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# On exponential mixing bounds and convergence rate for reciprocal Gamma diffusion processes

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**Abstract.** For a parametric class of "reciprocal gamma diffusion processes", certain exponential bounds for $\beta$-mixing and rate of convergence to stationary distribution are established.

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## 1. Introduction

In our recent paper [1] we considered a parametric class of 1D diffusion processes called Student diffusions. The reason for studying this class was a demand from stochastic financial applications; this study was initiated in [5]. The class of reciprocal gamma diffusions is another particular class which is interesting from the same point of view, [8]. Namely, in applied stochastic finance theory there is a need to have a description of parametric classes of processes with certain special properties, in particular, with heavy or light tails, "short" or "long" memory, and exponential or some other rate of mixing and convergence towards stationary distributions. For applications it is also highly desirable that the latter are known exactly. Hence, in this paper we investigate certain mixing properties and convergence rate to equilibrium distribution for this new particular class suggested in [8]. It turns out that the processes from this class possess polynomial tails and exponential mixing.

Although the model under consideration is rather specific, we apply the theory developed for general possibly non-symmetric and non-stationary processes. Notice that for strictly stationary symmetric processes there are other methods to study some other mixing coefficients, see, e.g., [4, section 2] about alpha-mixing; in particular, cf. examples in the end of that section. Our model here is symmetric; however, we establish an exponential decay of stronger and non-stationary beta-mixing, so there are various reasons why formally our results cannot be derived directly from [4].

The organisation of the paper is as follows. Section 2 relates to the definition of the gamma and reciprocal gamma density distribution.

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Section 3 is devoted to the presentation of the parametric class of ergodic stationary reciprocal gamma diffusions with a marginal inverse gamma distribution. Section 4 presents main results, section 5 contains auxiliary lemmae. Main results are proved briefly in section 6.

## 2. Gamma and reciprocal gamma distribution

Let random variable $Y$ have a gamma distribution with probability density function of the form

$$\mathfrak{g}(x) = \begin{cases} \dfrac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1} e^{-\alpha x}, & x > 0, \\ 0, & x \le 0, \end{cases} \tag{1}$$

where $\alpha > 0$ is a scale parameter and $\beta > 1$ is a shape parameter. Then random variable $X = \dfrac{1}{Y}$ has a reciprocal gamma distribution with probability density function,

$$\mathfrak{rg}(x) = \begin{cases} \dfrac{\alpha^\beta}{\Gamma(\beta)} x^{-\beta-1} e^{-\alpha/x}, & x > 0, \\ 0, & x \le 0, \end{cases} \tag{2}$$

with the same parameters $\alpha$ and $\beta$. In literature those distributions are denoted as $Y \sim \mathfrak{G}(\alpha, \beta)$ and $X \sim \mathfrak{RG}(\alpha, \beta)$. The moments of $k$-th order of a reciprocal gamma random variable are given by the following expression

$$E[X^k] = \frac{\alpha^k}{\prod_{j=1}^k (\beta - j)} = \alpha^k \frac{\Gamma(\beta - k)}{\Gamma(\beta)}, \qquad k < \beta. \tag{3}$$

In particular, expectation and variance of random variable $X \sim \mathfrak{RG}(\alpha, \beta)$ read

$$E[X] = \frac{\alpha}{\beta - 1}, \quad Var(X) = \frac{\alpha^2}{(\beta-1)^2(\beta-2)}. \tag{4}$$

An important feature of a reciprocal gamma distribution is a scaling property, i.e., if $\mathfrak{G}(\alpha, \beta)$ and $\mathfrak{RG}(\alpha, \beta)$ are gamma and reciprocal gamma random variables, respectively, with the same parameters $\alpha$ and $\beta$, then – a bit abusing notations – it is known that

$$\mathfrak{RG}(\alpha, \beta) = \frac{1}{\mathfrak{G}(\alpha, \beta)} = \frac{\alpha}{\mathfrak{G}(1, \beta)} = \alpha\, \mathfrak{RG}(1, \beta). \tag{5}$$

For references see [8].

