Math. Commun., Vol. **14**, No. 2, pp. 365-378 (2009)

Common fixed points of generalized contractive multivalued mappings in cone metric spaces

Mujahid Abbas[1,*], B. E. Rhoades[2] and Talat Nazir[1]

[1] *Department of Mathematics, Lahore University of Management Sciences, Lahore-54 792, Pakistan.*
[2] *Department of Mathematics, Indiana University, Bloomington, Indiana-47 405, U. S. A.*

Received July 21, 2009; accepted September 20, 2009

Abstract. In this paper we obtain some sufficient conditions for the existence of common fixed points of multivalued mappings satisfying generalized contractive conditions in non normal cone metric spaces. These results establish some of the most general common fixed point theorems for two multivalued maps in cone metric spaces.

AMS subject classifications: 47H10, 54H25, 54C60, 46B40

Key words: common fixed point, multivalued mapping, cone metric spaces, non normal cone

1. Introduction and premilinaries

The study of fixed points for multivalued contractions and nonexpansive maps using the Hausdorff metric was initiated by Markin [9]. Later, an interesting and rich fixed point theory for such maps was developed. The theory of multivalued maps has application in control theory, convex optimization, differential equations and economics. Recently, Huang and Zhang [4] introduced the concept of a cone metric space, replacing the set of positive real numbers by an ordered Banach space. They obtained some fixed point theorems in cone metric spaces using the normality of the cone, which induces an order in Banach spaces (see also, [1], [2]). Rezapour and Hamlbarani Haghi [10] showed the existence of a non normal cone metric space and also obtained some fixed point theorems in cone metric spaces (see also [5], [7], [11], [12], [13] and [16]). Wardowski [15] introduced the concept of multivalued contractions in cone metric spaces and, using the notion of normal cones, obtained fixed point theorems for such mappings. The aim of this paper is to prove some common fixed points results for multivalued mappings taking closed values in cone metric spaces. It is worth mentioning that our results do not require the assumption of a normal cone. Our results extend and unify various comparable results in the literature ([6], [8] and [14]).

Let E be a topological vector space. A subset P of E is called a cone if and only if:

*Corresponding author. *Email addresses:* `mujahid@lums.edu.pk` (M. Abbas), `rhoades@indiana.edu` (B. E. Rhoades), `talat@lums.edu.pk` (T. Nazir)

(a) P is closed, nonempty and $P \neq \{0\}$;

(b) $a, b \in R$, $a, b \geq 0$, $x, y \in P$ imply that $ax + by \in P$;

(c) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. A cone P is said to be normal in a normed space E if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K \|y\|.$$

The least positive number satisfying the above inequality is called the normal constant of P, while $x \ll y$ stands for $y - x \in intP$ (interior of P).
Rezapour and Hamlbarani Haghi [10] proved that there is no normal cone with normal constant $K < 1$ and for each $k > 1$ there is a cone with normal constant $K > k$.

Definition 1. *Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies:*

(d1) $0 \leq d(x, y)$ *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$;

(d2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

The concept of a cone metric space is more general than that of a metric space.

Definition 2. *Let (X, d) be a cone metric space, $\{x_n\}$ a sequence in X and $x \in X$. For every $c \in E$ with $0 \ll c$, we say that $\{x_n\}$ is*

(i) a Cauchy sequence if there is a natural number N such that for all $n, m > N$, $d(x_n, x_m) \ll c$.

(ii) a convergent sequence if there is a natural number N such that for all $n > N$, $d(x_n, x) \ll c$ for some x in X.

Remark 1. *If $a \leq ha$, for some $a \in P$ and $h \in (0, 1)$, then $a = 0$. A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. It is known that $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$. A set A in a cone metric space X is closed if for every sequence $\{x_n\}$ in A which converges to some x in X implies that $x \in A$.*

Let X be a cone metric space. We denote by $P(X)$ the family of all nonempty subsets of X, and by $P_{cl}(X)$ the family of all nonempty closed subsets of X. A point x in X is called a fixed point of a multivalued mapping $T : X \to P_{cl}(X)$ provided $x \in Tx$. The collection of all fixed points of T is denoted by $F(T)$.

2. Common fixed point results

Kannan [6] proved a fixed point theorem for a single valued self mapping T of a metric space X satisfying the property

$$d(Tx,Ty) \le h\{d(x,Tx) + d(y,Ty)\}$$

for all x, y in X and for a fixed $h \in [0, \frac{1}{2})$. Latif and Beg [8] introduced the notion of a $K-$ multivalued mapping, which is the extension of Kannan mappings to multivalued mappings. Recently, Rus [14] coined the term $R-$ multivalued mapping, which is a generalization of a $K-$ multivalued mapping.
In this section we obtain common fixed point theorems for two multivalued mappings on a cone metric space without using the condition of a normal cone.
Given the fact that in a cone one has only a partial ordering, it is doubtful whether the following theorem can be further generalized.

Theorem 1. *Let (X,d) be a complete cone metric space and $T_1, T_2 : X \to P_{cl}(X)$ two multivalued mappings such that for $i, j \in \{1,2\}$ with $i \neq j$ and for each $x, y \in X$, $u_x \in T_i(x)$, there exists $u_y \in T_j(y)$ such that*

$$d(u_x, u_y) \le hu(x, y; u_x, u_y), \tag{1}$$

where $h \in (0,1)$ is a constant and

$$u(x,y;u_x,u_y) \in \{d(x,y), d(x,u_x), d(y,u_y), \frac{d(x,u_x)+d(y,u_y)}{2}, \frac{d(x,u_y)+d(y,u_x)}{2}\}.$$

Then $F(T_1) = F(T_2) \neq \emptyset$. Also $F(T_1) = F(T_2) \in P_{cl}(X)$.

