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A class of superordination-preserving convex integral operator

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Abstract. If $H(\mathrm{U})$ denotes the space of analytic functions in the unit disk U, for the integral operator $A^h_{\alpha,\beta,\gamma,\delta} : \mathcal{K} \to H(\mathrm{U})$, with $\mathcal{K} \subset H(\mathrm{U})$, defined by

$$A^h_{\alpha,\beta,\gamma,\delta}[f](z) = \left[\frac{\beta+\gamma}{z^\gamma}\int_0^z f^\alpha(t)h(t)t^{\delta-1}\,\mathrm{d}\,t\right]^{1/\beta}, \ \Big(\alpha,\beta,\gamma,\delta \in \mathbb{C} \text{ and } h \in H(\mathrm{U})\Big),$$

we will determine sufficient conditions on g_1, g_2, α, β and γ such that

$$zh(z)\left[\frac{g_1(z)}{z}\right]^\alpha \prec zh(z)\left[\frac{f(z)}{z}\right]^\alpha \prec zh(z)\left[\frac{g_2(z)}{z}\right]^\alpha$$

implies

$$z\left[\frac{A^h_{\alpha,\beta,\gamma,\delta}[g_1](z)}{z}\right]^\beta \prec z\left[\frac{A^h_{\alpha,\beta,\gamma,\delta}[f](z)}{z}\right]^\beta \prec z\left[\frac{A^h_{\alpha,\beta,\gamma,\delta}[g_2](z)}{z}\right]^\beta.$$

In addition, both of the subordinations are sharp, since the left-hand side will be the largest function, and the right-hand side will be the smallest function so that the above implication has been held for all f functions satisfying the double differential subordination of the assumption.
The results generalize those of the last author from [3], obtained for the special case $\alpha = \beta$ and $h \equiv 1$.
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1. Introduction

Let $H(\mathrm{U})$ be the space of all analytical functions in the unit disk $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$. If $f, F \in H(\mathrm{U})$ and F is univalent in U, we say that the function f is subordinate to F, or F is superordinate to f, written $f(z) \prec F(z)$, if $f(0) = F(0)$ and $f(\mathrm{U}) \subseteq F(\mathrm{U})$.

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For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$, where $\mathbb{N}^*$ is the set of all positive integers, we denote

$$H[a,n] = \{f \in H(\mathrm{U}) : f(z) = a + a_n z^n + \cdots\}.$$

Letting $\varphi : \mathbb{C}^3 \times \overline{\mathrm{U}} \to \mathbb{C}$, $h \in H(\mathrm{U})$ and $q \in H[a,n]$, in [10] Miller and Mocanu determined conditions on φ such that

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z) \quad \text{implies} \quad q(z) \prec p(z),$$

for all p functions that satisfy the above superordination. Moreover, they found sufficient conditions so that the q function is the largest function with this property called the best subordinant of this superordination.

For the integral operator $A_{\beta,\gamma} : \mathcal{K}_{\beta,\gamma} \to H(\mathrm{U})$, $\mathcal{K}_{\beta,\gamma} \subset H(\mathrm{U})$, defined by

$$A_{\beta,\gamma}[f](z) = \left[\frac{\beta+\gamma}{z^\gamma}\int_0^z f^\beta(t)t^{\gamma-1}\,\mathrm{d}\,t\right]^{1/\beta}, \quad \beta,\gamma \in \mathbb{C}, \tag{1}$$

the third author determined in [3], in conjunction with [1] and [2], conditions on g_1, g_2, β and γ so that

$$z\left[\frac{g_1(z)}{z}\right]^\beta \prec z\left[\frac{f(z)}{z}\right]^\beta \prec z\left[\frac{g_2(z)}{z}\right]^\beta$$

implies

$$z\left[\frac{A_{\beta,\gamma}[g_1](z)}{z}\right]^\beta \prec z\left[\frac{A_{\beta,\gamma}[f](z)}{z}\right]^\beta \prec z\left[\frac{A_{\beta,\gamma}[g_2](z)}{z}\right]^\beta,$$

and that all the results are sharp.

