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Solving index form equations in two parametric families of biquadratic fields*

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Abstract. In this paper we find a minimal index and determine all integral elements with the minimal index in two families of totally real bicyclic biquadratic fields

$$K_c = \mathbb{Q}\left(\sqrt{(c-2)\,c}, \sqrt{(c+2)\,c}\right) \text{ and } L_c = \mathbb{Q}\left(\sqrt{(c-2)\,c}, \sqrt{(c+4)\,c}\right).$$

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1. Introduction

Consider an algebraic number field K of degree n with a ring of integers $\mathcal{O}_K$. It is a classical problem in algebraic number theory to decide if K admits power integral bases, that is, integral bases of the form $\left\{1, \alpha, ..., \alpha^{n-1}\right\}$. If there exist power integral bases in K, then $\mathcal{O}_K$ is a simple ring extension $\mathbb{Z}[\alpha]$ of $\mathbb{Z}$ and it is called monogenic. Let $\alpha \in \mathcal{O}_K$ be a primitive element of K, that is, $K = \mathbb{Q}(\alpha)$. Index of α is defined by

$$I(\alpha) = \left[\mathcal{O}_K^+ : \mathbb{Z}[\alpha]^+\right],$$

where $\mathcal{O}_K^+$ and $\mathbb{Z}[\alpha]^+$ denote the additive groups of $\mathcal{O}_K$ and the polynomial ring $\mathbb{Z}[\alpha]$, respectively. Therefore, the primitive element $\alpha \in \mathcal{O}_K$ generates a power integral basis if and only if $I(\alpha) = 1$. The minimal index $\mu(K)$ of K is the minimum of the indices of all primitive integers in the field K. The greatest common divisor of indices of all primitive integers of K is called the field index of K, and will be denoted by $m(K)$. Monogenic fields have both $\mu(K) = 1$ and $m(K) = 1$, but $m(K) = 1$ is not sufficient for the monogenity.

For any integral basis $\{1, \omega_2, ..., \omega_n\}$ of K let

$$L_i(\underline{X}) = X_1 + \omega_2^{(i)} X_2 + ... + \omega_n^{(i)} X_n, \quad i = 1, ..., n,$$

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where superscripts denote the conjugates. Then

$$\prod_{1\le i<j\le n}\left(L_i\left(\underline{X}\right)-L_j\left(\underline{X}\right)\right)^2=\left(I\left(X_2,...,X_n\right)\right)^2 D_K,$$

where D_K denotes the discriminant of K and $I\left(X_2,...,X_n\right)$ is a homogenous polynomial in $n-1$ variables of degree $n\left(n-1\right)/2$ with rational integer coefficients. This form is called the *index form* corresponding to the integral basis $\left\{1,\omega_2,...,\omega_n\right\}$. It can be shown that if the primitive integer $\alpha\in\mathcal{O}_K$ is represented by an integral basis as $\alpha=x_1+x_2\omega_2+...+x_n\omega_n$, then the index of α is just $I\left(\alpha\right)=\left|I\left(x_2,...,x_n\right)\right|$. Consequently, the minimal $\mu\in\mathbb{N}$ for which the equation $I\left(x_2,...,x_n\right)=\pm\mu$ is solvable in $x_2,...,x_n\in\mathbb{Z}$ is a minimal index $\mu\left(K\right)$. Further results on power integral bases, index form equations and related topics can be found in I. Gaál [11].

Biquadratic fields $K=\mathbb{Q}\left(\sqrt{m},\sqrt{n}\right)$ (where m,n are distinct square-free integers) were considered by several authors. K. S. Williams [26] gave an explicit formula for the integral basis and the discriminant of these fields. Necessary and sufficient conditions for biquadratic fields being monogenic were given by M. N. Gras and F. Tanoe [19]. T. Nakahara [23] proved that infinitely many fields of this type are monogenic but the minimal index of such fields can be arbitrary large. I. Gaál, A. Pethő and M. Pohst [16] gave an algorithm for determining the minimal index and all generators of integral bases in the totally real case by solving systems of simultaneous Pellian equations. G. Nyul [22] gave a complete characterization of power integral bases in the monogenic totally complex fields of this type. The field indices of biquadratic fields was determined by I. Gaál, A. Pethő and M. Pohst [14]. $P-$adic index form equations in biquadratic fields were studied by I. Gaál and G. Nyul [18]. Index form equations in general quartic fields were completely solved by I. Gaál, A. Pethő and M. Pohst [17]. In [21] we have determined the minimal index and all elements with the minimal index for an infinite family of totally real bicyclic biquadratic fields of the form $K=\mathbb{Q}\left(\sqrt{\left(4c+1\right)c},\sqrt{\left(c-1\right)c}\right)$ using theory of continued fractions. In the present paper, we will do the same for the following two infinite families of totally real bicyclic biquadratic fields

$$\begin{aligned}K_c&=\mathbb{Q}\left(\sqrt{\left(c-2\right)c},\sqrt{c\left(c+2\right)}\right)\\&=\mathbb{Q}\left(\sqrt{\left(c+2\right)\left(c-2\right)},\sqrt{c\left(c-2\right)}\right)\\&=\mathbb{Q}\left(\sqrt{\left(c+2\right)\left(c-2\right)},\sqrt{\left(c+2\right)c}\right)\end{aligned}\tag{1}$$

and

$$\begin{aligned}L_c&=\mathbb{Q}\left(\sqrt{\left(c-2\right)c},\sqrt{c\left(c+4\right)}\right)\\&=\mathbb{Q}\left(\sqrt{\left(c+4\right)\left(c-2\right)},\sqrt{c\left(c-2\right)}\right)\\&=\mathbb{Q}\left(\sqrt{\left(c+4\right)\left(c-2\right)},\sqrt{\left(c+4\right)c}\right).\end{aligned}\tag{2}$$

The main results of the present paper are given in the following theorems:

Theorem 1. *Let $c \geq 3$ be an odd integer such that c, $c-2$, $c+2$ are square-free integers. Then (1) is a totally real bicyclic biquadratic field and*

i) its field index is $m(K_c) = 1$ for all c,

ii) the minimal index of K_c is $\mu(K_c) = 4$ for all c,

iii) all integral elements with the minimal index are given by

$$x_1 + x_2\sqrt{c(c-2)} + x_3\frac{\sqrt{c(c-2)} + \sqrt{c(c+2)}}{2} + x_4\frac{1+\sqrt{(c-2)(c+2)}}{2},$$

where $x_1 \in \mathbb{Z}$ and $(x_2, x_3, x_4) = \pm(0, \pm 1, 1), \pm(1, 1, -1), \pm(-1, -1, 1)$.

Theorem 2. *Let $c \geq 3$ be an integer such that $c \equiv 1$ or $3 \pmod 6$ and c, $c-2$, $c+4$ are square-free integers. Then (2) is a totally real bicyclic biquadratic field and*

i) its field index is $m(L_c) = 1$ for all c,

ii) the minimal index of L_c is $\mu(L_c) = 12$ if $c \geq 7$ and $\mu(L_c) = 1$ if $c = 3$,

iii) all integral elements with the minimal index are given by

$$x_1 + x_2\sqrt{(c-2)(c+4)} + x_3\frac{\sqrt{(c-2)(c+4)} + \sqrt{(c-2)c}}{2} + x_4\frac{1+\sqrt{c(c+4)}}{2},$$

where $x_1 \in \mathbb{Z}$, $(x_2, x_3, x_4) = \pm(0, 1, 1), \pm(0, 1, -1), \pm(1, -1, -1), \pm(1, -1, 1)$ if $c \geq 7$ and $(x_2, x_3, x_4) = \pm(-1, 1, 0), \pm(0, 1, 0)$ if $c = 3$.

2. Preliminaries

Let m, n be distinct square-free integers, $l = \gcd(m, n)$, and define m_1, n_1 by $m = lm_1$, $n = ln_1$. Under these conditions the quartic field $K = \mathbb{Q}(\sqrt{m}, \sqrt{n})$ has three distinct quadratic subfields, namely $\mathbb{Q}(\sqrt{m})$, $\mathbb{Q}(\sqrt{n})$, $\mathbb{Q}(\sqrt{m_1 n_1})$ and Galois group V_4 (the Klein four group).

