $\frac{k_{n-1}}{k_{n-2}}, \frac{k_{n-3}}{k_{n-4}}, ..., \frac{k_2}{k_1}$ of the curvatures, are constants. This statement is related with the Lancret Theorem for generalized helices in $\mathbb{E}^3$ (the ratio of torsion to curvature is constant).

More recently, Izumiya and Takeuchi defined a new kind of helix (slant helix) and gave a characterization of slant helices in Euclidean 3−space E^3 [5]. Followingly Kula and Yaylı investigated spherical images; the tangent indicatrix and binormal indicatrix of a slant helix [6]. Morever, they gave a characterization for slant helices

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in E^3: "For involute of a curve γ, γ is a slant helix if and only if its involute is a general helix". In [7], the curves in $\mathbb{E}^n$ for which all the ratios $\frac{k_{n-1}}{k_{n-2}}, \frac{k_{n-3}}{k_{n-4}}, ..., \frac{k_2}{k_1}$ are constants were called *ccr* curves. In the same reference it is shown that in the even dimensional case a curve has constant curvature ratios if and only if its tangent indicatrix is a geodesic in the flat torus. In 2008, Önder et al. defined a new kind of a slant helix in Euclidean 4−space E^4 which they called a B_2−slant helix and gave characterizations of this slant helix in Euclidean 4−space E^4 [8].

In this study we define a new kind of a slant helix in Euclidean n−space E^n, where we use the constant angle φ in between a fixed direction X and the nth Frenet vector field V_n of the curve, that is,

$$\langle V_n, X\rangle = \cos\varphi,\ \varphi \neq \frac{\pi}{2},\ \varphi = constant.$$

Since the n-th Frenet vector field V_n of a curve makes a constant angle with a fixed direction X, we call this curve a V_n−slant helix in Euclidean n−space E^n. Firstly, we give a generalization of Hacısalihoğlu's harmonic curvature functions [9]. Next, we define a new Darboux vector field D and give new characterizations for V_n−slant helices. Finally, we show that Önder et al. study [8] is a special case for $n = 4$ of our study, and that one of the theorems in their paper is not correct, and we present a counter-example.

2. Preliminaries

Let $\alpha : I \subset \mathbb{R} \longrightarrow E^n$ be an arbitrary curve in E^n. Recall that the curve α is said to be a unit speed curve (or parameterized by arclength functions) if $\langle\alpha'(s), \alpha'(s)\rangle = 1$, where $\langle .,.\rangle$ denotes the standard inner product of $\mathbb{R}^n$ given by

$$\langle X, Y\rangle = \sum_{i=1}^{n} x_i y_i$$

for each $X = (x_1, x_{2,...,}x_n), Y = (y_1, y_{2,...,}y_n) \in \mathbb{R}^n$. In particular, the norm of a vector $X \in \mathbb{R}^n$ is given by $\|X\| = \sqrt{\langle X, X\rangle}$. Let $\{V_1, V_2, ..., V_n\}$ be the moving Frenet frame along the unit speed curve α, where V_i $(i = 1, 2, ..., n)$ denotes the ith Frenet vector field. Then Frenet formulas are given by

$$\begin{bmatrix} V_1' \\ V_2' \\ V_3' \\ \vdots \\ V_{n-2}' \\ V_{n-1}' \\ V_n' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \dots & 0 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \dots & 0 & 0 & 0 \\ 0 & -k_2 & 0 & 0 \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \dots & 0 & k_{n-2} & 0 \\ 0 & 0 & 0 & 0 \dots & -k_{n-2} & 0 & k_{n-1} \\ 0 & 0 & 0 & 0 \dots & 0 & -k_{n-1} & 0 \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_{n-2} \\ V_{n-1} \\ V_n \end{bmatrix}$$

where $k_i(i = 1, 2, ..., n)$ denotes the ith curvature function of the curve [2], [3]. If all of the curvatures k_i $(i = 1, 2, ..., n)$ of the curve vanish nowhere in $I \subset \mathbb{R}$, the curve is called a non-degenerate curve.

Definition 1. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^n$ be a curve in E^n with arc-length parameter s, and let X be a unit constant vector of E^n. For all $s \in I$, if*

$$\langle \alpha'(s), X\rangle = \cos\varphi,\ \varphi \neq \frac{\pi}{2},\ \varphi = constant,$$

then the curve α is called a general helix or inclined curve in E^n; where $\alpha'(s)$ is the unit tangent vector of α at its point $\alpha(s)$, and φ is a constant angle between the vectors α' and X [9].