Proof. Let $x^* \in T_1(x^*)$. Then there exists an $x \in T_2(x^*)$ such that

$$d(x^*, x) \le hu(x^*, x^*; x^*, x)$$

where

$$\begin{aligned} u(x^*,x^*;x^*,x) \in\ & \{d(x^*,x^*), d(x^*,x^*), d(x,x^*), \\ & \frac{d(x^*,x^*)+d(x^*,x)}{2}, \frac{d(x^*,x)+d(x^*,x^*)}{2}\} \\ =\ & \{0, d(x,x^*), \frac{d(x^*,x)}{2}\}. \end{aligned}$$

Now $u(x^*,x^*;x^*,x) = 0$ implies that $x^* = x$ and, $u(x^*,x^*;x^*,x) = d(x^*,x)$ gives $d(x^*,x) \le hd(x^*,x)$, which by Remark 1 implies that $x^* = x$. Similarly, for $u(x^*,x^*;x^*,x) = d(x^*,x)/2$, we obtain $x^* = x$. Thus $F(T_1) \subseteq F(T_2)$. Also, $F(T_2) \subseteq F(T_1)$ and therefore $F(T_1) = F(T_2)$.
Suppose that x_0 is an arbitrary point of X. For $i, j \in \{1,2\}$ with $i \neq j$ and $x_1 \in T_i(x_0)$, there exists $x_2 \in T_j(x_1)$ such that

$$d(x_1, x_2) \le hu(x_0, x_1; x_1, x_2),$$

where

$$\begin{aligned} u(x_0,x_1;x_1,x_2) &\in \{d(x_0,x_1),d(x_0,x_1),d(x_1,x_2),\frac{d(x_0,x_1)+d(x_1,x_2)}{2},\\ &\quad \frac{d(x_0,x_2)+d(x_1,x_1)}{2}\}\\ &= \{d(x_0,x_1),d(x_1,x_2),\frac{d(x_0,x_1)+d(x_1,x_2)}{2},\frac{d(x_0,x_2)}{2}\}. \end{aligned}$$

Now, $u(x_0,x_1;x_1,x_2)=d(x_0,x_1)$ implies that $d(x_1,x_2)\le hd(x_0,x_1)$. If

$$u(x_0,x_1;x_1,x_2)=d(x_1,x_2)$$

then $d(x_1,x_2)\le hd(x_1,x_2)$, which by Remark 1, implies that $x_1=x_2$. Also, if

$$u(x_0,x_1;x_1,x_2)=\frac{d(x_0,x_1)+d(x_1,x_2)}{2},$$

then we obtain

$$\begin{aligned} d(x_1,x_2) &\le \frac{h}{2}d(x_0,x_1)+\frac{h}{2}d(x_1,x_2)\\ &\le \frac{h}{2}d(x_0,x_1)+\frac{1}{2}d(x_1,x_2) \end{aligned}$$

and $d(x_1,x_2)\le hd(x_0,x_1)$. Finally, for $u(x_0,x_1;x_1,x_2)=\dfrac{d(x_0,x_2)}{2}$ we get

$$\begin{aligned} d(x_1,x_2) &\le \frac{h}{2}d(x_0,x_2)\le \frac{h}{2}d(x_0,x_1)+\frac{h}{2}d(x_1,x_2)\\ &\le \frac{h}{2}d(x_0,x_1)+\frac{1}{2}d(x_1,x_2), \end{aligned}$$

which also implies that $d(x_1,x_2)\le hd(x_0,x_1)$. Continuing this process, for $x_{2n}\in T_j(x_{2n-1})$, there exists $x_{2n+1}\in T_i\,(x_{2n})$ such that

$$d(x_{2n},x_{2n+1})\le hu(x_{2n-1},x_{2n};x_{2n},x_{2n+1}),$$

where

$$\begin{aligned} u(x_{2n-1},x_{2n};x_{2n},x_{2n+1}) &\in \{d(x_{2n-1},x_{2n}),d(x_{2n-1},x_{2n}),\\ &\quad d(x_{2n},x_{2n+1}),\frac{d(x_{2n-1},x_{2n})+d(x_{2n},x_{2n+1})}{2},\\ &\quad \frac{d(x_{2n-1},x_{2n+1})+d(x_{2n},x_{2n})}{2}\}\\ &= \{d(x_{2n-1},x_{2n}),d(x_{2n},x_{2n+1}),\\ &\quad \frac{d(x_{2n-1},x_{2n})+d(x_{2n},x_{2n+1})}{2},\frac{d(x_{2n-1},x_{2n+1})}{2}\}. \end{aligned}$$