The integral basis and the discriminant of K were described by K. S. Williams [26] in terms of m, n, m_1, n_1, l. He distinguished five cases according to the congruence behavior of m, n, m_1, n_1 modulo 4. In [14], I. Gaál, A. Pethő and M. Pohst described the corresponding index forms $I(x_2, x_3, x_4)$. They showed that in all five cases the index form is a product of three quadratic factors. For $x_2, x_3, x_4 \in \mathbb{Z}$ the quadratic factors of the index form admit integral values. If we fix the order of the factors in the index form and if we denote the absolute value of the first, second and third factor by $F_1 = F_1(x_2, x_3, x_4)$, $F_2 = F_3(x_2, x_3, x_4)$, $F_3 = F_3(x_2, x_3, x_4)$, respectively, then finding the minimal index $\mu(K)$ is equivalent to finding integers x_2, x_3, x_4 such that the product $F_1 F_2 F_3$ is minimal. It can be easily shown that $\pm F_1$, $\pm F_2$, $\pm F_3$ are not independent, i.e. that they are related, according to five possible cases, by relations given in [16, Lemma 1]. Biquadratic field $K = Q(\sqrt{m}, \sqrt{n})$ is either totally complex or totally real (there are no mixed fields of this type). In the totally real case the index form is the product of three factors F_1, F_2, F_3, of

"Pellian type", i.e. by equating the first, second and third factor by $\pm F_1$, $\pm F_2$ and $\pm F_3$, respectively, we obtain a system of three Pellian equations such that only two of them are independent. In this case I. Gaál, A. Pethő and M. Pohst [16] gave the following algorithm for finding the minimal index and all elements with the minimal index. Consider the system of equations obtained by equating the first quadratic factor of the index form with $\pm F_1$ and the second factor with $\pm F_2$. The system of these two equations can be written as

$$Ax^2 - By^2 = C \tag{3}$$
$$Dx^2 - Fz^2 = G \quad \text{in} \quad x, y, z \in \mathbb{Z}, \tag{4}$$

where the values of A, B, C, D, F, G and the new variables x, y, z, according to five possible cases, are listed in the table (see [16, p. 104]). In each particular case, first we find the field index $m(K)$ which can be easily calculated from [14, Theorem 4]. We proceed with $\mu = \nu \cdot m(K)$ $(\nu = 1, 2, ...)$. For each such μ we try to find positive integers F_1, F_2, F_3 with $\mu = F_1F_2F_3$ satisfying the corresponding relation of [16, Lemma 1]. If there exist such F_1, F_2, F_3, then we calculate all such triples. For each such triple we determine all solutions of the corresponding system (3) and (4). If none of these systems of equations have solutions, then we proceed to the next ν, otherwise μ is the minimal index and collecting all solutions of systems of equations corresponding to valid factors F_1, F_2, F_3 of μ we get all solutions of the equation

$$I(x_2, x_3, x_4) = \pm\mu,$$

i.e. we obtain all integral elements with the minimal index in K.

3. Minimal index of the field K_c

Let $c \geq 3$ be an integer such that $c, c-2, c+2$ are square-free integers relatively prime in pairs. Let $m = m_1 l$, $n = n_1 l$, where $m_1, n_1, l \in \{c, c-2, c+2\}$ are distinct integers. Then field (1) is a totally real bicyclic biquadratic field.

First note that $c, c-2, c+2$ are integers relatively prime in pairs if and only if c is an odd integer. Furthermore, by [10], there are infinitely many positive integers c for which $c(c-2)(c+2)$ is square-free. Therefore, there are infinitely many positive integers c for which c, $c-2, c+2$ are square-free integers relatively prime in pairs, which again implies that there are infinitely many totally real bicyclic biquadratic fields of the form (1). Thus, in Theorem 1 we need the assumptions: $c \geq 3$ is an odd integer such that $c, c-2, c+2$ are square-free integers. But it is important to note that in almost all intermediate results in this section we do not have the assumption that $c, c-2, c+2$ are square-free integers, since this assumption is not necessary for proving them.

In order to prove Theorem 1 we will use a method of I. Gaál, A. Pethő and M. Pohst [16] given in the previous section. Let $n_1 = c-2$, $m_1 = c+2$ and $l = c$. We have to observe the following cases:

i) If $c \equiv 1 \pmod 4$, then $n_1 \equiv 3 \pmod 4$, $m_1 \equiv 3 \pmod 4$, $l \equiv 1 \pmod 4$ which implies $m = m_1 l \equiv 3 \pmod 4$ and $n = n_1 l \equiv 3 \pmod 4$;

ii) If $c \equiv 3 \pmod 4$, then $n_1 \equiv 1 \pmod 4$, $m_1 \equiv 1 \pmod 4$, $l \equiv 3 \pmod 4$ which implies $m = m_1 l \equiv 3 \pmod 4$ and $n = n_1 l \equiv 3 \pmod 4$.

Since in both cases we have $(m, n) \equiv (3, 3) \pmod 4$, by equating the first, second and third quadratic factor of the corresponding index form with $\pm F_1$, $\pm F_2$ and $\pm F_3$, respectively, according to [16], we obtain the system

$$cU^2 - (c-2)V^2 = \pm F_1 \tag{5}$$

$$cZ^2 - (c+2)V^2 = \pm F_2 \tag{6}$$

$$(c-2)Z^2 - (c+2)U^2 = \pm 4F_3, \tag{7}$$

where

$$U = 2x_2 + x_4, \ V = x_4, \ Z = x_3, \tag{8}$$

and from [16, Lemma 1] we have that

$$\pm (c+2) F_1 \pm (c-2) F_2 = \pm 4cF_3 \tag{9}$$

must hold. In this case the integral basis of K_c is

$$\left\{1, \ \sqrt{c(c+2)}, \ \frac{\sqrt{c(c+2)} + \sqrt{c(c-2)}}{2}, \ \frac{1 + \sqrt{(c-2)(c+2)}}{2}\right\}$$

and its discriminant is $D_{K_c} = (4c(c-2)(c+2))^2$.

Now we will prove statement *i)* of Theorem 1. First we form differences $d_1 = m_1 - l$, $d_2 = n_1 - l$, $d_3 = m_1 - n_1$. We have $d_1 = 2$, $d_2 = -2$, $d_3 = 4$. Since neither 3 nor 4 divides all three differences d_1, d_2, d_3, according to [14, Theorem 4], we conclude $m(K_c) = 1$.

Now we will formulate our strategy of searching the minimal index $\mu(K_c) := \mu(c)$ and all elements with the minimal index. Finding the minimal index $\mu(c)$ is equivalent to finding the system of the above form with minimal product $F_1F_2F_3$ which has a solution.

Observe that if $(\pm F_1, \pm F_2, \pm F_3) = (2, -2, 1)$, then system (5), (6) and (7) has solutions $(U, V, Z) = (\pm 1, \pm 1, \pm 1)$ which implies that $\mu(c) \leq 4$ for all c.

For $c = 3$ and $c = 5$ we have discriminant $D_{K_c} < 10^6$. In [16] I. Gaál, A. Pethő and M. Pohst determined minimal indices and all elements with the minimal index in all 196 fields and totally real bicyclic biquadratic fields with discriminant $< 10^6$. It can be found there that $\mu(3) = \mu(5) = 4$ and all elements with the minimal index are given by $(x_2, x_3, x_4) = \pm(0, \pm 1, 1), \pm(1, 1, -1), \pm(-1, -1, 1)$.

First suppose that (U, V, Z) is a nonnegative integer solution of the system of equations (5), (6) and (7) with $F_1F_2F_3 \leq 4$. Observe that if one of the integers U, V, Z is equal to zero, then (5), (6) and (7) imply that the other two integers are not equal to zero.

i) If $V = 0$, then (5) and (6) imply $cU^2 = \pm F_1, cZ^2 = \pm F_2$. Therefore we have $F_1F_2 = c^2Z^2U^2 \leq 4$. Since $c \geq 3$ and U, $Z \neq 0$, we obtain a contradiction.

ii) If $Z=0$, then (6) and (7) imply $-(c+2)\,V^2=\pm F_2, -(c+2)\,U^2=\pm 4F_3$. Therefore we have $F_2F_3=\frac{(c+2)^2}{4}U^2V^2\leq 4$. Since $c\geq 3$ and $U,\ V\neq 0$, we obtain a contradiction.

iii) If $U=0$, then (5), (6) and (7) imply $-(c-2)\,V^2=\pm F_1, cZ^2-(c+2)\,V^2=\pm F_2$ and $(c-2)\,Z^2=\pm 4F_3$. Therefore we have $F_1F_3=\frac{(c-2)^2}{4}V^2Z^2\leq 4$ and Z is an even integer. Since $V\neq 0$ and $Z^2\geq 4$, we obtain a contradiction if $c\neq 3$. If $c=3$, then $F_1F_3=\frac{1}{4}V^2Z^2\leq 4$ which implies $(V,Z)=(1,2)$. Additionally, we have

$$F_1F_2F_3=\left|cZ^2-(c+2)\,V^2\right|\cdot\frac{(c-2)^2}{4}\cdot V^2Z^2\leq 4. \tag{10}$$

Now, for $c=3$ and $(V,Z)=(1,2)$ inequality (10) gives a contradiction.