3. V_n-slant helix and its harmonic curvature functions

In this section we give characterizations for a V_n-slant helix in Euclidean $n-$space E^n by using harmonic curvature functions in terms of V_n of the curve.

Definition 2. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^n$ be a unit speed curve with nonzero curvatures $k_i (i = 1, 2, ..., n)$ in E^n and let $\{V_1, V_2, ..., V_n\}$ denote the Frenet frame of the curve α. We call α a V_n-slant helix, if the nth unit vector field V_n makes a constant angle φ with a fixed direction X, that is,*

$$\langle V_n, X\rangle = \cos\varphi,\ \varphi \neq \frac{\pi}{2},\ \varphi = constant$$

along the curve, where X is unit vector field in E^n.

Definition 3. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^n$ be a unit speed curve with nonzero curvatures $k_i (i = 1, 2, ..., n)$ in E^n. Harmonic curvature functions in terms of V_n of α are defined by $H_i^* : I \subset \mathbb{R} \longrightarrow \mathbb{R}$,*

$$H_i^* = \begin{cases} 0, & i = 0, \\ \frac{k_{n-1}}{k_{n-2}}, & i = 1, \\ \left\{ k_{n-i} H_{i-2}^* - H_{i-1}^{*'} \right\} \frac{1}{k_{n-(i+1)}}, & i = 2, 3, ..., n-2. \end{cases} \tag{1}$$

Proposition 1. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^n$ be an arc-lengthed curve in E^n, $\{H_1^*, H_2^*, \ldots, H_{n-2}^*\}$ the harmonic curvature functions of the curve α and $\left\{H_1^{*'}, H_2^{*'}, \ldots, H_{n-2}^{*'}\right\}$ the differentiation of $\left\{H_1^*, H_2^*, \ldots, H_{n-2}^*\right\}$, then we may write*

$$\begin{bmatrix} H_1^{*'} \\ H_2^{*'} \\ H_3^{*'} \\ \vdots \\ H_{n-4}^{*'} \\ H_{n-3}^{*'} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -k_{n-3} & 0 & 0 & \ldots & 0 & 0 & 0 \\ k_{n-3} & 0 & -k_{n-4} & 0 & \ldots & 0 & 0 & 0 \\ 0 & k_{n-4} & 0 & -k_{n-5} & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & -k_2 & 0 \\ 0 & 0 & 0 & 0 & \ldots & k_2 & 0 & -k_1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} H_1^* \\ H_2^* \\ H_3^* \\ \vdots \\ H_{n-4}^* \\ H_{n-3}^* \\ H_{n-2}^* \end{bmatrix}.$$

Proof. It is obvious from Definition 3. $\square$

Proposition 2. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^n$ be an arc-lengthed parameter curve in E^n and X a unit constant vector field of $\mathbb{R}^n$. $\{V_1, V_2, ..., V_n\}$ denote the Frenet frame of the curve α and $\left\{H_1^*, H_2^*, ..., H_{n-2}^*\right\}$ denote the harmonic curvature functions of the curve α. If $\alpha : I \subset \mathbb{R} \longrightarrow E^n$ is a V_n−slant helix with X as its axis, then we have for all $i = 0, 1, ..., n-2$*

$$\left\langle V_{n-(i+1)}, X\right\rangle = H_i^* \left\langle V_n, X\right\rangle . \tag{2}$$

Proof. We apply the induction method for the proof .
The case of $i = 1$:

Since X is a unit constant vector field such that $\langle V_n(s), X\rangle = \cos\varphi$, for all $s \in I$, then differentiating this, with respect to s, we obtain $\left\langle V_n^{'}(s), X\right\rangle = 0$ or from Serret Frenet formulas $\langle -k_{n-1}V_{n-1}, X\rangle = 0$, where $k_{n-1} \neq 0$, then

$$\langle V_{n-1}, X\rangle = 0. \tag{3}$$

Again, differentiating (3), with respect to s, and by using the Serret-Frenet equations we have

$$\begin{aligned} \left\langle V_{n-1}^{'}, X\right\rangle &= 0, \\ \langle -k_{n-2}V_{n-2} + k_{n-1}V_n, X\rangle &= 0, \\ -k_{n-2}\langle V_{n-2}, X\rangle + k_{n-1}\langle V_n, X\rangle &= 0 \end{aligned}$$

and so (1) gives us

$$\begin{aligned} \langle V_{n-2}, X\rangle &= \frac{k_{n-1}}{k_{n-2}}\langle V_n, X\rangle , \\ \langle V_{n-2}, X\rangle &= H_1^* \langle V_n, X\rangle . \end{aligned}$$