If $u(x_{2n-1},x_{2n};x_{2n},x_{2n+1})=d(x_{2n-1},x_{2n})$, then $d(x_{2n},\,x_{2n+1})\le h\;d(x_{2n-1},\,x_{2n})$. For $u(x_{2n-1},x_{2n};x_{2n},x_{2n+1})=d(x_{2n},x_{2n+1})$, $d(x_{2n},\;x_{2n+1})\le h\;d(x_{2n},x_{2n+1})$, which by Remark 1 gives $x_{2n}=x_{2n+1}$. When

$$u(x_{2n-1},x_{2n};x_{2n},x_{2n+1})=\frac{d(x_{2n-1},x_{2n})+d(x_{2n},x_{2n+1})}{2},$$

we obtain

$$d(x_{2n},x_{2n+1}) \leq \frac{h}{2}[d(x_{2n-1},x_{2n})+d(x_{2n},x_{2n+1})] \\ \leq \frac{h}{2}d(x_{2n-1},x_{2n})+\frac{1}{2}d(x_{2n},x_{2n+1})$$

and

$$d(x_{2n},x_{2n+1}) \leq hd(x_{2n-1},x_{2n}).$$

Finally, $u(x_{2n-1},x_{2n};x_{2n},x_{2n+1}) = d(x_{2n-1},x_{2n+1})/2$ gives that

$$d(x_{2n},x_{2n+1}) \leq \frac{h}{2}d(x_{2n-1},x_{2n+1}) \leq \frac{h}{2}[d(x_{2n-1},x_{2n})+d(x_{2n},x_{2n+1})] \\ \leq \frac{h}{2}d(x_{2n-1},x_{2n})+\frac{1}{2}d(x_{2n},x_{2n+1})$$

and

$$d(x_{2n},x_{2n+1}) \leq hd(x_{2n-1},x_{2n}).$$

In a similar manner, for $x_{2n+1} \in T_i(x_{2n})$, there exists $x_{2n+2} \in T_i\,(x_{2n+1})$ so that

$$d(x_{2n+1},x_{2n+2}) \leq hd(x_{2n},x_{2n+1}).$$

Therefore

$$d(x_n,x_{n+1}) \leq hd(x_{n-1},x_n) \leq h^2 d(x_{n-2},x_{n-2}) \\ \leq .... \leq h^n d(x_0,x_1)$$

for all $n \geq 1$ and so for $m > n$ we have $d(x_n,x_m) \leq h^n d(x_0,x_1)/(1-h)$.

Let $0 \ll c$ be given. Choose a symmetric open neighborhood V of 0 such that $c+V \subseteq P$. Also, choose a natural number N_1 such that $h^n d(x_0,x_1)/(1-h) \in V$ for all $n \geq N_1$ which implies that $h^n d(x_0,x_1)/(1-h) \ll c$ for all $n > N_1$. Hence $d(x_n,x_m) \ll c$ for all $n,m > N_1$. Therefore $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists an element $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Let $0 \ll c$ be given and $0 < \delta < \min\{\frac{1}{4}, 1-h\}$. Choose a natural number N such that $d(x_m,x^*) \ll \delta c$ for all $m \geq N$. Let $n \geq N$ be given. Then, for $x_{2n} \in T_j(x_{2n-1})$, there exists $u_n \in T_i(x^*)$ such that

$$d(x_{2n},u_n) \leq hu(x_{2n-1},x^*;x_{2n},u_n),$$

where

$$u(x_{2n},x^*;x_{2n+1},u_n) \in \{d(x_{2n-1},x^*), d(x_{2n-1},x_{2n}), d(x^*,u_n), \\ \frac{d(x_{2n-1},x_{2n})+d(x^*,u_n)}{2}, \frac{d(x_{2n-1},u_n)+d(x^*,x_{2n})}{2}\}.$$

Note that

$$d(u_n,x^*) \leq d(u_n,x_{2n})+d(x_{2n},x^*) \\ \leq hu(x_{2n},x^*;x_{2n+1},u_n)+d(x_{2n},x^*).$$

Now, $u(x_{2n}, x^*; x_{2n+1}, u_n) = d(x_{2n-1}, x^*)$ implies that

$$\begin{aligned} d(u_n, x^*) &\leq hd(x_{2n-1}, x^*) + d(x_{2n}, x^*) \\ &\ll h(\frac{c}{2h}) + \frac{c}{2} = c. \end{aligned}$$

If $u(x_{2n}, x^*; x_{2n+1}, u_n) = d(x_{2n-1}, x_{2n})$, then

$$\begin{aligned} d(u_n, x^*) &\leq hd(x_{2n-1}, x_{2n}) + d(x_{2n}, x^*) \\ &\leq hd(x_{2n-1}, x^*) + hd(x^*, x_{2n}) + d(x_{2n}, x^*) \\ &\leq h(\frac{c}{3h}) + h(\frac{c}{3h}) + \frac{c}{3} = c. \end{aligned}$$

In case $u(x_{2n}, x^*; x_{2n+1}, u_n) = d(x^*, u_n)$, then

$$d(u_n, x^*) \leq hd(x^*, u_n) + d(x_{2n}, x^*)$$

and so

$$d(u_n, x^*) \leq \frac{1}{1-h} d(x_{2n}, x^*) \ll c.$$

If $u(x_{2n}, x^*; x_{2n+1}, u_n) = \dfrac{d(x_{2n-1}, x_{2n}) + d(x^*, u_n)}{2}$, we get

$$\begin{aligned} d(u_n, x^*) &\leq \frac{h}{2}[d(x_{2n-1}, x_{2n}) + d(x^*, u_n)] + d(x_{2n}, x^*) \\ &\leq \frac{h}{2}[d(x_{2n-1}, x^*) + d(x^*, x_{2n})] + \frac{1}{2}d(x^*, u_n) + d(x_{2n}, x^*) \end{aligned}$$

and so

$$\begin{aligned} d(u_n, x^*) &\leq h[d(x_{2n-1}, x^*) + d(x^*, x_{2n})] + 2d(x^*, x_{2n}) \\ &\ll c. \end{aligned}$$