## 3. Reciprocal gamma diffusion

Consider a stochastic differential diffusion equation

$$dX_t = -\theta(X_t - \frac{\alpha}{\beta - 1})\, dt + \sqrt{\frac{2\theta}{\beta - 1} X_t^2}\, dW_t, \quad t \ge 0, \tag{6}$$

with some initial data $X_0$. Here $\theta > 0$, $\alpha > 0$, $\beta > 1$, and $(W_t,\, t \ge 0)$ is a standard Brownian motion (BM). Equivalently,

$$X_t = X_0 - \theta \int_0^t (X_s - \frac{\alpha}{\beta - 1})\, ds + \int_0^t \sqrt{\frac{2\theta}{\beta-1} X_s^2}\, dW_s, \quad t \ge 0,$$

where $X_0$ is a random variable independent of Brownian motion $W_t$; in particular a non-random value is, of course, allowed. Due to a global Lipschitz condition on both coefficients, the stochastic differential equation above has a unique strong solution which is a strong Markov process. Moreover, solution $X = \{X_t, t \ge 0\}$ is ergodic with invariant reciprocal gamma probability density function (2), see [8]. Notice that this invariant density, of course, does not depend on $\theta > 0$.

## 4. Main results

Let $F^X_{\le s} = \sigma(X_u,\, u \le s)$. Notations $E_x$ and $P_x$ ($E_{st}$ and $P_{st}$) are used for the processes with initial data $x$ (stationary distribution as initial data, independent of the BM). We recollect the definition of two mixing coefficients, $\alpha(t)$ and $\beta(t)$:
Strong mixing coefficient or the Rosenblatt coefficient

$$\alpha^x(t) = \sup_{s\ge 0} \sup_{A \in F^X_{\le s}, B\in F^X_{\ge t+s}} |P_x(AB) - P_x(A)P(B)|;$$

Complete regularity condition or the Kolmogorov coefficient

$$\beta^x(t) = \sup_{s\ge 0} E_x \operatorname*{var}_{B\in F_{\ge t+s}X} (P(B|F_s) - P(B)).$$

Denote $\alpha^{st}(t)$ and $\beta^{st}(t)$ the versions of both coefficients for stationary distributed initial data $X_0$, respectively. Denote by $\mu^x(t)$ the distribution of $X_t$ with initial data $x$, and by $\mu^{st}$ the invariant measure for $X$.

**Theorem 1.** *For any $\beta > 1$ and $m < \beta$, there exist constants $C, c, \lambda > 0$ such that*

$$\beta^x(t) \le K(x) e^{-ct}, \quad K(x) = C\left(\frac{1}{|x|^\lambda} + |x|^m\right), \quad t \ge 0, \tag{7}$$

*and for stationary regime,*

$$\beta^{st}(t) \le Ce^{-ct}, \quad t \ge 0. \tag{8}$$

**Theorem 2.** *Under the assumptions of Theorem 1,*

$$var(\mu^x(t) - \mu^{st}) \le K(x)\, e^{-ct}, \quad t \ge 0. \tag{9}$$

Notice that the bounds obtained above allow us to apply the Central Limit Theorem for functionals of the process, see e.g., [6]. We do not pursue this goal here, see e.g., [1, Corollary 1] and [10]. In particular, the latter references contain hints as to how apply the method from [6] to non-stationary processes.

## 5. Preliminary results

For any $R > 1$, let $\tau_1 := \inf(t \geq 0 : \ X_t \leq R)$, $\tau_2 := \inf(t \geq 0 : \ X_t \geq R^{-1})$.

**Lemma 1.** *For any $\alpha, \theta > 0$, $\beta > 1$ and any $m < \beta$, there exists a constant $\alpha_1 > 0$ such that for any $R$ large enough and $x > R$,*

$$E_x \exp(\alpha_1 \tau_1) \leq |x|^m.$$

**Proof**. Consider a Lyapunov function $f(t,x) = \exp(\alpha_1 t) x^m$. Then, the stochastic differentiation of this function along the trajectory $X$ reads