The case of $i = 2$:

Differentiating the last equation of $i = 1$ with respect to s, we have

$$\begin{aligned} \left\langle V_{n-2}^{'}, X\right\rangle &= H_1^{*'}\langle V_n, X\rangle \\ \langle -k_{n-3}V_{n-3} + k_{n-2}V_{n-1}, X\rangle &= H_1^{*'}\langle V_n, X\rangle \\ -k_{n-3}\langle V_{n-3}, X\rangle + k_{n-2}\langle V_{n-1}, X\rangle &= H_1^{*'}\langle V_n, X\rangle \end{aligned}$$

and by using (3) and Proposition 1, we have $\langle V_{n-3}, X\rangle = H_2^* \langle V_n, X\rangle$. Let us assume that Proposition 2 is true for the case $i-1$. This means that

$$\langle V_{n-i}, X\rangle = H_{i-1}^* \langle V_n, X\rangle . \tag{4}$$

Differentiating (4) with respect to s, we have $\left\langle V_{n-i}^{'}, X\right\rangle = H_{i-1}^{*'}\langle V_n, X\rangle$ or from Serret Frenet formulas

$$\begin{aligned} \langle -k_{n-i-1}V_{n-i-1} + k_{n-i}V_{n-i+1}, X\rangle &= H_{i-1}^{*'}\langle V_n, X\rangle , \qquad (5) \\ -k_{n-i-1}\langle V_{n-i-1}, X\rangle + k_{n-i}\langle V_{n-i+1}, X\rangle &= H_{i-1}^{*'}\langle V_n, X\rangle . \end{aligned}$$

Let us assume that Proposition (2) is true for the case $i-2$. This means that

$$\langle V_{n-i+1}, X\rangle = H_{i-2}^{*}\langle V_n, X\rangle .$$

If we substitute this in (5) we have

$$\left\langle V_{n-(i+1)}, X\right\rangle = \frac{1}{k_{n-(i+1)}}\left\{k_{n-i}H_{i-2}^{*} - H_{i-1}^{*'}\right\}\langle V_n, X\rangle$$

and then using (1) we obtain $\left\langle V_{n-(i+1)}, X\right\rangle = H_i^{*}\langle V_n, X\rangle$ which completes the proof. □

Corollary 1. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^n$ be an arc-lengthed parameter curve in E^n and X a unit constant vector field of $\mathbb{R}^n$. $\{V_1, V_2, ..., V_n\}$ denote the Frenet frame of the curve α and $\{H_1^*, H_2^*, ..., H_{n-2}^*\}$ denote the harmonic curvature functions of the curve α. If the axis of a V_n-slant helix α is X, then we write*

$$X = \left\{H_{n-2}^{*}V_1 + H_{n-3}^{*}V_2 + \cdots + H_1^{*}V_{n-2} + V_n\right\}\langle V_n, X\rangle$$

or

$$X = \left\{H_{n-2}^{*}V_1 + H_{n-3}^{*}V_2 + \cdots + H_1^{*}V_{n-2} + V_n\right\}\cos\varphi.$$

Proof. If the axis of a V_n-slant helix α in E^n is X, then we can write $X = \sum\limits_{i=1}^{n}\lambda_i V_i$, Then by using Proposition 2

$$\begin{aligned}
\lambda_1 &= \langle V_1, X\rangle = H_{n-2}^{*}\langle V_n, X\rangle ,\\
\lambda_2 &= \langle V_2, X\rangle = H_{n-3}^{*}\langle V_n, X\rangle ,\\
&\vdots\\
\lambda_{n-2} &= \langle V_{n-2}, X\rangle = H_1^{*}\langle V_n, X\rangle ,\\
\lambda_{n-1} &= 0,\\
\lambda_n &= \langle V_n, X\rangle .
\end{aligned}$$

Thus it is easy to obtain $X = \left\{H_{n-2}^{*}V_1 + H_{n-3}^{*}V_2 + \cdots + H_1^{*}V_{n-2} + V_n\right\}\langle V_n, X\rangle .$ □

Definition 4. *Let α be a unit speed curve in E^n. $\{V_1, V_2, ..., V_n\}$ denote the Frenet frame of the curve and $\{H_1^*, H_2^*, ..., H_{n-2}^*\}$ denote the harmonic curvature functions. The vector*

$$D = H_{n-2}^{*}V_1 + H_{n-3}^{*}V_2 + \cdots + H_1^{*}V_{n-2} + V_n$$

is called a Darboux vector of the V_n-slant helix α.