Finally, if $u(x_{2n}, x^*; x_{2n+1}, u_n) = \dfrac{d(x_{2n-1}, u_n) + d(x^*, x_{2n})}{2}$, then

$$\begin{aligned} d(u_n, x^*) &\leq \frac{h}{2}[d(x_{2n-1}, u_n) + d(x^*, x_{2n})] + d(x_{2n}, x^*) \\ &\leq \frac{h}{2}[d(x_{2n-1}, x^*) + d(x^*, u_n)] + \frac{h}{2}d(x^*, x_{2n}) + d(x_{2n}, x^*) \\ &\leq \frac{1}{2}d(x_{2n-1}, x^*) + \frac{1}{2}d(x^*, u_n) + \frac{3}{2}d(x_{2n}, x^*) \end{aligned}$$

and so

$$\begin{aligned} d(u_n, x^*) &\leq d(x_{2n-1}, x^*) + 3d(x_{2n}, x^*) \\ &\ll c. \end{aligned}$$

Thus $u_n \to x^*$ as $n \to \infty$. Since $T_i(x^*)$ is closed, $x^* \in F(T_j) = F(T_i)$.

Now, we prove that $F(T_i)$ is closed. Let $\{p_n\}$ be a sequence in $F(T_j) = F(T_i)$ such that $p_n \to p$ as $n \to \infty$. Since $p_n \in T_i(p_n)$, there exists $q_n \in T_j(p)$ such that

$$d(p_n, q_n) \leq hu(p_n, p; p_n, q_n),$$

where

$$\begin{aligned} u(p_n, p; p_n, q_n) \in \; & \{d(p_n, p), d(p_n, p_n), d(p, q_n), \frac{d(p_n, p_n) + d(p, q_n)}{2}, \\ & \frac{d(p_n, q_n) + d(p, p_n)}{2}\} \\ = \; & \{d(p_n, p), 0, d(p, q_n), \frac{d(p, q_n)}{2}, \frac{d(p_n, q_n) + d(p, p_n)}{2}\}. \end{aligned}$$

Now we show that $q_n \to p$ as $n \to \infty$. Let $0 \ll c$ be given and $0 < \delta < \min\{\frac{1}{4}, 1-h\}$. Choose a natural number N_2 such that $d(p_m, p) \ll \delta c$ for all $m \geq N_2$ is given. If $u(p_n, p; p_n, q_n) = d(p_n, p)$ for some n, then

$$\begin{aligned} d(q_n, p) & \leq d(q_n, p_n) + d(p_n, p) \\ & \leq hd(p_n, p) + d(p_n, p) \\ & \ll c. \end{aligned}$$

If $u(p_n, p; p_n, q_n) = 0$, then our claim follows immediately. If $u(p_n, p; p_n, q_n) = d(p, q_n)$, then

$$\begin{aligned} d(q_n, p) & \leq d(q_n, p_n) + d(p_n, p) \\ & \leq hd(p, q_n) + d(p_n, p) \end{aligned}$$

and so

$$d(q_n, p) \leq \frac{1}{1-h} d(p_n, p) \ll c.$$

If $u(p_n, p; p_n, q_n) = d(p, q_n)/2$, then

$$\begin{aligned} d(q_n, p) & \leq d(q_n, p_n) + d(p_n, p) \\ & \leq \frac{h}{2} d(p, q_n) + d(p_n, p) \\ & \leq \frac{d(p, q_n)}{2} + d(p_n, p) \end{aligned}$$

and so

$$d(q_n, p) \leq 2d(p_n, p) \ll c.$$

If $u(p_n, p; p_n, q_n) = [d(p_n, q_n) + d(p, p_n)]/2$, then

$$\begin{aligned} d(q_n, p) & \leq d(q_n, p_n) + d(p_n, p) \\ & \leq \frac{h}{2}[d(p_n, q_n) + d(p, p_n)] + d(p_n, p) \\ & \leq \frac{h}{2}[d(p_n, p) + d(p, q_n) + d(p, p_n)] + d(p_n, p) \\ & \leq \frac{1}{2} d(p, q_n) + 2d(p, p_n) \end{aligned}$$

and so

$$d(q_n, p) \leq 4d(p_n, p) \ll c.$$

Thus $q_n \to p$ as $n \to \infty$. Since $q_n \in T_j(p)$ for each $n \geq 1$ and $T_j(p)$ is closed, $p \in T_j(p)$. Therefore, $F(T_j) = F(T_i) \in P_{cl}(X)$. □

The following theorem generalizes [14, Theorem 3.4] to cone metric spaces.

Theorem 2. *Let (X, d) be a complete cone metric space and $T_1, T_2 : X \to P_{cl}(X)$ two multivalued mappings such that for $i, j \in \{1, 2\}$ with $i \neq j$ and for each $x, y \in X$, $u_x \in T_i(x)$, there exists $u_y \in T_j(y)$ such that*

$$d(u_x, u_y) \leq \alpha d(x, y) + \beta d(x, u_x) + \gamma d(y, u_y), \tag{2}$$

where $\alpha, \beta, \gamma \geq 0$ and $\alpha+\beta+\gamma < 1$. Then $F(T_1) = F(T_2) \neq \phi$ and $F(T_1) = F(T_2) \in P_{cl}(X)$.