Let (U,V,Z) be a positive integer solution of the system of Pellian equations

$$cU^2-(c-2)\,V^2=\lambda_1, \tag{11}$$
$$cZ^2-(c+2)\,V^2=\lambda_2, \tag{12}$$

where λ_1 and λ_2 are integers such that $|\lambda_1|\leq 4$ and $|\lambda_2|\leq 4$. We have

$$\begin{aligned}\left|\sqrt{\frac{c}{c-2}}-\frac{V}{U}\right| &= \left|\frac{c}{c-2}-\frac{V^2}{U^2}\right|\cdot\left|\sqrt{\frac{c}{c-2}}+\frac{V}{U}\right|^{-1}\\ &< \frac{|\lambda_1|}{(c-2)\,U^2}\cdot\sqrt{\frac{c-2}{c}}\leq\frac{4}{\sqrt{c\,(c-2)}U^2}\leq\begin{cases}\frac{3}{U^2}, \text{ if } c=3\\ \frac{2}{U^2}, \text{ if } c=5\\ \frac{1}{U^2}, \text{ if } c\geq 7\end{cases}.\end{aligned}$$

Similarly,

$$\begin{aligned}\left|\sqrt{\frac{c+2}{c}}-\frac{Z}{V}\right| &= \left|\frac{c+2}{c}-\frac{Z^2}{V^2}\right|\cdot\left|\sqrt{\frac{c+2}{c}}+\frac{Z}{V}\right|^{-1}\\ &< \frac{|\lambda_2|}{cV^2}\cdot\sqrt{\frac{c}{c+2}}\leq\frac{4}{\sqrt{c\,(c+2)}V^2}\leq\begin{cases}\frac{2}{V^2}, \text{ if } c=3\\ \frac{1}{V^2}, \text{ if } c\geq 5\end{cases}.\end{aligned}$$

The simple continued fraction expansion of a quadratic irrational $\alpha=\frac{a+\sqrt{d}}{b}$ is periodic. This expansion can be obtained using the following algorithm. Multiplying the numerator and the denominator by b, if necessary, we may assume that $b|(d-a^2)$. Let $s_0=a$, $t_0=b$ and

$$a_n=\left\lfloor\tfrac{s_n+\sqrt{d}}{t_n}\right\rfloor,\quad s_{n+1}=a_nt_n-s_n,\quad t_{n+1}=\tfrac{d-s_{n+1}^2}{t_n}\quad\text{for}\quad n\geq 0 \tag{13}$$

(see [24, Chapter 7.7]). If $(s_j,t_j)=(s_k,t_k)$ for $j<k$, then

$$\alpha=[a_0,\ldots,a_{j-1},\overline{a_j,\ldots,a_{k-1}}].$$

Applying this algorithm to the quadratic irrational $\sqrt{\frac{c+2}{c}} = \frac{\sqrt{c(c+2)}}{c}$ we find that

$$\sqrt{\frac{c+2}{c}} = \left[1, \overline{c, 2}\right], \text{ where } (s_0, t_0) = (0, c),$$
$$(s_1, t_1) = (c, 2), (s_2, t_2) = (c, c), (s_3, t_3) = (c, 2).$$

The expansion of the quadratic irrational $\sqrt{\frac{c}{c-2}}$ can be obtained from the expansion of $\sqrt{\frac{c+2}{c}}$ by replacing c by $c-2$.

Let p_n/q_n denote the nth convergent of α. The following result of Worley [27] and Dujella [5] extends classical results of Legendere and Fatou concerning Diophantine approximations of the form $\left|\alpha - \frac{a}{b}\right| < \frac{1}{2b^2}$ and $\left|\alpha - \frac{a}{b}\right| < \frac{1}{b^2}$.

Theorem 3 (Worley [27], Dujella [5])**.** *Let α be a real number and a and b coprime nonzero integers, satisfying the inequality*

$$\left|\alpha - \frac{a}{b}\right| < \frac{M}{b^2},$$

where M is a positive real number. Then $(a,b) = (rp_{n+1} \pm up_n, rq_{n+1} \pm uq_n)$, for some $n \geq -1$ and nonnegative integers r and u such that $ru < 2M$.

We would like to apply Theorem 3 in order to determine all values of λ_1 with $|\lambda_1| \leq 4$, for which equation (11) has solutions in coprime integers and all values of λ_2 with $|\lambda_2| \leq 4$ for which equation (12) has solutions in coprime integers. An explicit version of Theorem 3 for $M = 2$, was given by Worley [27, Corollary, p. 206]. Recently, Dujella and Ibrahimpašić [6, Propositions 2.1 and 2.2] extended Worley's work and gave explicit and sharp versions of Theorem 3 for $M = 3, 4, ..., 12$. We need the following lemma (see [8, Lemma 1]).

Lemma 1. *Let $\alpha\beta$ be a positive integer which is not a perfect square, and let p_n/q_n denote the nth convergent of a continued fraction expansion of $\sqrt{\frac{\alpha}{\beta}}$. Let the sequences (s_n) and (t_n) be defined by (13) for the quadratic irrational $\frac{\sqrt{\alpha\beta}}{\beta}$. Then*

$$\alpha(rq_{n+1} + uq_n)^2 - \beta(rp_{n+1} + up_n)^2 = (-1)^n(u^2 t_{n+1} + 2rus_{n+2} - r^2 t_{n+2}). \quad (14)$$

Since the period length of the continued fraction expansions of both $\sqrt{\frac{c+2}{c}}$ and $\sqrt{\frac{c}{c-2}}$ is equal to 2, according to Lemma 1, we have to consider only the fractions $(rp_{n+1} + up_n)/(rq_{n+1} + uq_n)$ for $n = 0$ and $n = 1$. By checking all possibilities, it is now easy to prove the following results.

Proposition 1. *Let $c \geq 3$ be an odd integer and λ_1 an integer such that $|\lambda_1| \leq 4$ and such that equation (11) has a solution in relatively prime integers U and V.*

i) If $c \geq 7$, then $\lambda_1 \in A_1(c) = \{2\}$.

ii) If $c = 5$, then $\lambda_1 \in A_1(5) = \{2, 2-c, 32-7c\} = \{2, -3\}$.

iii) If $c=3$, *then* $\lambda_1 \in A_1(3) = \{2, c, 2-c, 8-3c, 18-5c\} = \{2, 3, -1\}$.

Proposition 2. *Let* $c \geq 3$ *be an odd integer and* λ_1 *an integer such that* $|\lambda_1| \leq 4$ *and such that equation (12) has a solution in relatively prime integers* V *and* Z.

i) If $c \geq 5$, *then* $\lambda_2 \in A_2(c) = \{-2\}$.

ii) If $c = 3$, *then* $\lambda_2 \in A_2(c) = \{-2, c, 7c-18\} = \{-2, 3\}$.

Corollary 1. *Let* $c \geq 3$ *be an odd integer.*

i) Let (U, V) *be a positive integer solution of equation (5) such that* $\gcd(U, V) = d$ *and* $F_1 \leq 4d^2$. *Then*

$$\pm F_1 \in \left\{\lambda_1 d^2 : \lambda_1 \in A_1(c)\right\},$$

where the sets $A_1(c)$ *are given in Proposition 1.*

ii) Let (V, Z) *be a positive integer solution of equation (6)such that* $\gcd(V, Z) = g$ *and* $F_2 \leq 4g^2$. *Then*

$$\pm F_2 \in \left\{\lambda_2 g^2 : \lambda_2 \in A_2(c)\right\},$$

where the sets $A_2(c)$ *are given in Proposition 2.*

Proof. Directly from Propositions 1 and 2. □

Proposition 3. *Let* $c \geq 3$ *be an odd integer. Let* (U, V, Z) *be a positive integer solution of the system of Pellian equations (5) and (6), where* $\gcd(U, V) = d$, $\gcd(V, Z) = g$ *and* $F_1, F_2 \leq 4$. *Then*

i) $(\pm F_1, \pm F_2) \in B(c) \times D(c)$, *where* $B(c) = B_0 \cup B_1(c)$, $D(c) = D_0 \cup D_1(c)$ *and* $B_0 = \{2\}, D_0 = \{-2\}, B_1(5) = \{-3\}, B_1(3) = \{3, -1, -4\}, B_1(c) = \emptyset, c \geq 7$, *and* $D_1(3) = \{3\}, D_1(c) = \emptyset, c \geq 5$,

ii) Additionally, if $F_1 F_2 \leq 4$, *then* $(\pm F_1, \pm F_2) \in S(c)$, *where* $S(c) = S_0 \cup S_1(c)$ *and* $S_0 = \{(2, -2)\}, S_1(3) = \{(-1, -2), (-1, 3)\}$, $S_1(c) = \emptyset$ *for* $c \geq 5$.

Proof. Consider *i*): From Corollary 1 we have $\pm F_1 \in \left\{\lambda_1 d^2 : \lambda_1 \in A_1(c)\right\}$ and $\pm F_2 \in \left\{\lambda_2 g^2 : \lambda_2 \in A_2(c)\right\}$, where sets $A_1(c)$ and $A_2(c)$ are given in Propositions 1 and 2, respectively.

a) For all $c \geq 3$ we have $\pm F_1 = 2d^2$. Additionally, we have $\pm F_1 = -3d^2$ if $c = 5$ and $\pm F_1 = 3d^2$, $-d^2$ if $c = 3$. Since $F_1 \leq 4$, we obtain:

   i. $F_1 = 2d^2 \leq 4$ implies $d = 1$, i.e. $\pm F_1 = 2$;
   ii. $F_1 = 3d^2 \leq 4$ implies $d = 1$. Thus, $\pm F_1 = -3$ for $c = 5$ and $\pm F_1 = 3$ for $c = 3$;
   iii. $F_1 = d^2 \leq 4$ implies $d = 1, 2$. Thus, $\pm F_1 = -1, -4$ for $c = 3$.