$$\begin{aligned} df(t,X_t) &= \alpha_1 e^{\alpha_1 t} X_t^m dt + m X_t^{m-1} e^{\alpha_1 t} (dX_t) + \frac{1}{2}\, m\,(m-1) X_t^{m-2} e^{\alpha_1 t} (dX_t)^2 \\ &= \alpha_1 e^{\alpha_1 t} X_t^m dt + \frac{1}{2} e^{\alpha_1 t} \frac{2\theta}{\beta - 1}\, m\,(m-1) X_t^m dt - \theta m X_t^{m-1} e^{\alpha_1 t} X_t dt \\ &\quad + \frac{\alpha\theta m}{\beta-1} X_t^{m-1} e^{\alpha_1 t} dt + \sqrt{\frac{2\theta}{\beta-1} X_t^2}\, dW_t \\ &= e^{\alpha_1 t} X_t^m \left[ \alpha_1 + \frac{\theta}{\beta-1}\, m\,(m-1) + \frac{\alpha\theta m}{\beta-1} X_t^{-1} - \theta m \right] dt \\ &\quad + \sqrt{\frac{2\theta}{\beta-1} X_t^2}\, dW_t. \end{aligned}$$

Due to the assumption we have $m\theta - \frac{m(m-1)\theta}{\beta-1} > 0$. Let $0 < \alpha_1 < m\theta - \frac{m(m-1)\theta}{\beta-1}$. Then, for $R$ large enough,

$$\left[ \alpha_1 + \frac{\theta}{\beta-1}\, m\,(m-1) + \frac{\alpha\theta m}{\beta-1} X_t^{-1} - \theta m \right] < c \leq 0.$$

Hence, taking expectations, we obtain,

$$E_x f(t \wedge \tau_1, X_{t\wedge\tau_1}) - f(0,x) \leq c E_x \int_0^{t\wedge\tau_1} e^{\alpha_1 s} X_s^m ds \leq 0.$$

Due to the Fatou Lemma, as $t \to \infty$, Lemma 1 follows. $\square$

Notice that to avoid any question about expectations of the stochastic integral above, as usual, we can apply a standard localization procedure.

**Lemma 2.** *For any $\alpha, \theta > 0$, $\beta > 1$ and any $\lambda > 0$, there exists a constant $\alpha_2 > 0$, such that for any $R$ large enough and $x < \dfrac{1}{R}$,*

$$E_x \exp(\alpha_2 \tau_2) \leq \frac{1}{x^\lambda}.$$

**Proof**. Here the initial condition is very near to the origin. Notice that our diffusion can never reach zero if it starts from positive values. Hence, it is convenient to transform the space and consider another process $Y_t := \lambda \ln X_t \equiv \ln X_t^{\lambda}$. Its stochastic differential reads

$$\frac{1}{\lambda}\, dY_t = d(\ln X_t) = \frac{dX_t}{X_t} - \frac{1}{2}\frac{dX_t^2}{X_t^2} = \left(-(\theta + \frac{\theta}{\beta-1}) + \frac{\alpha\theta}{\beta-1}\exp(-Y_t)\right) dt + \sqrt{\frac{2\theta}{\beta-1}} dW_t = \left[-\frac{\theta\beta}{\beta-1} + \frac{\alpha\theta}{\beta-1}\exp(-Y_t)\right] dt + \sqrt{\frac{2\theta}{\beta-1}} dW_t.$$

Let us consider a Lyapunov function $f(t,y) = \exp(-y + \alpha_2 t)$. We have

$$\begin{aligned} df(t,Y_t) &= \alpha_2 f(t,Y_t)dt \,-\, f(t,Y_t)dY_t \,+\, \frac{1}{2} f(t,Y_t)(dY_t)^2 \\ &= \alpha_2 f(t,Y_t)dt \,-\, f(t,Y_t)\lambda\left[-\frac{\theta\beta}{\beta-1} + \frac{\alpha\theta}{\beta-1}\exp(-Y_t)\right] dt \\ &\quad - f(t,Y_t)\lambda\left[\frac{2\theta}{\beta-1}\right]^{1/2} dW_t + \frac{1}{2} f(t,Y_t)\,\frac{2\lambda^2\theta}{\beta-1} dt \\ &= f(t,Y_t)\left[\alpha_2 + \frac{\lambda^2\theta}{\beta-1} + \frac{\lambda\theta\beta}{\beta-1} - \frac{\lambda\alpha\theta}{\beta-1}\exp(-Y_t)\right] dt \\ &\quad - \lambda f(t,Y_t)\sqrt{\frac{2\theta}{\beta-1}} dW_t. \end{aligned}$$