Proof. Suppose that x_0 is an arbitrary point of X. For $i, j \in \{1, 2\}$ with $i \neq j$, take $x_1 \in T_i(x_0)$. Then there exists $x_2 \in T_j(x_1)$ such that

$$d(x_1, x_2) \leq \alpha d(x_0, x_1) + \beta d(x_0, x_1) + \gamma d(x_1, x_2),$$

which implies that

$$d(x_1, x_2) \leq kd(x_0, x_1),$$

where $0 < k = (\alpha + \beta)/(1 - \gamma) < 1$. Now for $x_2 \in T_j(x_1)$ there exists $x_3 \in T_i(x_2)$ such that $d(x_2, x_3) \leq kd(x_1, x_2)$. Continuing this process we obtain a sequence $\{x_n\}$ in X with $x_{2n-1} \in T_i(x_{2n-2})$, $x_{2n} \in T_j(x_{2n-1})$ such that $d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n)$ which further implies that $d(x_n, x_{n+1}) \leq k^n d(x_0, x_1)$ for all $n \geq 1$ Then, for $m > n$,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m+1}, x_m) \\ &\leq [k^n + k^{n+1} + ... + k^{m-1}]d(x_1, x_0) \\ &\leq k^n d(x_0, x_1)/(1 - k). \end{aligned}$$

Let $0 \ll c$ be given. Choose a symmetric open neighborhood V of 0 such that $c + V \subseteq P$. Also, choose a natural number N_1 such that $k^n d(x_0, x_1)/(1 - k) \in V$ for all $n \geq N_1$ which implies that $k^n d(x_0, x_1)/(1 - k) \ll c$ for all $n > N_1$ and hence $d(x_n, x_m) \ll c$ for all $n, m > N_1$. Therefore $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists an element $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Let $0 \ll c$ be given. Now for $x_{2n} \in T_j(x_{2n-1})$, there exists $u_n \in T_i(x^*)$ such that

$$d(x_{2n}, u_n) \leq \alpha d(x_{2n-1}, x^*) + \beta d(x_{2n-1}, x_{2n}) + \gamma d(x^*, u_n),$$

which further gives

$$d(x^*, u_n) \leq d(x^*, x_{2n}) + \alpha d(x_{2n-1}, x^*) + \beta d(x_{2n-1}, x_{2n}) + \gamma d(x^*, u_n),$$

and so

$$d(x^*, u_n) \leq \frac{1}{1-\gamma}[d(x^*, x_{2n}) + \alpha d(x_{2n-1}, x^*) + \beta d(x_{2n-1}, x_{2n})] \ll c$$

for a sufficiently large n which shows that $u_n \to x^*$ as $n \to \infty$. Since $T_i(x^*)$ is closed, $x^* \in F(T_i)$ and so $F(T_i) \neq \phi$. Let $x^* \in X$ be a fixed point of T_1. Then, by hypothesis, there exists $x \in T_2 x^*$ such that

$$\begin{aligned} d(x^*, x) &\leq \alpha d(x^*, x^*) + \beta d(x^*, x^*) + \gamma d(x, x^*) \\ &= \gamma d(x, x^*), \end{aligned}$$

which by using Remark1, implies that $d(x^*, x) = 0$, and so $x^* = x$. Thus, $F(T_1) \subseteq F(T_2)$. Similarly, $F(T_2) \subseteq F(T_1)$.
Now we prove that $F(T_i)$ is closed. Let $\{x_n\}$ be a sequence in $F(T_j) = F(T_i)$ such that $x_n \to x$ as $n \to \infty$. Since $x_n \in T_i(x_n)$, there exists $v_n \in T_j(x)$ such that

$$d(x_n, v_n) \leq \alpha d(x_n, x) + \beta d(x_n, x_n) + \gamma d(x, v_n),$$

and so

$$d(x, v_n) \leq d(x, x_n) + \alpha d(x_n, x) + \gamma d(x, v_n).$$

Thus,

$$d(x, v_n) \leq \frac{1+\alpha}{1-\gamma} d(x, x_n) \ll c$$

for a sufficiently large n. Thus $d(x, v_n) \to 0$ as $n \to \infty$. Since $v_n \in T_j(x)$ for each $n \in \mathbb{N}$ and $T_j(x)$ is closed, $x \in T_j(x)$. Hence, $x \in F(T_j) = F(T_i)$. $\square$

Example 1. *Let $X = [0,1]$, $E = \mathbb{R}^2$ and $P = \{(x, y) \in E : x, y \geq 0\}$. Let $d : X \times X \to E$ be defined by*

$$d(x, y) = (|x-y|, h|x-y|),$$

where $h \geq 0$. Define T_1, $T_2 : X \to P_{cl}(X)$ by

$$T_1 x = [0, \frac{x}{4}] \text{ and } T_2 x = [0, \frac{x}{3}].$$

Note that for $x = y = 0$, (2) is satisfied as $u_x = u_y = 0$. For $x = y \neq 0$ and $u_x \in T_1 x$, take $u_y = 0$. Then

$$\begin{aligned} d(u_x, u_y) &= (u_x, h u_x) \leq (\frac{x}{4}, h\frac{x}{4}) \\ &\leq \frac{1}{6}(0,0) + \frac{2}{6}(\frac{3x}{4}, h\frac{3x}{4}) + \frac{2}{6}(x, hx) \\ &\leq \frac{1}{6}(x-y, h(x-y)) + \frac{2}{6}(x-u_x, h(x-u_x)) + \frac{2}{6}(x, hx) \\ &= \alpha d(x, y) + \beta d(x, u_x) + \gamma d(y, u_y), \end{aligned}$$

and so (2) is satisfied with $\alpha = 1/6$, $\beta = \gamma = 2/6$. Now when $x = 0$, $y \neq 0$, then (2) is satisfied for $u_y = 0$. If $x \neq 0$, $y = 0$, then since $u_y = 0$, for any $u_x \in T_1 x$ we have

$$\begin{aligned} d(u_x, u_y) &\leq (\frac{x}{4}, h\frac{x}{4}) \\ &\leq \alpha d(x, y) + \beta d(x, u_x) + \gamma d(y, u_y). \end{aligned}$$