- Therefore, we obtain sets $B(c)$.

b) For all $c \geq 3$ we have $\pm F_2 = -2g^2$. Additionally, we have $\pm F_2 = 3g^2$ if $c = 3$. Since $F_2 \leq 4$, we obtain:

   i. $F_2 = 2g^2 \leq 4$ implies $g = 1$, i.e. $\pm F_2 = -2$;

   ii. $F_2 = 3g^2 \leq 4$ implies $g = 1$. Thus, $\pm F_2 = 3$ for $c = 3$.

- Therefore, we get sets $D(c)$.

Proof of *ii)* follows directly from *i)* since $S(c) = \{(s,t) \in B(c) \times D(c) : |s| \cdot |t| \leq 4\}$. □

If system (5), (6) and (7) has a solution for some positive integers F_1, F_2, F_3, $F_1F_2F_3 \leq 4$, then $(\pm F_1, \pm F_2) \in S(c)$, where the set $S(c)$ is given in Proposition 3 and triple $(\pm F_1, \pm F_2, \pm F_3)$ satisfies one of the equations in (9). First, for each pair $(\pm F_1, \pm F_2) \in S(c)$ we check if there exists $F_3 \in \mathbb{N}$, $F_1F_2F_3 \leq 4$, such that some of the equations (9) holds. For all pairs of the form $(\pm F_1, \pm F_2) = (s,t)$ condition $F_1F_2F_3 \leq 4$ is satisfied if $F_3 \in F(s,t) = \{k \in \mathbb{N} : k\,|s|\,|t| \leq 4\}$. Therefore, for each pair $(s,t) \in S(c)$ and for each $k \in F(s,t)$, we have to check if some of these four equations

$$s(c+2) + t(c-2) = \pm 4kc \quad or \quad s(c+2) - t(c-2) = \pm 4kc \tag{15}$$

holds. For example, if $c \geq 3$, then $(\pm F_1, \pm F_2) = (2,-2) \in S(c)$. From (15) we obtain

$$8 = \pm 4kc \quad or \quad 4c = \pm 4kc.$$

Since $k \in F(2,-2) = \{1\}$, the only possibility is $\pm F_3 = 1$. We proceed similarly for $(-1,-2), (-1,3) \in S(3)$. The only triple obtained in this way is $(\pm F_1, \pm F_2, \pm F_3) = (2,-2,1)$ and the corresponding system is

$$cU^2 - (c-2)V^2 = 2 \tag{16}$$

$$cZ^2 - (c+2)V^2 = -2 \tag{17}$$

$$(c-2)Z^2 - (c+2)U^2 = 4. \tag{18}$$

Since this system has solution $(U,V,Z) = (\pm 1, \pm 1, \pm 1)$, we have $\mu(c) = 4$ for all c.

The next step is finding all elements with the minimal index. Therefore we have to solve the above system.

In [20], Ibrahimpašić found all primitive solutions of the Thue inequality

$$\left|x^4 - 2cx^3y + 2x^2y^2 + 2cxy^3 + y^4\right| \leq 6c + 4,$$

where $c \geq 3$ is an integer. He showed that solving the above inequality reduces to solving the systems of Pellian equations of the form

$$cU^2 - (c-2)V^2 = 2\mu, cZ^2 - (c+2)V^2 = -2\mu \quad \text{with} \quad |\mu| \leq 6c + 4, \tag{19}$$

where $U = -x^2 + 2xy + y^2$, $V = x^2 + y^2$, $Z = x^2 + 2xy - y^2$. First, using continued fractions (i.e. using Theorem 3), Ibrahimpašić found the set of all values of μ for which system (19) has solutions. Then, for all obtained values of μ, he solved corresponding systems by using the method given in [7]. Note, if we put $\mu = 1$ in (19), then we obtain equations (16) and (17). In [20, Section 3], Ibrahimpašić showed the following result

Lemma 2. *Let $c \geq 3$ be an integer. The only solutions of system (16) and (17) are* $(U, V, Z) = (\pm 1, \pm 1, \pm 1)$.

Therefore we have the following proposition which finishes the proof of Theorem 1.

Proposition 4. *Let $c \geq 3$ be an odd integer such that c, $c+2$, $c-2$ are square-free integers. Then all integral elements with the minimal index in the field* $K_c = \mathbb{Q}\left(\sqrt{(c-2)\,c}\,,\sqrt{c\,(c+2)}\right)$ *are given by* $(x_2, x_3, x_4) = \pm(0, \pm 1, 1), \pm(1, 1, -1), \pm(-1, 1, 1)$.

Proof. By Lemma 2, all solutions of system (16), (17) and (18) are given by $(U, V, Z) = (\pm 1, \pm 1, \pm 1)$ and since we have $U = 2x_2 + x_4$, $V = x_4$, $Z = x_3$, we obtain

$$2x_2 + x_4 = \pm 1,\ \ x_4 = \pm 1,\ \ x_3 = \pm 1,$$

which implies $(x_2, x_3, x_4) = \pm(0, \pm 1, 1), \pm(1, 1, -1), \pm(-1, 1, 1)$. □

4. Minimal index of the field L_c

Let $c \geq 3$ be an integer such that $c, c-2, c+4$ are square-free integers relatively prime in pairs. Then field (2) is a totally real bicyclic biquadratic field.

Note that $c, c-2, c+4$ are integers relatively prime in pairs if and only if $c \equiv 1$ or $3 (\mathrm{mod}\, 6)$. Furthermore, by [10], there are infinitely many positive integers c for which $c\,(c-2)\,(c+4)$ is a square-free integer. Therefore, there are infinitely many positive integers c for which $c, c-2, c+4$ are square-free integers relatively prime in pairs, which again implies that there are infinitely many totally real bicyclic biquadratic fields of the form (2). Hence, in Theorem 2 we need the assumptions: $c \geq 3$ is an integer such that $c \equiv 1$ or $3\ (\mathrm{mod}\, 6)$ and c, $c-2, c+2$ are square-free integers. But, similarly to Section 3, in almost all intermediate results in this section we do not have the assumption that $c, c-2, c+4$ are square-free integers, since this assumption is not necessary for proving them.

In order to prove Theorem 2 we will use a method of I. Gaál, A. Pethő and M. Pohst [16] again. We have to observe the following cases:

i) If $c \equiv 1\,(\mathrm{mod}\, 4)$, $l = c-2$, $m_1 = c+4$ and $n_1 = c$, then $n_1 \equiv 1\,(\mathrm{mod}\, 4)$, $m_1 \equiv 1\,(\mathrm{mod}\, 4)$, $l \equiv 3\,(\mathrm{mod}\, 4)$ which implies $m = m_1 l \equiv 3\,(\mathrm{mod}\, 4)$ and $n = n_1 l \equiv 3\,(\mathrm{mod}\, 4)$;

ii) Let $c \equiv 3\,(\mathrm{mod}\, 4)$, $l = c-2$, $m_1 = c+4$ and $n_1 = c$. Then $l \equiv 1\,(\mathrm{mod}\, 4)$, $m_1 \equiv 3\,(\mathrm{mod}\, 4)$, $n_1 \equiv 3\,(\mathrm{mod}\, 4)$ which implies $m = m_1 l \equiv 3\,(\mathrm{mod}\, 4)$ and $n = n_1 l \equiv 3\,(\mathrm{mod}\, 4)$.

Since in both cases we have $(m,n) \equiv (3,3) \pmod 4$, similarly to Section 3, according to [16], we obtain the system

$$(c-2)\,U^2 - cV^2 = \pm F_1 \tag{20}$$
$$(c-2)\,Z^2 - (c+4)\,V^2 = \pm F_2 \tag{21}$$
$$cZ^2 - (c+4)\,U^2 = \pm 4F_3, \tag{22}$$

where

$$U = 2x_2 + x_3,\ V = x_4,\ Z = x_3, \tag{23}$$

and from [16, Lemma 1] we obtain that

$$\pm(c+4)\,F_1 \pm cF_2 = \pm 4\,(c-2)\,F_3 \tag{24}$$

must hold. In this case the integral basis of L_c is

$$\left\{1,\ \sqrt{(c-2)\,(c+4)},\ \frac{\sqrt{(c-2)\,(c+4)}+\sqrt{(c-2)\,c}}{2},\ \frac{1+\sqrt{c\,(c+4)}}{2}\right\}$$

and its discriminant is $D = (4c\,(c-2)\,(c+4))^2$.

Now we will calculate the field index $m\,(L_c)$ of L_c. We form differences $d_1 = m_1 - l = 6$, $d_2 = n_1 - l = 2$, $d_3 = m_1 - n_1 = 4$. Since neither 3 nor 4 divides all three differences d_1, d_2, d_3, according to [14, Theorem 4], we conclude $m\,(L_c) = 1$. Therefore, we have proved statement *i)* of Theorem 2.