For $R$ large enough and $t < \tau_2$, we have

$$\left[\alpha_2 + \frac{\theta(\lambda\beta + \lambda^2)}{\beta-1} - \frac{\lambda\alpha\theta}{\beta-1}\exp(-Y_t)\right] \le \kappa \le 0.$$

Hence, taking expectation, we get $Ef(t\wedge\tau_2, Y_{t\wedge\tau_2}) - f(Y_0, 0) \le 0$. So, using the Fatou Lemma as $t \to \infty$, we get $E_x \exp(\alpha_2\tau_2) \le \frac{1}{x^{\lambda}}$. Lemma 2 is proved. □

**Lemma 3.** *For any $\alpha, \theta > 0$, $\beta > 1$, any $\lambda > 0$ and any $m < \beta$,*

$$E_{st} X^m + E_{st}\frac{1}{X^{\lambda}} < \infty.$$

**Proof**. This is straightforward due to the convergence,

$$\int_0^{\infty} x^m x^{-\beta-1} e^{-\frac{\alpha}{x}} dx + \int_0^{\infty} \frac{1}{x^{\lambda}} x^{-\beta-1} e^{-\frac{\alpha}{x}} dx < \infty.$$

□

In what follows, we shall consider recurrence properties of a couple of independent Markov processes satisfying the same equation (6). Consider a direct product of two identical probability spaces where two independent copies of our Markov process are defined, say $(Z_t,\, t \geq 0)$ and $(Z'_t,\, t \geq 0)$, with the initial values $Z_0 = z$, $Z'_0 = z'$, respectively. Define a new function $\phi_R(z) \in C^2$ as follows:

$$\phi_R(z) = \begin{cases} z^m, & \text{if } z \geq R, \\ \text{any } C^2 \text{ function}, & \text{if } \dfrac{1}{R} \leq z \leq R, \\ \lambda \ln z, & \text{if } z \leq \dfrac{1}{R}. \end{cases}$$

For $R_1 \geq R$, define a stopping time $\gamma$,

$$\gamma := \inf(t \geq 0 : Z_t \in [\frac{1}{R_1}, R_1] \,\&\, Z'_t \in [\frac{1}{R_1}, R_1]).$$

**Lemma 4.** *For any $\alpha, \theta > 0$, any $\lambda > 0$ and any $m < \beta$, there exist $C, \alpha_3 > 0$ such that for $R_1 (\geq R)$ large enough and for any $z, z' > 0$ as initial starting points for $Z$ and $Z'$,*

$$E_{z,z'} \exp(\alpha_3 \gamma) \leq C \left(\phi_{R_1}(z) + \phi_{R_1}(z')\right).$$

**Proof**. Consider a Lyapunov function with $\alpha_3 > 0$,

$$f(t, \phi(z), \phi(z')) \,=\, \exp(\alpha_3\, t)(\phi(z) + \phi(z')).$$

Due to Itô's formula, we have

$$\begin{aligned} df(t, \phi(Z_t), \phi(Z'_t)) = {} & \alpha_3 \exp(\alpha_3 t)(\phi(Z_t) + \phi(Z'_t))dt + \exp(\alpha_3 t) L\phi(Z_t)dt \\ & + \exp(\alpha_3 t) L(\phi(Z'_t))dt + \sqrt{\frac{2\theta}{\beta - 1} Z_t^2}\, \phi'(Z_t) dW_t \\ & + \sqrt{\frac{2\theta}{\beta - 1} Z_t^2}\, \phi'(Z'_t) dW'_t. \end{aligned}$$

Notice that

$$\sup_{\frac{1}{R} \leq z \leq R} L\phi_R(z) \,\vee\, 0 := C^* < \infty,$$

and also

$$\sup_{z \notin [\frac{1}{R}, R]} L\phi_R(z) \leq 0.$$

Define $S_+ := \{z : L\phi_R(z) > 0\}$. Then in the integration $\int_0^{t\wedge\gamma} (\ldots)\, ds$, for every $s$, there are two main cases to be considered:

I: At time $s$, one process is either $(\geq R_1)$ or $(\leq 1/R_1)$, while the other process is in $(R_1 \setminus S_+)$. Then the contribution from both processes in the $ds$ term are negative, and one of them provides a large negative value, $\leq -K_{R_1}$ such that $-K_{R_1} + \alpha_3 < 0$ for any chosen $\alpha_3 > 0$, if $R_1$ is large enough.