Since $d(x,y)$ is symmetric in x and y and $\beta = \gamma$, it is sufficient to consider $0 < x < y$, $u_x \in T_1x$. Take $u_y = 0$. Then

$$\begin{aligned} d(u_x,u_y) &= (u_x,hu_x) \leq (\frac{x}{4},h\frac{x}{4}) \\ &\leq \frac{1}{6}(0,0)+\frac{2}{6}(\frac{3x}{4},h\frac{3x}{4})+\frac{2}{6}(y,hy) \\ &\leq \alpha d(x,y)+\beta d(x,u_x)+\gamma d(y,u_y). \end{aligned}$$

Now we show that for $x, y \in X$, $u_x \in T_2x$, there exists $u_y \in T_1y$ such that (2) is satisfied. For $x = y = 0$, (2) is satisfied as $u_x = u_y = 0$. For $x = y \neq 0$, $u_x \in T_2x$, take $u_y = 0$. Then we have

$$\begin{aligned} d(u_x,u_y) &= (u_x,hu_x) \leq (\frac{x}{3},h\frac{x}{3}) \\ &\leq \frac{1}{6}(0,0)+\frac{2}{6}(\frac{2x}{3},h\frac{2x}{3})+\frac{2}{6}(x,hx) \\ &\leq \alpha d(x,y)+\beta d(x,u_x)+\gamma d(y,u_y). \end{aligned}$$

Now when $x = 0$, $y \neq 0$, then (2) is satisfied for $u_y = 0$. If $x \neq 0$, $y = 0$, then for any $u_x \in T_2x$, we have $u_y = 0$, and

$$\begin{aligned} d(u_x,u_y) &= d(u_x,hu_x) \leq (\frac{x}{3},h\frac{x}{3}) \\ &\leq \alpha d(x,y)+\beta d(x,u_x)+\gamma d(y,u_y). \end{aligned}$$

For $0 < x < y$, $u_x \in T_2x$, take $u_y = 0$. Then we have

$$\begin{aligned} d(u_x,u_y) &= (u_x,hu_x) \leq (\frac{x}{3},h\frac{x}{3}) \\ &\leq \frac{1}{6}(0,0)+\frac{2}{6}(\frac{2x}{3},h\frac{2x}{3})+\frac{2}{6}(y,hy) \\ &\leq \alpha d(x,y)+\beta d(x,u_x)+\gamma d(y,u_y), \end{aligned}$$

with $\alpha+\beta+\gamma = 5/6$. Note that $F(T_1) = F(T_2) \neq \phi$. Moreover, $F(T_1) = F(T_2) \in P_{cl}(X)$.

Now we present another example with a different topological vector space as a range.

Example 2. *Let $E = C_R[0,\infty)$, $P = \{f \in E : f(x) \geq 0,\ x \in [0,\infty)\}$, $X = [0,1]$ with a usual metric and with a cone metric $d : X \times X \to E$ defined by $d(x,y) = f_{x,y}$, where $f_{x,y}(t) = t\,|x-y|$ ([3]). Define $T_1, T_2 : X \to P_{cl}(X)$ as*

$$T_1x = [\frac{x}{7},\frac{x}{5}] \text{ and } T_2x = [\frac{x}{6},\frac{x}{3}].$$

Now if $x = y$, then (2) is satisfied with $u_x = u_y$, $\alpha = \beta = 4/10$ and $\gamma = 1/10$. Also the following cases arise:

Case (i): $x = 0$, $y > 0$. *Then* $u_x = 0 \in T_1 x$, *choosing* $u_y = y/6 \in T_2 y$ *we have*

$$\begin{aligned}
[d(u_x, u_y)](t) &= f_{u_x,u_y}(t) = tu_y = t\frac{y}{6} \\
&\le t[\frac{4}{10}(y) + \frac{4}{10}(0) + \frac{1}{10}(\frac{5}{6}y)] \\
&\le t[\frac{4}{10}(y-x) + \frac{4}{10}(x-u_x) + \frac{1}{10}(y-u_y)] \\
&= [\alpha d(x,y) + \beta d(x,u_x) + \gamma d(y,u_y)](t),
\end{aligned}$$

with $\alpha = \beta = 4/10$, $\gamma = 1/10$ *and for all* $t \in [0,\infty)$.
Case (ii): $x > 0$, $y = 0$. *Then, for* $u_x \in T_1 x$, *choosing* $u_y = 0 \in T_2 y$ *we have*

$$\begin{aligned}
[d(u_x, u_y)](t) &= f_{u_x,u_y}(t) = tu_x \le t\frac{x}{5} \\
&\le t[\frac{4}{10}(x) + \frac{4}{10}(\frac{4x}{5}) + \frac{1}{10}(0)] \\
&\le t[\frac{4}{10}(x-y) + \frac{4}{10}(x-u_x) + \frac{1}{10}(y-u_y)] \\
&= [\alpha d(x,y) + \beta d(x,u_x) + \gamma d(y,u_y)](t).
\end{aligned}$$