We will apply the same strategy of searching the minimal index and all elements with the minimal index as in the previous case. Observe that if

$$(\pm F_1, \pm F_2, \pm 4F_3) = (-2, -6, -4),$$

then system (20), (21) and (22) has solutions $(U,V,Z) = (\pm1, \pm1, \pm1)$ which implies that $\mu\,(L_c) := \mu\,(c) \leq 12$.

Also, if $c = 3$ and $(\pm F_1, \pm F_2, \pm 4F_3) = (1, 1, -4)$, then system (20), (21) and (22) has solutions $(U,V,Z) = (\pm1, 0, \pm1)$ which implies that $\mu\,(3) = 1$, i.e. field L_3 is monogenic. In [16, p. 109] it can be found that $\mu\,(3) = 1$ and all elements with the minimal index are given by $(x_2, x_3, x_4) = \pm(-1,1,0), \pm(0,1,0)$.

4.1. Finding the minimal index

Let $c \geq 7$ be an integer such that $c \equiv 1$ or $3 \pmod 6$. First suppose that (U,V,Z) is a nonnegative integer solution of the system of equations (20), (21) and (22) with $F_1F_2F_3 \leq 12$. If one of the integers U, V, Z is equal to zero, then (20), (21) and (22) imply that the other two integers are not equal to zero. Thus we have:

i) If $V = 0$, then (20) and (21) imply $(c-2)\,U^2 = \pm F_1, (c-2)\,Z^2 = \pm F_2$. Therefore we have $F_1F_2 = (c-2)^2\,Z^2U^2 \leq 12$. If $c \geq 7$ and $U,\ Z \neq 0$, we obtain a contradiction.

ii) If $Z=0$, then (21) and (22) imply $-(c+4)\,V^2=\pm F_2, -(c+4)\,U^2=\pm 4F_3$. Therefore we have $F_2F_3=\frac{(c+4)^2}{4}U^2V^2\leq 12$. Since $c\geq 7$ and $U,\ V\neq 0$, we obtain a contradiction.

iii) If $U=0$, then (20) and (22) imply $-cV^2=\pm F_1, cZ^2=\pm 4F_3$. Therefore we have $F_1F_3=\frac{c^2}{4}V^2Z^2\leq 12$ and Z is an even integer. Since $c\geq 7$, $V\neq 0$ and $Z^2\geq 4$ we obtain a contradiction.

To find triple $(\pm F_1,\pm F_2,\pm 4F_3)$ for which the system of the form (20), (21) and (22), with $F_1F_2F_3\leq 12$, has positive solutions we will use the same method as in Section 3. Since quadratic irrational $\sqrt{\frac{c+4}{c-2}}$ has a quite irregular continued fraction expansion, we will observe the system of equations (20) and (22) instead of (20) and (21).

Let (U,V,Z) be a positive integer solution of the system of Pellian equations

$$(c-2)\,U^2-cV^2=\lambda_1, \tag{25}$$

$$cZ^2-(c+4)\,U^2=\lambda_3, \tag{26}$$

where λ_1 and λ_3 are integers such that $|\lambda_1|\leq 12$ and $|\lambda_3|\leq 48$. We have

$$\left|\sqrt{\frac{c}{c-2}}-\frac{U}{V}\right|=\left|\frac{c}{c-2}-\frac{U^2}{V^2}\right|\cdot\left|\sqrt{\frac{c}{c-2}}+\frac{U}{V}\right|^{-1}$$

$$<\frac{|\lambda_1|}{(c-2)\,V^2}\cdot\sqrt{\frac{c-2}{c}}\leq\frac{12}{\sqrt{c\,(c-2)}V^2}\leq\begin{cases}\frac{3}{V^2}, \text{ if } c=7\\ \frac{2}{V^2}, \text{ if } c=9,13\\ \frac{1}{V^2}, \text{ if } c\geq 15\end{cases}$$

and

$$\left|\sqrt{\frac{c+4}{c}}-\frac{Z}{U}\right|=\left|\frac{c+4}{c}-\frac{Z^2}{U^2}\right|\cdot\left|\sqrt{\frac{c+4}{c}}+\frac{Z}{U}\right|^{-1}$$

$$<\frac{|\lambda_3|}{cU^2}\cdot\sqrt{\frac{c}{c+4}}\leq\frac{48}{\sqrt{c\,(c+4)}U^2}\leq\frac{M}{U^2},$$

where $M=1$ if $c\geq 49$, $M=2$ if $25\leq c\leq 45$, $M=3$ if $15\leq c\leq 21$, $M=4$ if $c=13$, $M=5$ if $c=9$ and $M=6$ if $c=7$.

Applying algorithm (13) to quadratic irrational $\sqrt{\frac{c+4}{c}}=\frac{\sqrt{c(c+4)}}{c}$ we find that if $c>1$ is an odd integer, then

$$\sqrt{\frac{c+4}{c}}=\left[1,\overline{\frac{c-1}{2},1,2c+2,1,\frac{c-1}{2},2}\right]$$

and $(s_0,t_0)=(0,c)\,,(s_1,t_1)=(c,4)\,,(s_2,t_2)=(c-2,2c-1)\,,(s_3,t_3)=(c+1,1)\,,$ $(s_4,t_4)=(c+1,2c-1)\,,(s_5,t_5)=(c-2,4)\,,(s_6,t_6)=(c,c)\,,(s_7,t_7)=(c,4)$.

Note that the continued fraction expansion of the quadratic irrational $\sqrt{\frac{c}{c-2}}$ was obtained in Section 3.

Now we will apply Theorem 3 and Lemma 1 in order to determine all values of λ_1 with $|\lambda_1| \leq 12$, for which equation (11) has solutions in relatively prime integers and all values of λ_3 with $|\lambda_3| \leq 48$ for which equation (12) has solutions in relatively prime integers.

Since the period length of the continued fraction expansion of $\sqrt{\frac{c+4}{c}}$ is equal to 6 if $c > 1$ is odd, according to Lemma 1, we have to consider only the fractions $(rp_{n+1} + up_n)/(rq_{n+1} + uq_n)$ for $n = 0, 1, ..., 5$.

Since the period length of the continued fraction expansion of $\sqrt{\frac{c}{c-2}}$ is equal to 2, according to Lemma 1, we have to consider only the fractions $(rp_{n+1}+up_n)/(rq_{n+1}+uq_n)$ for $n = 0, 1$.

By checking all possibilities, it is now easy to prove the following results.

Proposition 5. *Let $c \geq 7$ be an integer such that $c \equiv 1$ or $3 \pmod 6$ and λ_1 an integer such that $|\lambda_1| \leq 12$ and equation (25) has a solution in relatively prime integers U and V.*

i) *If $c \geq 15$, then* $\lambda_1 \in A_1(c) = \{-2\}$.

ii) *If $c = 13$, then* $\lambda_1 \in A_1(13) = \{-2, c-2\} = \{-2, 11\}$.

iii) *If $c = 9$, then* $\lambda_1 \in A_1(9) = \{-2, -c, c-2\} = \{-2, -9, 7\}$.

iv) *If $c = 7$, then* $\lambda_1 \in A_1(7) = \{-2, -c, c-2, 11c-72\} = \{-2, -7, 5\}$.

Proposition 6. *Let $c \geq 7$ be an integer such that $c \equiv 1$ or $3 \pmod 6$ and λ_3 an integer such that $|\lambda_3| \leq 48$ and equation (26) has a solution in relatively prime integers V and Z.*

i) *If $c \geq 49$, then* $\lambda_3 \in A_3(c) = \{-1, -4\}$.

ii) *If $c = 45$, then* $\lambda_3 \in A_3(c) = \{-1, -4, c\} = \{-1, -4, 45\}$.

iii) *If $25 \leq c \leq 43$, then* $\lambda_3 \in A_3(c) = \{-1, -4, -c-4, c\}$.

iv) *If $c = 21$, then* $\lambda_3 \in A_3(c) = \{-1, -4, 2c-1, -c-4, c\} = \{-1, -4, 41, -25, 21\}$.

v) *If $c = 19$, then* $\lambda_3 \in A_3(c) = \{-1, -4, -2c-9, 2c-1, -c-4, c\} = \{-1, -4, -47, 37, -23, 19\}$.

vi) *If $c = 15$, then* $\lambda_3 \in A_3(c) = \{-1, -4, -2c-9, 2c-1, -c-4, 3c-4, c\} = \{-1, -4, -39, 29, -19, 41, 15\}$.

vii) *If $c = 13$, then* $\lambda_3 \in A_3(c) = \{-1, -4, 4c-9, -2c-9, 12c-121, 14c-169, 16c-225, 2c-1, -c-4, 3c-4, 11c-100, 13c-144, 15c-196, c\} = \{-1, -4, 43, -35, 35, 13, -17, 25\}$.

viii) *If $c = 9$, then* $\lambda_3 \in A_3(c) = \{-1, -4, 4c-9, -2c-9, 4c, 6c-25, 8c-49, 10c-81, 12c-121, 14c-169, 16c-225, 2c-1, -c-4, 3c-4, 5c-16, 7c-36, 9c-64, 11c-100, -3c-16, 13c-144, c\} = \{-1, -4, 27, -27, 36, 29, 23, 9, -13, -43, 17\}$.

ix) *If* $c = 7$, *then* $\lambda_3 \in A_3(c) = \{-1, -4, 4c-9, -2c-9, 4c, 6c-25, 8c-49, 10c-81, 12c-121, 6c-1, -4c-16, 2c-1, -c-4, 3c-4, 5c-16, 7c-36, 9c-64, 11c-100, -3c-16, 5c-16, 9c-64, 11c-100, -3c-16, 5c-16, c\} = \{-1, -4, 19, -23, 28, 17, 7, -11, -37, 41, -44, 13\}$.