In this paper we will consider the integral operator $A^h_{\alpha,\beta,\gamma,\delta} : \mathcal{K} \to H(\mathrm{U})$ with $\mathcal{K} \subset H(\mathrm{U})$ defined by

$$A^h_{\alpha,\beta,\gamma,\delta}[f](z) = \left[\frac{\beta+\gamma}{z^\gamma}\int_0^z f^\alpha(t)h(t)t^{\delta-1}\,\mathrm{d}\,t\right]^{1/\beta}, \tag{2}$$

where $\alpha,\beta,\gamma,\delta \in \mathbb{C}$ and $h \in H(\mathrm{U})$ (all powers are principal ones).

We will generalize all these previous results in order to give sufficient conditions on the g_1 and g_2 functions and on the α, β, γ and δ parameters, such that the next sandwich-type result holds:

$$zh(z)\left[\frac{g_1(z)}{z}\right]^\alpha \prec zh(z)\left[\frac{f(z)}{z}\right]^\alpha \prec zh(z)\left[\frac{g_2(z)}{z}\right]^\alpha$$

implies

$$z\left[\frac{A^h_{\alpha,\beta,\gamma,\delta}[g_1](z)}{z}\right]^\beta \prec z\left[\frac{A^h_{\alpha,\beta,\gamma,\delta}[f](z)}{z}\right]^\beta \prec z\left[\frac{A^h_{\alpha,\beta,\gamma,\delta}[g_2](z)}{z}\right]^\beta.$$

Moreover, the functions from the left-hand side and the right-hand side are the best subordinant and the best dominant, respectively.

2. Preliminaries

Let $c \in \mathbb{C}$ with $\operatorname{Re} c > 0$, let $n \in \mathbb{N}^*$ and let

$$C_n = C_n(c) = \frac{n}{\operatorname{Re} c}\left[|c|\sqrt{1 + 2\operatorname{Re}\left(\frac{c}{n}\right)} + \operatorname{Im} c\right].$$

If R is the univalent function $R(z) = \dfrac{2C_n z}{1-z^2}$, then the open door function $R_{c,n}$ is defined by

$$R_{c,n}(z) = R\left(\frac{z+b}{1+\bar{b}z}\right),\ z \in \mathrm{U},$$

where $b = R^{-1}(c)$.

Remark that $R_{c,n}$ is univalent in U, $R_{c,n}(0) = c$ and $R_{c,n}(\mathrm{U}) = R(\mathrm{U})$ is the complex plane slit along the half-lines $|\operatorname{Im} w| \geq C_n$ and $\operatorname{Re} w = 0$.

Moreover, if $c > 0$, then $C_{n+1} > C_n$ and $\lim\limits_{n\to\infty} C_n = \infty$, hence $R_{c,n} \prec R_{c,n+1}$ and $\lim\limits_{n\to\infty} R_{c,n}(\mathrm{U}) = \mathbb{C}$. We will use the notation $R_c \equiv R_{c,1}$.

Let denote the class of functions

$$A_n = \{f \in H(\mathrm{U}) : f(z) = z + a_{n+1}z^{n+1} + \cdots\},$$

and let $A \equiv A_1$.

Lemma 1 (Integral Existence Theorem, see [7, 8]). *Let $\phi, \Phi \in H[1,n]$ with $\phi(z) \neq 0$, $\Phi(z) \neq 0$ for $z \in \mathrm{U}$. Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\beta \neq 0$, $\alpha + \delta = \beta + \gamma$ and $\operatorname{Re}(\alpha + \delta) > 0$. If the function $f(z) = z + a_{n+1}z^{n+1} + \cdots \in A_n$ and if it satisfies*

$$\alpha\frac{zf'(z)}{f(z)} + \frac{z\phi'(z)}{\phi(z)} + \delta \prec R_{\alpha+\delta,n}(z)$$

then

$$F(z) = \left[\frac{\beta+\gamma}{z^\gamma \Phi(z)}\int_0^z f^\alpha(t)\phi(t)t^{\gamma-1}\,\mathrm{d}\,t\right]^{1/\beta} = z + b_{n+1}z^{n+1} + \cdots \in A_n,$$

$$\frac{F(z)}{z} \neq 0,\ z \in \mathrm{U},$$

and

$$\operatorname{Re}\left[\beta\frac{zF'(z)}{F(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right] > 0,\ z \in \mathrm{U}.$$

(All powers are principal ones).