Case (iii): $0 < y < x$. *Then for* $u_x \in T_1 x$, *taking* $u_y = y/6$, *we have*

$$\begin{aligned}
[d(u_x, u_y)]\,t &= f_{u_x,u_y}(t) \le t(\frac{x}{5} - \frac{y}{6}) \\
&\le t[\frac{4}{10}(0) + \frac{4}{10}(\frac{4x}{5}) + \frac{1}{10}(\frac{5}{6}y)] \\
&\le t[\frac{4}{10}(x-y) + \frac{4}{10}(x-u_x) + \frac{1}{10}(y-u_y)] \\
&= [\alpha d(x,y) + \beta d(x,u_x) + \gamma d(y,u_y)]t.
\end{aligned}$$

Case (iv): $0 < x < y$. *Then, for* $u_x \in T_1 x$, *taking* $u_y = y/5$, *yields*

$$\begin{aligned}
[d(u_x, u_y)](t) &= f_{u_x,u_y}(t) \le t(\frac{y}{5} - \frac{x}{7}) \\
&\le t[\frac{4}{10}(y-x) + \frac{4}{10}(\frac{4x}{5}) + \frac{1}{10}(\frac{4}{5}y)] \\
&\le t[\frac{4}{10}(y-x) + \frac{4}{10}(x-u_x) + \frac{1}{10}(y-u_y)] \\
&= [\alpha d(x,y) + \beta d(x,u_x) + \gamma d(y,u_y)](t).
\end{aligned}$$

Now we show that for $x, y \in X$, $u_x \in T_2 x$, *there exists* $u_y \in T_1 y$ *such that* (2) *is satisfied. We consider the following cases.*
Case (i): $x = 0$, $y > 0$. *Then with* $u_x = 0 \in T_2 x$, $u_y = y/7 \in T_1 y$ *we have*

$$\begin{aligned}
[d(u_x, u_y)](t) &= f_{u_x,u_y}(t) = tu_y = t\frac{y}{7} \\
&\le t[\frac{4}{10}(y) + \frac{4}{10}(0) + \frac{1}{10}(\frac{6}{7}y)] \\
&\le t[\frac{4}{10}(y-x) + \frac{4}{10}(x-u_x) + \frac{1}{10}(y-u_y)] \\
&= [\alpha d(x,y) + \beta d(x,u_x) + \gamma d(y,u_y)](t).
\end{aligned}$$

Case (ii): $x > 0$, $y = 0$. Then, for $u_x \in T_2x$, $u_y = 0 \in T_1y$ we have

$$\begin{aligned}
[d(u_x, u_y)](t) &= f_{u_x,u_y}(t) = tu_x \leq t\frac{x}{3} \\
&\leq t[\frac{4}{10}(x) + \frac{4}{10}(\frac{2x}{3}) + \frac{1}{10}(0)] \\
&\leq t[\frac{4}{10}(x-y) + \frac{4}{10}(x-u_x) + \frac{1}{10}(0)] \\
&= [\alpha d(x, y) + \beta d(x, u_x) + \gamma d(y, u_y)](t).
\end{aligned}$$

Case (iii): $0 < y < x$. Then, for $u_x \in T_2x$, take $u_y = \dfrac{8y}{50}$ to obtain

$$\begin{aligned}
[d(u_x, u_y)](t) &= f_{u_x,u_y}(t) = t(u_x - \frac{8y}{50}) \leq t(\frac{x}{3} - \frac{8y}{50}) \\
&\leq t[\frac{4}{10}(x-y) + \frac{4}{10}(\frac{2x}{3}) + \frac{1}{10}(\frac{42}{50}y)] \\
&\leq t[\frac{4}{10}(x-y) + \frac{4}{10}(x-u_x) + \frac{1}{10}(y-u_y)] \\
&= [\alpha d(x, y) + \beta d(x, u_x) + \gamma d(y, u_y)](t).
\end{aligned}$$

Case (iv): $0 < x < y$. Then, for $u_x \in T_2x$, taking $u_y = y/6$ yields

$$\begin{aligned}
[d(u_x, u_y)](t) &= f_{u_x,u_y}(t) \leq t\left|\frac{x}{3} - \frac{y}{6}\right| \\
&\leq t[\frac{4}{10}(y-x) + \frac{4}{10}(\frac{2x}{3}) + \frac{1}{10}(\frac{5}{6}y)] \\
&\leq t[\frac{4}{10}(y-x) + \frac{4}{10}(x-u_x) + \frac{1}{10}(y-u_y)] \\
&= [\alpha d(x, y) + \beta d(x, u_x) + \gamma d(y, u_y)](t).
\end{aligned}$$

Also note that $F(T_1) = F(T_2) \neq \phi$. Moreover, $F(T_1) = F(T_2) \in P_{cl}(X)$.

Other examples in the support of Theorem 2 can be constructed by taking $E = \ell^p$ ($p > 0$), $P = \{\{x_n\}_{n\geq 1} \in E \mid x_n \geq 0\}$, and $d(x, y) = \{(\rho(x, y)/2^n)^{1/p}\}_{n\geq 1}$, where ρ is a metric on any nonempty set X.