Corollary 2. *Let* $c \geq 7$ *be an integer such that* $c \equiv 1$ *or* $3 \pmod 6$.

i) *Let* (U, V) *be a positive integer solution of equation (20) such that* $\gcd(U, V) = d$ *and* $F_1 \leq 12d^2$. *Then* $\pm F_1 \in \left\{\lambda_1 d^2 : \lambda_1 \in A_1(c)\right\}$, *where the sets* $A_1(c)$ *are given in Proposition 5.*

ii) *Let* (V, Z) *be a positive integer solution of equation (21) such that* $\gcd(V, Z) = g$ *and* $4F_3 \leq 48g^2$. *Then* $\pm 4F_3 \in \left\{\lambda_3 g^2 : \lambda_3 \in A_3(c)\right\}$, *where the sets* $A_3(c)$ *are given in Proposition 6.*

Proof. Directly from Propositions 5 and 6. □

Proposition 7. *Let* $c \geq 7$ *be an integer such that* $c \equiv 1$ *or* $3 \pmod 6$. *Let* (U, V, Z) *be a positive integer solution of the system of Pellian equations (20) and (21) where* $\gcd(U, V) = d$, $\gcd(V, Z) = g$ *and* $F_1, F_3 \leq 12$. *Then*

i) $(\pm F_1, \pm 4F_3) \in B(c) \times D(c)$, *where* $B(c) = B_0 \cup B_1(c)$, $D(c) = D_0 \cup D_1(c)$ *and* $B_0 = \{-2, -8\}$, $D_0 = \{-4, -16, -36\}$, $B_1(7) = \{5, -7\}$, $B_1(9) = \{7, -9\}$, $B_1(13) = \{11\}$, $B_1(c) = \emptyset, c \geq 15$, *and* $D_1(7) = \{28, -44\}$, $D_1(9) = \{36\}$, $D_1(c) = \emptyset, c \geq 13$.

ii) *Additionally, if* $F_1 F_3 \leq 12$, *then* $(\pm F_1, \pm 4F_3) \in S(c)$, *where* $S(c) = S_0 \cup S_1(c)$ *and* $S_0 = \{(-2, -4), (-2, -16), (-8, -4)\}$, $S_1(7) = \{(5, -4), (-7, -4)\}$, $S_1(9) = \{(7, -4), (-9, -4)\}$, $S_1(13) = \{(11, -4)\}$ *and* $S_1(c) = \emptyset$ *for* $c \geq 15$.

Proof. Let us consider *i*). From Corollary 2 we have $\pm F_1 \in \left\{\lambda_1 d^2 : \lambda_1 \in A_1(c)\right\}$ and $\pm 4F_3 \in \left\{\lambda_3 g^2 : \lambda_3 \in A_3(c)\right\}$, where sets $A_1(c)$ and $A_3(c)$ are given in Propositions 5 and 6, respectively.

a) For all $c \geq 7$ we have $\pm F_1 = -2d^2$. Additionally, we have $\pm F_1 = (c-2)d^2$ if $c \leq 13$ and $\pm F_1 = -cd^2$ if $c \leq 9$. Since $F_1 \leq 12$, we obtain:

   i. $F_1 = 2d^2 \leq 12$ implies $d = 1, 2$, i.e. $\pm F_1 = -2, -8$;

   ii. $F_1 = (c-2)d^2 \leq 12$ implies $d \leq \sqrt{\frac{12}{c-2}} < 2$. Thus, $\pm F_1 = 5$ for $c = 7$, $\pm F_1 = 7$ for $c = 9$ and $\pm F_1 = 11$ for $c = 13$;

   iii. $F_1 = cd^2 \leq 12$ implies $d \leq \sqrt{\frac{12}{c}} < 2$. Thus, $\pm F_1 = -7$ for $c = 7$ and $\pm F_1 = -9$ for $c = 9$.

- Therefore, we obtain sets $B(c)$.

b) For all $c \geq 7$ we have $\pm 4F_3 = -g^2, -4g^2$. Since $F_3 \leq 12$, we obtain:

   i. $4F_3 = g^2 \leq 48$ implies $g = 2, 4, 6$, i.e. $\pm 4F_3 = -4, -16, -36$;

ii. $4F_3 = 4g^2 \leq 48$ implies $g = 1, 2, 3$. Thus, $\pm 4F_3 = -4, -16, -36$.

- Additionally, we have $\pm 4F_3 = cg^2$ if $c \leq 45$, $\pm 4F_3 = (-c-4)\,g^2$ if $c \leq 43$, $\pm 4F_3 = (2c-1)\,g^2$ if $c \leq 21$, $\pm 4F_3 = (-2c-9)\,g^2$ if $c \leq 19$, $\pm 4F_3 = (3c-4)\,g^2$ if $c \leq 15$, $\pm 4F_3 = (4c-9)\,g^2$, $(12c-121)\,g^2$, $(11c-100)\,g^2$ if $c \leq 13$, $\pm 4F_3 = 36g^2, 29g^2, 23g^2, -43g^2$ if $c = 9$ and $\pm 4F_3 = 28g^2, 17g^2, 41g^2, -44g^2$ if $c = 7$. Similarly, since $F_3 \leq 12$, we obtain:

  iii. $4F_3 = cg^2 \leq 48$ implies $g = 2$ if $c = 7, 9$, i.e. $\pm 4F_3 = 28$ if $c = 7$ and $\pm 4F_3 = 36$ if $c = 9$;

  iv. $4F_3 = (c+4)\,g^2 \leq 48$ implies $g = 2$ if $c = 7$, i.e. $\pm 4F_3 = -44$ if $c = 7$;

  v. $4F_3 = |11c-100|\,g^2 \leq 48$ implies $g = 2$ if $c = 9$, i.e. $\pm 4F_3 = -4$ if $c = 9$;

  vi. $4F_3 = 36g^2 \leq 48$ implies $g = 1$, i.e. $\pm 4F_3 = 36$ if $c = 9$;

  vii. $4F_3 = 28g^2 \leq 48$ implies $g = 1$, i.e. $\pm 4F_3 = 28$ if $c = 7$;

  viii. $4F_3 = 44g^2 \leq 48$ implies $g = 1$, i.e. $\pm 4F_3 = -44$ if $c = 7$.

- All other cases give a contradiction. Therefore, we get sets $D\,(c)$.

The proof of *ii)* follows directly from *i)* since $S\,(c) = \{(s,t) \in B\,(c) \times D\,(c) : |s| \cdot |t| \leq 48\}$.

□

If system (20), (21) and (22) has a solution for some positive integers F_1, F_2, F_3, $F_1F_2F_3 \leq 12$, then $(\pm F_1, \pm 4F_3) \in S\,(c)$, where the set $S\,(c)$ is given in Proposition 7 and the triple $(\pm F_1, \pm F_2, \pm 4F_3)$ satisfies one of the equations in (24). First, for each pair $(\pm F_1, \pm 4F_3) \in S\,(c)$ we check if there exist $F_2 \in \mathbb{N}$, $F_1F_2F_3 \leq 12$ such that any of the equations (24) holds. For all pairs of the form $(\pm F_1, \pm 4F_3) = (s,t)$ condition $F_1F_2F_3 \leq 12$ is satisfied if $F_2 \in F\,(s,t) = \{k \in \mathbb{N} : k\,|s|\,|t| \leq 48\}$. Therefore, for each pair $(s,t) \in S\,(c)$ and for each $k \in F\,(s,t)$, we have to check if any of these four equations

$$(c+4)\,s + (c-2)\,t = \pm ck \quad \textit{or} \quad (c+4)\,s - (c-2)\,t = \pm ck \tag{27}$$

holds. For example, if $c \geq 7$, then $(\pm F_1, \pm 4F_3) = (-2, -4) \in S\,(c)$. From (27) we obtain

$$-6c = \pm ck \quad \textit{or} \quad 2c - 16 = \pm ck.$$

Since $k \in F\,(-2,-4) = \{1,2,3,4,5,6\}$, the only possibility is $\pm F_2 = -6$. We proceed similarly for every element from set $S\,(c)$, $c \geq 7$. The only triple obtained in this way is $(\pm F_1, \pm F_2, \pm 4F_3) = (-2, -6, -4)$ and the corresponding system is

$$(c-2)\,U^2 - cV^2 = -2 \tag{28}$$
$$(c-2)\,Z^2 - (c+4)\,V^2 = -6 \tag{29}$$
$$cZ^2 - (c+4)\,U^2 = -4. \tag{30}$$

Since this system has solution $(U,V,Z) = (\pm 1, \pm 1, \pm 1)$, we have $\mu\,(c) = 12$ if $c \geq 7$.