Theorem 1. *Let α be a unit speed curve in E^n. $\{V_1, V_2, ..., V_n\}$ denote the Frenet frame of the curve and $\{H_1^*, H_2^*, ..., H_{n-2}^*\}$ denote the harmonic curvature functions. Then α is a V_n-slant helix if and only if D is a constant vector field.*

Proof. Let α be a V_n-slant helix in E^n and X the axis of α. From Corollary 1, we have

$$X = \left\{H_{n-2}^* V_1 + H_{n-3}^* V_2 + \cdots + H_1^* V_{n-2} + V_n\right\} \cos\varphi,$$

where $\cos\varphi$ is a constant and hence D is a constant vector field.

Conversely, if D is a constant vector field, then we have

$$\begin{aligned} \langle D, V_n\rangle &= 1, \\ \|D\| \, \|V_n\| \cos\varphi &= 1, \\ \|D\| \cos\varphi &= 1. \end{aligned}$$

Thus we get $\cos\varphi = 1/\|D\|$, where φ is a constant angle between D and V_n. In this case we can define a unique axis of the V_n-slant helix as $X = \cos\varphi D$, where $\langle X, V_n\rangle = 1/\|D\| = \cos\varphi$. Thus X is a constant vector and α is a V_n-slant helix. This completes the proof. □

Theorem 2. *Let α be a unit speed curve in E^n. $\{V_1, V_2, ..., V_n\}$ denote the Frenet frame of the curve and $\left\{H_1^*, H_2^*, ..., H_{n-2}^*\right\}$ denote the harmonic curvature functions. Then α is a V_n-slant helix if and only if*

$$H_{n-2}^{*'} - k_1 H_{n-3}^* = 0. \tag{6}$$

Proof. If we differentiate D along the curve α, we get

$$D' = H_{n-2}^{*'} V_1 + H_{n-2}^* V_1^{'} + H_{n-3}^{*'} V_2 + H_{n-3}^* V_2^{'} + ... + H_1^{*'} V_{n-2} + H_1^* V_{n-2}^{'} + V_n^{'}.$$

The Serret-Frenet formulas and Proposition 1 give

$$D' = \left\{H_{n-2}^{*'} - k_1 H_{n-3}^*\right\} V_1. \tag{7}$$

Since α is a V_n-slant helix, D is a constant vector field. Thus we can write $D' = 0$ or $H_{n-2}^{*'} - k_1 H_{n-3}^* = 0$.

Conversely, if (6) is zero, that is, $H_{n-2}^{*'} - k_1 H_{n-3}^* = 0$, we can easily see that $D' = 0$ or D is a constant vector field, and then from Theorem 1 we have that α is a V_n-slant helix in E^n. This completes the proof. □

Corollary 2. *Let α be a unit speed curve in E^3. $\{T, N, B\}$ denote the Frenet frame and $\{k_1, k_2\}$ nonzero curvature functions of the curve. Then α is a $B-$slant helix if and only if α is a generalized helix.*

Proof. Let α be a $B-$slant helix in E^3. Then from Theorem 2 we have $H_1^{*'} - k_1 H_0^* = 0$. By using Definition 3

$$\left(\frac{k_2}{k_1}\right)' = 0, \tag{8}$$

and then $\frac{k_2}{k_1} = constant$, and thus $\frac{k_1}{k_2} = constant$, so α is a generalized helix.

Conversely, let α be a generalized helix. So, $\frac{k_1}{k_2}$ and $\frac{k_2}{k_1} = constant$. Thus, $\left(\frac{k_2}{k_1}\right)' = 0$ or $H_1^{*'} - k_1 H_0^* = 0$. Then from Theorem 2 α is a $B-$slant helix. This completes the proof. □

Corollary 3. *Let α be a unit speed curve in E^4. $\{V_1, V_2, V_3, V_4\}$ denote the Frenet frame and $\{k_1, k_2, k_3\}$ curvature functions of the curve. Then α is a V_4-slant helix (B_2-slant helix) if and only if*

$$\left[\frac{1}{k_1}\left(\frac{k_3}{k_2}\right)'\right]' + k_1\frac{k_3}{k_2} = 0. \tag{9}$$

Proof. Let α be a V_4-slant helix in E^4. Then from Theorem 2 we have $H_2^{*'} - k_1H_1^* = 0$. By using Definition 3 we have

$$\left[\frac{1}{k_1}\left(\frac{k_3}{k_2}\right)'\right]' + k_1\frac{k_3}{k_2} = 0.$$

Conversely, we assume that the equation $\left[\frac{1}{k_1}\left(\frac{k_3}{k_2}\right)'\right]' + k_1\frac{k_3}{k_2} = 0$ holds. Then from Theorem 2 and Definition 3 we easily obtain that α is a V_4-slant helix. □