The following corollary extends Theorem 4.1 of [8] to the case of two mappings on cone metric spaces.

Corollary 1. *Let (X, d) be a complete cone metric space and P a non normal cone. If $T_1, T_2 : X \to P_{cl}(X)$ are two multivalued mappings such that for $i, j \in \{1, 2\}$ with $i \neq j$, $x, y \in X$ and $u_x \in T_i(x)$, there exists $u_y \in T_j(y)$ such that*

$$d(u_x, u_y) \leq h[d(x, u_x) + d(y, u_y)],$$

where $\alpha, \beta, \gamma \geq 0$ are fixed constants with $\alpha + \beta + \gamma < 1$. Then $F(T_1) = F(T_2) \neq \phi$ and $F(T_1) = F(T_2) \in P_{cl}(X)$.

The following corollary extends Theorem 4.1 of [8] to cone metric spaces.

Corollary 2. *Let (X,d) be a complete cone metric space and P a non normal cone. If $T : X \to P_{cl}(X)$ is a multivalued mapping such that for each $x, y \in X$ and $u_x \in T(x)$, there exists $u_y \in T(y)$ such that*

$$d(u_x, u_y) \leq h[d(x, u_x) + d(y, u_y)],$$

where $0 \leq h < \frac{1}{2}$. Then $F(T) \neq \phi$ and $F(T) \in P_{cl}(X)$.

Corollary 3. *Let (X,d) be a complete cone metric space and P a non normal cone. If $T : X \to P_{cl}(X)$ is a multivalued mapping such that for each $x, y \in X$ and $u_x \in T(x)$, there exists $u_y \in T(y)$ such that*

$$d(u_x, u_y) \leq \alpha d(x, y),$$

where $0 \leq \alpha < 1$. Then $F(T) \neq \phi$ and $F(T) \in P_{cl}(X)$.

Proof. Take $\beta = \gamma = 0$, and $T_1 = T_2 = T$ in Corollary 1.

Corollary 4. *Let (X,d) be a complete cone metric space and P a non normal cone. If $T : X \to P_{cl}(X)$ is a multivalued mapping such that for each $x, y \in X$ and $u_x \in T(x)$, there exists $u_y \in T(y)$ such that*

$$d(u_x, u_y) \leq \alpha d(x, y) + \beta d(x, u_x) + \gamma d(y, u_y),$$

where $\alpha, \beta, \gamma \geq 0$ are fixed constants with $\alpha + \beta + \gamma < 1$, then T has a fixed point.

The above corollary is an extension of Theorem 3.1 of [14] to cone metric spaces.

Acknowledgement

The authors are grateful to the referees for the critical remarks which improved the work.

References

[1] M. ABBAS, B. E. RHOADES, *Fixed and periodic point results in cone metric spaces*, Applied Mathematics Letters **22**(2009), 511–515.

[2] M. ABBAS, G. JUNGCK, *Common fixed point results of noncommuting mappings without continuity in cone metric spaces*, J. Math. Anal. Appl. **341**(2008), 418–420.

[3] R. H. HAGHI, SH. REZAPOUR, *Fixed points of multifunctions on regular cone metric spaces*, Expositiones Mathematicae, to appear.

[4] L. G. HUANG, X. ZHANG, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Anal. Appl. **332**(2007), 1467–1475.

[5] D. ILIĆ, V. RAKOČEVIĆ, *Common fixed points for maps on cone metric space*, J. Math. Anal. Appl. **341**(2008), 876–882.

[6] R. KANNAN, *Some results on fixed points*, Bull. Calcutta. Math. Soc. **60**(1968), 71–76.

[7] D. KILM, D. WARDOWSKI, *Dynamic processes and fixed points of set-valued nonlinear contractions in cone metric spaces*, Nonlinear. Anal. **71**(2009), 5170–5175.

[8] A. LATIF, I. BEG, *Geometric fixed points for single and multivalued mappings*, Demonstratio Math. **30**(1997), 791–800.

[9] J. T. Markin, *Continuous dependence of fixed point sets*, Proc. Amer. Math. Soc. **38**(1973), 545–547.

[10] Sh. Rezapour. R. H. Haghi, *Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings"*, J. Math. Anal. Appl. **345**(2008), 719–724.

[11] Sh. Rezapour. R. H. Haghi, *Two results about fixed point of multifunctions*, Bull. Iranian Mathematical Society, to appear.

[12] Sh. Rezapour. R. H. Haghi, *Fixed point of multifunctions on cone metric spaces*, Numer. Funct. Anal. and Opt. **30**(2009), 825–832.

[13] Sh. Rezapour, M. Derafshpour, *Some common fixed point results in cone metric spaces*, Nonlinear Convex Analysis, to appear.

[14] I. A. Rus, A. Petrusel, A. Sintamarian, *Data dependence of fixed point set of some multivalued weakly Picard operators*, Nonlinear Analysis **52**(2003), 1944–1959.

[15] D. Wardowski, *End points and fixed point of set valued contractions in cone metric spaces*, Nonlinear Analysis: Theory, Methods & Applications **71**(2009), 512–516.

[16] K. Wlodarczyk, R. Plebaniak, C. Obczynski, *Convergence theorems, best approximation and best proximity for set-valued dynamic systems of relatively quasi asymptotic contractions in cone uniform spaces*, Nonlinear Analysis: Theory, Methods & Applications, to appear.