The next step is to find all elements with the minimal index. Therefore we have to solve system (28), (29) and (30). It will be done in the next subsection.

4.2. Finding all elements with the minimal index

Now, we have to solve system (28), (29) and (30) obtained in the previous subsection. That system is very suitable for application of the method given in [7]. We will prove the following result

Theorem 4. *Let $c \geq 7$ be an integer. The only solutions to system (28) and (30) are $(U, V, Z) = (\pm 1, \pm 1, \pm 1)$.*

Therefore we have the following corollary which finishes the proof of Theorem 2.

Corollary 3. *Let $c \geq 7$ be an integer such that $c \equiv 1$ or $3 \pmod 6$ and c, $c-2$, $c+4$ are square-free integers. Then all integral elements with the minimal index in field (2) are given by $(x_2, x_3, x_4) = \pm(0,1,1), \pm(0,1,-1), \pm(1,-1,-1), \pm(1,-1,1)$.*

Proof. Since all solutions of system (28), (29) and (30) are given by $(U, V, Z) = (\pm 1, \pm 1, \pm 1)$ and since in this case we have $U = 2x_2 + x_3$, $V = x_4$, $Z = x_3$, we obtain

$$x_4 = \pm 1,\ 2x_2 + x_3 = \pm 1,\ x_3 = \pm 1,$$

which implies $(x_2, x_3, x_4) = \pm(0,1,1), \pm(0,1,-1), \pm(1,-1,-1), \pm(1,-1,1)$. □

In order to prove Theorem 4, first we will find a lower bound for solutions of this system using the "congruence method" introduced in [9]. The comparison of this lower bound with an upper bound obtained from a theorem of Bennett [4] on simultaneous approximations of algebraic numbers finishes the proof for $c \geq 292023$. For $c \leq 292022$ we use a theorem of Baker and Wüstholz [3] and a version of the reduction procedure due to Baker and Davenport [2].

Lemma 3. *Let (U, V, Z) be a positive integer solution of the system of Pellian equations (28) and (30). Then there exist nonnegative integers m and n such that*

$$U = u_m = v_n,$$

where sequences (u_m), (v_n) are given by

$$u_0 = 1,\ u_1 = 2c - 1,\ u_{m+2} = (2c-2)\, u_{m+1} - u_m,\ m \geq 0, \tag{31}$$

$$v_0 = 1,\ v_1 = c + 1,\ v_{n+2} = (c+2)\, v_{n+1} - v_n,\ n \geq 0. \tag{32}$$

Proof. If (U, V) is a solution of equation (28), then there exist $m \geq 0$ such that $U = u_m$, where sequence (u_m) is given by (31) (see [7, Lemma 2]).
Let $Z_1 = cZ$, then equation (30) is equivalent to equation

$$Z_1^2 - c\,(c+4)\, U^2 = -4c. \tag{33}$$

It is obvious that $(a_1, b_1) = (c+2, 1)$ is the fundamental solution of the equation

$$A^2 - c\,(c+4)\, B^2 = 4.$$

By [25, Theorem 2], it follows that if (z_0, v_0) is the fundamental solution of a class of equation (33), then inequalities

$$0 < |z_0| \leq \sqrt{(a_1-2)\cdot c} = c$$
$$0 < v_0 \leq \frac{b_1}{\sqrt{(a_1-2)}}\sqrt{c} = 1$$

must hold. This implies that $(z_0, v_0) = (c, 1)$ and $(z_0', v_0') = (-c, 1)$ are possible fundamental solutions of equation (33). Since

$$z_0 v_0' \equiv z_0' v_0 \pmod{2c},$$

these solutions belong to the same class (see [25, Theorem 4]). Therefore we have only one fundamental solution $(z_0, v_0) = (c, 1)$. Now, all solutions (z, v) of equation (33) in positive integers are given by $(Z_1, U) = (z_n, v_n)$ where

$$z_n + v_n\sqrt{c\,(c+4)} = \left(c + \sqrt{c\,(c+4)}\right)\left(\frac{c+2+\sqrt{c\,(c+4)}}{2}\right)^n \tag{34}$$

and n is a nonnegative integer (see [25, Theorem 3]). From (34) we obtain that if (Z, U) is a solution of equation (28), then there exist $n \geq 0$ such that $U = v_n$, where sequence (v_n) is given by (32). □

Therefore, in order to prove Theorem 4, it suffices to show that $u_m = v_n$ implies $m = n = 0$.

Solving recurrences (31) and (32) we find

$$\begin{aligned} u_m = \frac{1}{2\sqrt{c-2}}\Big[&(\sqrt{c}+\sqrt{c-2})\left(c-1+\sqrt{c(c-2)}\right)^m \\ &-(\sqrt{c}-\sqrt{c-2})\left(c-1-\sqrt{c(c-2)}\right)^m\Big], \end{aligned} \tag{35}$$

$$\begin{aligned} v_n = \frac{1}{2\sqrt{c+4}}\Big[&(\sqrt{c}+\sqrt{c+4})\left(\frac{c+2+\sqrt{c\,(c+4)}}{2}\right)^n \\ &-(\sqrt{c}-\sqrt{c+4})\left(\frac{c+2-\sqrt{c\,(c+4)}}{2}\right)^n\Big]. \end{aligned} \tag{36}$$

4.2.1. Congruence relations

Now we will find a lower bound for nontrivial solutions using the congruence method.

Lemma 4. *Let sequences (u_m) and (v_n) be defined by (31) and (32), respectively. Then for all $m, n \geq 0$ we have*

$$u_m \equiv (-1)^{m-1}\left(m(m+1)c-1\right) \pmod{4c^2}, \tag{37}$$
$$v_n \equiv \frac{n(n+1)}{2}c + 1 \pmod{c^2}\ . \tag{38}$$

Proof. We have obtained congruence (37) in [7, Lemma 3]. Congruence (38) can be proved easily by induction. □

Suppose that m and n are positive integers such that $u_m = v_n$. Then, of course, $u_m \equiv v_n \,(\mathrm{mod}\; c^2)$. By Lemma 4, we have $(-1)^m \equiv 1 \pmod{c}$ and therefore m is even.

Assume that $n(n+1) < \frac{2}{3}c$. Since $m \le n$, we also have $m(m+1) < \frac{2}{3}c$. Furthermore, Lemma 4 implies

$$1 - m(m+1)c \equiv \frac{n(n+1)}{2}c + 1 \pmod{c^2}$$

and

$$-m(m+1) \equiv \frac{n(n+1)}{2} \pmod{c}. \tag{39}$$

Consider the positive integer

$$A = \frac{n(n+1)}{2} + m(m+1).$$

We have $0 < A < c$ and, by (39) $A \equiv 0 \,(\mathrm{mod}\; c)$, a contradiction.

Hence $n(n+1) \ge \frac{2}{3}c$ and it implies $n > \sqrt{0.703c} - 0.5$. Therefore we proved

Proposition 8. *If $u_m = v_n$ and $m \neq 0$, then $n > \sqrt{0.703c} - 0.5$.*

4.2.2. An application of a theorem of Bennett

It is clear that solutions of system (28) and (30) induce good rational approximations to the irrational numbers

$$\theta_1 = \sqrt{\frac{c-2}{c}} \quad \text{and} \quad \theta_2 = \sqrt{\frac{c+4}{c}}\,.$$

More precisely, we have

Lemma 5. *All positive integer solutions (U, V, Z) of the system of Pellian equations (28) and (30) satisfy*

$$|\theta_1 - \frac{V}{U}| < \frac{1}{\sqrt{c\,(c-2)}} \cdot U^{-2}, \quad |\theta_2 - \frac{Z}{U}| < \frac{2}{\sqrt{c\,(c+4)}} \cdot U^{-2}.$$

Proof. We have

$$\begin{aligned}\left|\sqrt{\frac{c-2}{c}} - \frac{V}{U}\right| &= \left|\frac{c-2}{c} - \frac{V^2}{U^2}\right| \cdot \left|\sqrt{\frac{c-2}{c}} + \frac{V}{U}\right|^{-1} \\ &< \frac{2}{cU^2} \cdot \frac{1}{2}\sqrt{\frac{c}{c-2}} = \frac{1}{\sqrt{c\,(c-2)}} \cdot U^{-2}\end{aligned}$$

and

$$\begin{aligned}\left|\sqrt{\frac{c+4}{c}} - \frac{Z}{U}\right| &= \left|\frac{c+4}{c} - \frac{Z^2}{U^2}\right| \cdot \left|\sqrt{\frac{c+4}{c}} + \frac{Z}{U}\right|^{-1} \\ &< \frac{4}{cU^2} \cdot \sqrt{\frac{c}{c+4}} = \frac{4}{\sqrt{c\,(c+4)}} \cdot U^{-2}\end{aligned}$$

□

Numbers θ_1 and θ_2 are square roots of rationals which are very close to 1. For simultaneous Diophantine approximations to such kind of numbers we will use the following theorem of Bennett [4, Theorem 3.2].