Remark 1. *Önder et al.* [8] *gave the following characterization for a B_2-slant helix (V_4-slant helix) by using curvature functions of the curve.*

Theorem 3. *A unit speed curve $\alpha : I \subset \mathbb{R} \longrightarrow E^4$ with nonzero curvature functions $k_1(s)$, $k_2(s)$, $k_3(s)$ is a B_2-slant helix (V_4-slant helix) if and only if the following condition is satisfied,*

$$\left(\frac{k_3}{k_2}\right)^2 + \frac{1}{k_1^2}\left(\left(\frac{k_3}{k_2}\right)'\right)^2 = \tan^2\varphi_3 = constant,$$

where φ_3 is the constant angle between the second binormal unit vector field B_2 and a constant unit vector U.

The above theorem is true for the necessity case but not true for the sufficiency case, because if we differentiate the formula in Theorem 3, we have

$$\left(\frac{k_3}{k_2}\right)'\left\{k_1\frac{k_3}{k_2} + \left[\frac{1}{k_1}\left(\frac{k_3}{k_2}\right)'\right]'\right\} = 0.$$

Therefore, Theorem 3 is true only if $\frac{k_3}{k_2}$ is not a constant. We can give the following Example 1 which demonstrates this fact.

Example 1. $\alpha(s) = \begin{pmatrix} a\cos\left(\frac{r}{\sqrt{a^2r^2+b^2}}s\right), a\sin\left(\frac{r}{\sqrt{a^2r^2+b^2}}s\right), \\ b\cos\left(\frac{1}{\sqrt{a^2r^2+b^2}}s\right), b\sin\left(\frac{1}{\sqrt{a^2r^2+b^2}}s\right) \end{pmatrix}$ *is a unit speed curve in E^4. If we denote*

$$\frac{1}{\sqrt{a^2r^2+b^2}} = m \;\; and \;\; \frac{1}{\sqrt{a^2r^4+b^2}} = n,$$

it is easy to obtain the Frenet vectors and curvature functions as follows:

$$\begin{aligned}
V_1 &= T = (-amr\sin(mrs), amr\cos(mrs), -bm\sin(ms), bm\cos(ms)),\\
V_2 &= N = (-anr^2\cos(mrs), -anr^2\sin(mrs), -bn\cos(ms), -bn\sin(ms)),\\
V_3 &= B_1 = (bm\sin(mrs), -bm\cos(mrs), -amr\sin(ms), amr\cos(ms)),\\
V_4 &= B_2 = (bn\cos(mrs), bn\sin(mrs), -anr^2\cos(ms), -anr^2\sin(ms)).
\end{aligned}$$

Let

$$\begin{aligned}
k_1 &= \frac{m^2}{n},\\
k_2 &= m^2nabr(r^2-1),\\
k_3 &= nr.
\end{aligned}$$

The Darboux vector field of the above curve is

$$D = -\left[\frac{1}{k_1}\left(\frac{k_3}{k_2}\right)'\right]V_1 + \frac{k_3}{k_2}V_2 + V_4.$$

Since $\left(\frac{k_3}{k_2}\right)' = 0$, we can write that $D = \frac{k_3}{k_2}V_2 + V_4$. Thus we obtain that $D' \neq 0$. Although $\left(\frac{k_3}{k_2}\right)^2 + \frac{1}{k_1^2}\left(\left(\frac{k_3}{k_2}\right)'\right)^2 = constant$, D is not a constant vector field. According to Theorem 1, α is not a V_4-slant helix (B_2-slant helix.)

Instead of Önder et al's Theorem 3, we can give the following one in E^n, $n \geq 3$.

Theorem 4. *Let α be a unit speed non-degenerate curve in E^n with Frenet vector fields $\{V_1, V_2, ..., V_n\}$ and the harmonic curvature functions $\{H_1^*, H_2^*, ..., H_{n-2}^*\}$. If α is a V_n-slant helix, then*

$$\sum_{i=1}^{n-2} {H_i^*}^2 = constant.$$

Proof. Let $\alpha : I \subset \mathbb{R} \longrightarrow E^n$ be a V_n-slant helix. By using Corollary 1, since the axis of α is a unit vector field, we have

$$\left\{{H_1^*}^2 + {H_2^*}^2 + \cdots + {H_{n-3}^*}^2 + {H_{n-2}^*}^2\right\}\cos^2\varphi + \cos^2\varphi = 1. \tag{10}$$