II: At time $s$, one process is either ($\geq R_1$) or ($\leq 1/R_1$), but the other process is in the domain ($S_+$). Then again the first process provides a large negative contribution $\leq -K_{R_1}$, so that $-K_{R_1} + C^* + \alpha_3 < 0$ for any chosen $\alpha_3 > 0$, if $R_1$ is large enough.

Here a brief mathematical representation of the above description is given:

$$
\begin{aligned}
&E\, f(Z_{t\wedge\gamma}, {Z'}_{t\wedge\gamma}, t\wedge\gamma) - f(z, z', 0)\\
&\leq \underbrace{E\int_0^{t\wedge\gamma} (....Z_s + ....{Z'}_s).1\left((|Z_s| > R_1 \vee |Z_s| < \frac{1}{R_1}) \text{ and } (\frac{1}{R} < |{Z'}_s| < R)\right) ds}_{\leq -K_{R_1}+C^*+\alpha_3 \leq 0}\\
&\quad + \underbrace{E\int_0^{t\wedge\gamma} (....Z_s + ....{Z'}_s).1\left((|Z_s| > R \vee |Z_s| < \frac{1}{R}) \text{ and } (\frac{1}{R} < |{Z'}_s| < R)\right) ds}_{\leq(-K_{R_1}+C^* \leq 0)}\\
&\quad + \underbrace{E\int_0^{t\wedge\gamma} (....\widetilde{Z}_s + ....{Z'}_s).1(\frac{1}{R_1} < |Z_s| < R_1 \text{ and } \frac{1}{R_1} < |{Z'}_s| < R_1}_{(\equiv 0 \,\text{since}\, \gamma=0)} ds.
\end{aligned}
$$

In both cases the term with $ds$ is negative for $s \leq \gamma$. Hence, we can finish the proof similarly to the proofs of Lemmae 1 and 2. Lemma 4 is proved. □

**Lemma 5.** *For any $\alpha, \theta > 0$ and for every $m < \beta$, there exist $\lambda > 0$ and $C > 0$ such that for every $x > 0$,*

$$\sup_{t\geq 0} E_x \left( X_t^m + \frac{1}{X_t^\lambda} \right) \leq C \left( 1 + x^m + \frac{1}{x^\lambda} \right). \tag{10}$$

Notice that the unit on the right-hand side here has been added just for simplicity: clearly it may be dropped. This is the only place were $\lambda > 0$ is to be chosen. Perhaps, by some improvement of the method one can show (10) with any $\lambda > 0$. Nevertheless, for our purposes some lambda is sufficient.

**Proof**. Proof of

$$\sup_{t\geq 0} E_x X_t^m \leq C(1 + x^m) \tag{11}$$

follows similarly to the proof of Lemma 2 from [1]. Hence, it suffices to show that there exists $\lambda > 0$ such that

$$\sup_{t\geq 0} E_x X_t^{-\lambda} \leq C\left(1 + x^{-\lambda}\right) \tag{12}$$

holds true for any $x > 0$. Consider transformation $Y_t = \ln X_t$ (it does not involve $\lambda$). In the new scale, the equation has a constant diffusion coefficient and the drift, say, $\tilde{b}(y)$, satisfying

$$y\tilde{b}(y) \leq -c < 0, \quad |y| \geq c_1.$$

It was shown in [9] that under such condition, there exists $\lambda > 0$ and $C > 0$ such that for any initial $y \in R$,

$$E_y \exp(\lambda |Y_t|) \le C \exp(\lambda |y|), \quad \forall \ t \ge 0. \tag{13}$$

Strictly speaking, only bounded coefficients were considered in [9]. However, our drift here is "better" in the domain of negative $y$ values in the sense that this drift is not just positive, but even it goes to $+\infty$ as $y \to -\infty$. Hence, e.g. a simple comparison theorem reduces our case to that considered in [9]. Finally, (13) implies (12). Lemma 5 is proved. □

## 6. Proof of Theorem 1 and 2

The proofs follow from Lemmae 4 and 5 similarly to the calculus in the proof of [1, Theorem 1], via a Harnack inequality. Notice that, of course, instead of using Harnack, it is possible to apply the intersection idea, similarly to [1, Proof of Theorem 1, step 3]. Theorems 1 and 2 are proved.

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