Theorem 5. *If a_i, p_i, q and N are integers for $0 \leq i \leq 2$, with $a_0 < a_1 < a_2$, $a_j = 0$ for some $0 \leq j \leq 2$, q nonzero and $N > M^9$, where*

$$M = \max_{0\leq i\leq 2}\{|a_i|\} \geq 3,$$

then we have

$$\max_{0\leq i\leq 2}\left\{\left|\sqrt{1+\frac{a_i}{N}} - \frac{p_i}{q}\right|\right\} > (130N\gamma)^{-1}q^{-\lambda}$$

where

$$\lambda = 1 + \frac{\log(32.04N\gamma)}{\log\left(1.68N^2\prod_{0\leq i<j\leq 2}(a_i-a_j)^{-2}\right)}$$

and

$$\gamma = \begin{cases} \frac{(a_2-a_0)^2(a_2-a_1)^2}{2a_2-a_0-a_1}, & \text{if } a_2-a_1 \geq a_1-a_0, \\ \frac{(a_2-a_0)^2(a_1-a_0)^2}{a_1+a_2-2a_0}, & \text{if } a_2-a_1 < a_1-a_0. \end{cases}$$

We will apply Theorem 5 with $a_0 = -2$, $a_1 = 0$, $a_2 = 4$, $N = c$, $M = 4$, $q = U$, $p_0 = V$, $p_1 = U$, $p_2 = Z$. If $c \geq 262\,145$, then the condition $N > M^9$ is satisfied and we obtain

$$(130 \cdot c \cdot \frac{288}{5})^{-1}U^{-\lambda} < \frac{4}{\sqrt{c\,(c+4)}} \cdot U^{-2}. \tag{40}$$

If $c \geq 281220$, then $2-\lambda > 0$ and (40) imply

$$\log U < \frac{10.082}{2-\lambda}. \tag{41}$$

Furthermore,

$$\frac{1}{2-\lambda} = \frac{1}{1-\frac{\log(32.04\cdot c\frac{288}{5})}{\log(1.68c^2\frac{1}{256})}} < \frac{\log\left(0.00657c^2\right)}{\log(0.00000355c)}.$$

On the other hand, from (36) we find that

$$v_n > 0.88\Big(\frac{c+2+\sqrt{c\,(c+4)}}{2}\Big)^n > (0.88c+0.88)^n,$$

and Proposition 8 implies that if $(m,n) \neq (0,0)$, then

$$U > (0.88c+0.88)^{\sqrt{0.703c}-0.5}.$$

Therefore,

$$\log U > (\sqrt{0.703c}-0.5)\log(0.88c+0.88). \tag{42}$$

Combining (41) and (42) we obtain

$$\sqrt{0.703c} - 0.5 < \frac{10.082 \log\left(0.00657c^2\right)}{\log(0.88c + 0.88)\log(0.00000355c)} \tag{43}$$

and (43) yields a contradiction if $c \geq 292023$. Therefore we proved

Proposition 9. *If c is an integer such that $c \geq 292023$, then the only solution of the equation $u_m = v_n$ is $(m, n) = (0, 0)$.*

4.2.3. The Baker-Davenport method

In this section we will apply the so called Baker-Davenport reduction method in order to prove Theorem 4 for $7 \leq c \leq 292022$.

Lemma 6. *If $u_m = v_n$ and $m \neq 0$, then*

$$\begin{aligned} 0 &< m \log\left(c - 1 + \sqrt{c(c-2)}\right) - n \log\left(\frac{c + 2 + \sqrt{c\,(c+4)}}{2}\right) \\ &\quad + \log \frac{\sqrt{c+4}(\sqrt{c} + \sqrt{c-2})}{\sqrt{c-2}(\sqrt{c} + \sqrt{c+4})} \\ &< 0.23912 \left(\frac{c + 2 + \sqrt{c\,(c+4)}}{2}\right)^{-2n}. \end{aligned}$$

Proof. In standard way (for e.g. see [7, Lemma 5]). □

Now we will apply the following theorem of Baker and Wüstholz [3]:

Theorem 6. *For a linear form $\Lambda \neq 0$ in logarithms of l algebraic numbers $\alpha_1, \ldots, \alpha_l$ with rational integer coefficients $b_1, \ldots, b_l$ we have*

$$\log \Lambda \geq -18(l+1)!\, l^{l+1} (32d)^{l+2} h'(\alpha_1) \cdots h'(\alpha_l) \log(2ld) \log B\,,$$

where $B = \max\{|b_1|, \ldots, |b_l|\}$, and where d is the degree of the number field generated by $\alpha_1, \ldots, \alpha_l$.

Here

$$h'(\alpha) = \frac{1}{d} \max\{h(\alpha), |\log\,\alpha|, 1\}\,,$$

and $h(\alpha)$ denotes the standard logarithmic Weil height of α. We will apply Theorem 6 to the form from Lemma 6. We have $l = 3$, $d = 4$, $B = n$,

$$\begin{aligned} \alpha_1 &= c - 1 + \sqrt{c(c-2)}, \\ \alpha_2 &= \frac{c + 2 + \sqrt{c\,(c+4)}}{2}, \\ \alpha_3 &= \frac{\sqrt{c+4}(\sqrt{c} + \sqrt{c-2})}{\sqrt{c-2}(\sqrt{c} + \sqrt{c+4})}. \end{aligned}$$

Under the assumption that $7 \le c \le 292022$ we find that

$$h'(\alpha_1) = \frac{1}{2}\log\alpha_1 < \frac{1}{2}\log 2c, \quad h'(\alpha_2) = \frac{1}{2}\log\alpha_2 < 6.2924.$$

Furthermore, $\alpha_3 < 1.2145$, and the conjugates of α_3 satisfy

$$\begin{aligned} |\alpha_3'| &= \frac{\sqrt{c+4}(\sqrt{c}-\sqrt{c-2})}{\sqrt{c-2}(\sqrt{c}+\sqrt{c+4})} < 1, \\ |\alpha_3''| &= \frac{\sqrt{c+4}(\sqrt{c}+\sqrt{c-2})}{\sqrt{c-2}(\sqrt{c+4}-\sqrt{c})} < 292025.51 \\ |\alpha_3'''| &= \frac{\sqrt{c+4}(\sqrt{c}-\sqrt{c-2})}{\sqrt{c-2}(\sqrt{c+4}-\sqrt{c})} < 1. \end{aligned}$$

Therefore,

$$h'(\alpha_3) < \frac{1}{4}\log\left[16\,(c-2)^2 \cdot 1.2145 \cdot 292025.51\right] < 10.181.$$

Finally,

$$\log\left[0.239\,12\left(\frac{c+2+\sqrt{c\,(c+4)}}{2}\right)^{-2n}\right] < -2n\log(2c)\,.$$

Hence, Theorem 6 implies

$$2n\log(2c) < 3.822 \cdot 10^{15} \cdot \frac{1}{2} \cdot \log(2c) \cdot 6.2924. \cdot 10.181\log n$$

and

$$\frac{n}{\log n} < 6.12122 \cdot 10^{16}. \tag{44}$$

which implies $n < 2.59542 \times 10^{18}$.

We may reduce this large upper bound using a variant of the Baker-Davenport reduction procedure [2]. The following lemma is a slight modification of [9, Lemma 5 a)]:

Lemma 7. *Assume that M is a positive integer. Let p/q be a convergent of the continued fraction expansion of κ such that $q > 10M$ and let $\varepsilon = \|\mu q\| - M \cdot \|\kappa q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality*

$$0 < m - n\kappa + \mu < AB^{-n}$$

in integers m and n with

$$\frac{\log(Aq/\varepsilon)}{\log B} \le n \le M\,.$$

We apply Lemma 7 with $\kappa = \frac{\log\alpha_2}{\log\alpha_1}, \mu = \frac{\log\alpha_3}{\log\alpha_1}, A = \frac{0.239\,12}{\log\alpha_1}, B = \left(\frac{c+2+\sqrt{c(c+4)}}{2}\right)^2$ and $M = 2.59542 \times 10^{18}$. If the first convergent such that $q > 10M$ does not satisfy the condition $\varepsilon > 0$, then we use the next convergent.

We performed the reduction from Lemma 7 for $7 \le c \le 292022$. The use of the second convergent was necessary in 3686 cases ($\approx 3.63\%$), the third convergent was used in 209 cases ($\approx 0.07\%$), the fourth in 37 cases, the fifth convergent is used in only one case: $c = 169901$. In all cases we obtained $n \le 7$. More precisely, we obtained $n \le 7$ for $c \ge 7$; $n \le 6$ for $c \ge 9$; $n \le 5$ for $c \ge 14$; $n \le 4$ for $c \ge 57$; $n \le 3$ for $c \ge 144$; $n \le 2$ for $c \ge 1442$. The next step of the reduction in all cases gives $n \le 1$, which completes the proof.

Therefore, we proved

Proposition 10. *If c is an integer such that $7 \le c \le 292022$, then the only solution of the equation $u_m = v_n$ is $(m, n) = (0, 0)$.*

Proof of Theorem 4: The statement follows directly from Propositions 9 and 10.

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