Thus we get

$$\sum_{i=1}^{n-2} {H_i^*}^2 = \frac{1-\cos^2\varphi}{\cos^2\varphi} = \tan^2\varphi \ = constant,$$

which completes the proof. □

In this case we can give the following corollary in E^4 :

Corollary 4. *If a unit speed curve $\alpha : I \subset \mathbb{R} \longrightarrow E^4$ with nonzero curvatures $k_1(s)$, $k_2(s)$, $k_3(s)$ is a V_4-slant helix (B_2-slant helix), then we have*

$$\left(\frac{k_3}{k_2}\right)^2 + \frac{1}{k_1^2}\left(\left(\frac{k_3}{k_2}\right)'\right)^2 = \tan^2\varphi = constant, \tag{11}$$

where φ is the constant angle between the second binormal unit vector field V_4 and the constant unit vector field X.

Proof. It is obvious from Theorem 4 for $n = 4$. □

Theorem 5. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^{2m+1}$ be a unit speed curve in E^{2m+1} and $\left\{H_1^*, H_2^*, ..., H_{2m-1}^*\right\}$ the harmonic curvature functions. If the ratios*

$$\frac{k_2}{k_1}, \frac{k_4}{k_3}, \frac{k_6}{k_5}, \ldots, \frac{k_{2m-2}}{k_{2m-3}}, \frac{k_{2m}}{k_{2m-1}}$$

are constant, then we have $H_{2i}^* = 0$, for $1 \leq i \leq m$ and

$$H_{2i-1}^* = \frac{k_{2m}}{k_{2m-1}}.\frac{k_{2m-2}}{k_{2m-3}}.\frac{k_{2m-4}}{k_{2m-5}}...\frac{k_{2m-(2i-3)}}{k_{2m+1-(2i-2)}}\frac{k_{2m+1-(2i-1)}}{k_{2m+1-2i}}.$$

Proof. We apply the induction method for the proof .
The case of $i = 1$:
From Definition 3 we may write

$$H_2^* = \left\{k_{2m-1}H_0^* - H_1^{*'}\right\}\frac{1}{k_{2m-2}},$$

or

$$H_2^* = -\frac{1}{k_{2m-2}}\left(\frac{k_{2m}}{k_{2m-1}}\right)',$$

where $\frac{k_{2m}}{k_{2m-1}} = constant$, so $H_2^* = 0$ and again Definition 3 gives us

$$H_3^* = \left\{k_{2m-2}H_1^* - H_2^{*'}\right\}\frac{1}{k_{2m-3}}.$$

By using $H_2^* = 0$ and Definition 3 we can write

$$H_3^* = \frac{k_{2m-2}}{k_{2m-3}}\frac{k_{2m}}{k_{2m-1}}.$$

Let us assume that Theorem 5 is true for the case $i = p$, then we may write

$$H_{2p}^* = 0$$

and

$$H_{2p-1}^* = \frac{k_{2m}}{k_{2m-1}}.\frac{k_{2m-2}}{k_{2m-3}}.\frac{k_{2m-4}}{k_{2m-5}}...\frac{k_{2m-(2p-3)}}{k_{2m+1-(2p-2)}}\frac{k_{2m+1-(2p-1)}}{k_{2m+1-2p}}.$$

Definition 3 gives $H_{2p+1}^{*}=\left\{k_{2m-2p}H_{2p-1}^{*}-H_{2p}^{*'}\right\}\frac{1}{k_{2m-2p-1}}$. By using $H_{2p}^{*}=0$ and Definition 3, we can write

$$H_{2p+1}^{*}=\frac{k_{2m}}{k_{2m-1}}.\frac{k_{2m-2}}{k_{2m-3}}.\frac{k_{2m-4}}{k_{2m-5}}...\frac{k_{2m+1-(2p-1)}}{k_{2m+1-2p}}.\frac{k_{2m+1-(2p+1)}}{k_{2m+1-(2p+2)}},$$

which completes the proof. □

Here we can give the following results:

Corollary 5. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^{2m+1}$ be a unit speed curve in E^{2m+1} and $\{k_1, k_2, ..., k_{2m}\}$ denote the curvature functions of the curve. If the ratios $\frac{k_2}{k_1}, \frac{k_4}{k_3}, \frac{k_6}{k_5}$ $\frac{k_{2m-2}}{k_{2m-3}}, \frac{k_{2m}}{k_{2m-1}}$ are constants, then the axis of a $V_{2m+1}-$slant helix α is*

$$D=H_{2m-1}^{*}V_1+H_{2m-3}^{*}V_3+\cdots+H_1^{*}V_{2m-1}+V_{2m+1}.$$

Proof. According to Definition 4, for $n=2m+1$ we have

$$D=H_{2m-1}^{*}V_1+H_{2m-2}^{*}V_2+\cdots+H_1^{*}V_{2m-1}+V_{2m+1}$$

where from Theorem 5 we get

$$D=H_{2m-1}^{*}V_1+H_{2m-3}^{*}V_3+\cdots+H_1^{*}V_{2m-1}+V_{2m+1}.$$

This completes the proof. □

Corollary 6. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^{2m+1}$ be a unit speed curve in E^{2m+1} and $\{k_1, k_2, ..., k_{2m}\}$ denote the curvature functions of the curve. If the ratios $\frac{k_2}{k_1}, \frac{k_4}{k_3}, \frac{k_6}{k_5}$ $\frac{k_{2m-2}}{k_{2m-3}}, \frac{k_{2m}}{k_{2m-1}}$ are constants, then α is a $V_{2m+1}-$slant helix.*

Proof. Let the ratios $\frac{k_2}{k_1}, \frac{k_4}{k_3}, \frac{k_6}{k_5}$ $,\ldots,\frac{k_{2m-2}}{k_{2m-3}}, \frac{k_{2m}}{k_{2m-1}}$ be constant. Then from Theorem 5 H_{2i-1}^{*} is constant for $1 \leq i \leq m$. In this case, according to Corollary 5, D is a constant vector field, then Theorem 1 give us that α is a $V_{2m+1}-$slant helix. This completes the proof. □

4. Geometrical means of the Darboux vector of the V_n-slant helix

Lemma 1 (see [10]). *The Darboux axis at the time s is determined by the kernel of the Frenet matrix $M_3(s)$ given with respect to the basis T, N, B,*

$$M_3(s)=\begin{bmatrix} 0 & k_1 & 0 \\ -k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \in \mathbb{R}_3^3.$$

Theorem 6. *Let α be a unit speed non-degenerate curve in $E^n(n=odd)$ and $\{k_1, k_2, ..., k_{n-1}\}$ denote the curvature functions of the curve, then the Frenet matrix $M_n(s)$ is given by*

$$M_n(s) = \begin{bmatrix} 0 & k_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -k_2 & 0 & k_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & k_{n-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & -k_{n-2} & 0 & k_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -k_{n-1} & 0 \end{bmatrix} \in \mathbb{R}^n_n.$$

Then α is a V_n-slant helix if and only if the vector $D = \left[H^*_{n-2}, H^*_{n-1}, ..., H^*_1, H^*_0, 1\right] \in \mathbb{R}^n$ satisfies the Frenet equations:

$$\frac{d}{ds}\left[H^*_{n-2}, H^*_{n-1}, ..., H^*_1, H^*_0, 1\right]^T = M_n(s)\left[H^*_{n-2}, H^*_{n-1}, ..., H^*_1, H^*_0, 1\right]^T. \qquad (12)$$

Proof. Direct substitution shows that

$$M_n(s)D^T = \begin{bmatrix} 0 & k_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -k_2 & 0 & k_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & k_{n-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & -k_{n-2} & 0 & k_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -k_{n-1} & 0 \end{bmatrix} \cdot \begin{bmatrix} H^*_{n-2} \\ H^*_{n-3} \\ H^*_{n-4} \\ \vdots \\ H^*_1 \\ H^*_0 \\ 1 \end{bmatrix}$$

$$= \left[k_1 H^*_{n-3}, H^{*'}_{n-3}, H^{*'}_{n-4}, ..., H^{*'}_1, H^{*'}_0, 1\right]^T$$

This equality can be written as:

$$M_n(s)D^T = \frac{d}{ds}\left[H^*_{n-2}, H^*_{n-1}, ..., H^*_1, H^*_0, 1\right]^T - (H^{*'}_{n-2} - k_1 H^*_{n-3})\left[1, 0, 0, ..., 0, 0, 0\right]^T.$$

Since α is a V_n-slant helix, Theorem 2 gives

$$\frac{d}{ds}\left[H^*_{n-2}, H^*_{n-1}, ..., H^*_1, H^*_0, 1\right]^T = M_n(s)\left[H^*_{n-2}, H^*_{n-1}, ..., H^*_1, H^*_0, 1\right]^T.$$

This completes the proof. □

Now, we can ask the question: When does the Darboux vector field D belong to the kernel of $M_n(s)$?

Case I: The question in $\mathbb{R}^3$ is that $\frac{d}{ds}\left[H^*_1, H^*_0, 1\right]^T = \frac{d}{ds}\left[\frac{k_2}{k_1}, 0, 1\right]^T = 0$, so the Darboux vector field D lies in the kernel of $M_3(s)$.

Case II: The question in $\mathbb{R}^4$ is that by using Example 1 we show that any ccr-curve is not a V_4-slant helix. So, the Darboux vector field D cannot be the kernel of $M_4(s)$.

Case III: When the curve satisfies that some quotients $\frac{k_i}{k_{i-1}}$ are constant for all $i = 2, 3, ..., n-1$ in $\mathbb{R}^n (n = \text{odd})$, then we have again that

$$\frac{d}{ds}\left[H^*_{n-2}, H^*_{n-1}, ..., H^*_1, H^*_0, 1\right]^T = 0.$$

Then, the Darboux vector field D belongs to the kernel of $M_n(s)$.

Case IV: The question for $n = \text{even}$ in $\mathbb{R}^n$, then $M_n(s)$ is a regular matrix and only zero vector is in the kernel of $M_n(s)$. Since the Darboux vector field D is not zero, it cannot be the kernel of $M_n(s)$.

Proposition 3 (see [10])**.** *The Darboux axis at the time s is determined by the kernel of the Frenet matrix $M_{2m+1}(s)$ given with respect to the basis $V_1, V_2, ..., V_{2m+1}$ in $\mathbb{R}^{2m+1}$, $m > 2$. Then the Darboux vector can be given as*

$$D = a_0 V_1 + a_1 V_3 + ... + a_m V_{2m+1},$$

where $a_0 = k_2 k_4 ... k_{2m}, a_1 = \frac{k_1}{k_2} a_0, a_2 = \frac{k_3}{k_4} a_1, \ldots a_i = \frac{k_{2i-1}}{k_{2i}} a_{i-1}, \ldots a_m = \frac{k_{2m-1}}{k_m} a_{m-1}$ $= k_1 k_3 ... k_{2m-1}$.

Definition 5. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^{2m+1}$ be a unit speed non-degenerate curve in E^{2m+1}. Let us assume that $\{k_1, k_2, ..., k_{2m}\}$ are the curvature functions of the curve α. If the ratios $\frac{k_2}{k_1}, \frac{k_4}{k_3}, \frac{k_6}{k_5}, ..., \frac{k_{2m-2}}{k_{2m-3}}, \frac{k_{2m}}{k_{2m-1}}$ are constants, then the curve α is called a $V_{2m+1}-$slant helix in the sense of Hayden .*

Proposition 4 (see [10])**.** *The Darboux vector $d = a_0 V_1 + a_1 V_3 + ... + a_m V_{2m+1}$ lies in the kernel of the Frenet matrix $M_{2m+1}(s)$ in E^{2m+1}, $m > 2$.*

Proposition 5. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^{2m+1}$ be a unit speed curve in E^{2m+1}. The Darboux vector D of the curve α lies in the kernel of $M_{2m+1}(s)$ if and only if the curve α is a $V_{2m+1}-$slant helix in the sense of Hayden.*

Proof. In this case we have $D = \frac{1}{a_m} d$, where $a_m = k_1 k_3 ... k_{2m-1}$ and since the Darboux vector D of the curve α lies in the kernel of $M_{2m+1}(s)$, then from (12) and Theorem 2 we easily obtain that the curve α is a $V_{2m+1}-$slant helix in the sense of Hayden.

Conversely, let us assume that the curve α is a $V_{2m+1}-$slant helix in the sense of Hayden, then from Definition 5 and Eq.(12) we can easily show that the Darboux vector D of the curve α lies in the kernel of $M_{2m+1}(s)$. This completes the proof. □

Corollary 7. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E^{2m+1}$ be a unit speed curve in E^{2m+1}. The curve α is a $V_{2m+1}-$slant helix in the sense of Hayden if and only if the curve α is a generalized helix in the sense of Hayden.*

Proof. Let α be a $V_{2m+1}-$slant helix in the sense of Hayden. According to Definition 5, the ratios $\frac{k_2}{k_1}, \frac{k_4}{k_3}, \frac{k_6}{k_5}$ $\frac{k_{2m-2}}{k_{2m-3}}, \frac{k_{2m}}{k_{2m-1}}$ are constants, hence the ratios $\frac{k_1}{k_2}, \frac{k_3}{k_4}, \frac{k_5}{k_6}$ $\frac{k_{2m-3}}{k_{2m-2}}, \frac{k_{2m-1}}{k_{2m}}$ are constants. Then, from [4] α is a generalized helix in the sense of Hayden.

A converse case is obvious. This completes the proof. □

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