A function $L(z;t) : \mathrm{U} \times [0,+\infty) \to \mathbb{C}$ is called a subordination (or a Loewner) chain if $L(\cdot;t)$ is analytic and univalent in U for all $t \geq 0$, $L(z;\cdot)$ is continuously differentiable on $[0,+\infty)$ for all $z \in \mathrm{U}$ and $L(z;s) \prec L(z;t)$ when $0 \leq s \leq t$.

Lemma 2 (see [12], p. 159). *The function $L(z;t) = a_1(t)z + a_2(t)z^2 + \ldots$, with $a_1(t) \neq 0$ for all $t \geq 0$ and $\lim\limits_{t\to+\infty}|a_1(t)| = +\infty$, is a subordination chain if and only if*

$$\operatorname{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] > 0,\ z \in \mathrm{U},\ t \geq 0.$$

The well-known class of convex functions of order α in U, $\alpha < 1$ will be denoted by $K(\alpha)$, and $K \equiv K(0)$ is the class of convex (and univalent) functions in U. Also, the class of starlike functions of order α in U, $\alpha < 1$, will be denoted by $S^*(\alpha)$, and $S^* \equiv S^*(0)$ is the class of starlike (and univalent) functions in U.

If $\beta > 0$ and $\beta+\gamma > 0$, for a given $\alpha \in \left[-\frac{\gamma}{\beta}, 1\right)$ we define the order of starlikeness of the class $A_{\beta,\gamma}$ by the largest number $\delta = \delta(\alpha; \beta, \gamma)$ such that $A_{\beta,\gamma}(S^*(\alpha)) \subset S^*(\delta)$, where $A_{\beta,\gamma}$ is given by (1).

Lemma 3 (see [11]). *Let $\beta > 0$, $\beta + \gamma > 0$. If $\alpha \in [\alpha_0, 1)$, where*

$$\alpha_0 = \max\left\{\frac{\beta - \gamma - 1}{2\beta}; -\frac{\gamma}{\beta}\right\},$$

then the order of starlikeness of the class $A^h_{\alpha,\beta,\gamma,\delta}(S^*(\alpha))$ is given by

$$\delta(\alpha; \beta, \gamma) = \frac{1}{\beta}\left[\frac{\beta + \gamma}{{}_2F_1(1, 2\beta(1 - \alpha), \beta + \gamma + 1; 1/2)} - \gamma\right],$$

where ${}_2F_1$ represents the (Gaussian) hypergeometric function.

Lemma 4 (see [6], Theorem 1). *Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and let $h \in H(\mathrm{U})$, with $h(0) = c$. If $\mathrm{Re}[\beta h(z) + \gamma] > 0$, $z \in \mathrm{U}$, then the solution of the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z), \tag{3}$$

with $q(0) = c$, is analytic in U and satisfies $\mathrm{Re}[\beta q(z) + \gamma] > 0$, $z \in \mathrm{U}$.

Let Q be the set of functions f that are analytic and injective on $\overline{\mathrm{U}} \setminus E(f)$, where

$$E(f) = \left\{\zeta \in \partial\mathrm{U} : \lim_{z\to\zeta} f(z) = \infty\right\},$$

and such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathrm{U} \setminus E(f)$ (see [10]).

Lemma 5 (see [10], Theorem 7). *Let $q \in H[a, 1]$, let $\chi : \mathbb{C}^2 \to \mathbb{C}$ and set $\chi(q(z), zq'(z)) \equiv h(z)$. If $L(z, t) = \chi(q(z), tzq'(z))$ is a subordination chain and $p \in H[a, 1] \cap Q$, then*

$$h(z) \prec \chi(p(z), zp'(z)) \quad \textit{implies} \quad q(z) \prec p(z).$$

Furthermore, if $\chi(q(z), zq'(z)) = h(z)$ has a univalent solution $q \in Q$, then q is the best subordinant.

Like in [5] and [9], let $\Omega \subset \mathbb{C}$, $q \in Q$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ is the class of those functions $\psi : \mathbb{C}^3 \times \mathrm{U} \to \mathbb{C}$ that satisfy the admissibility condition

$$\psi(r, s, t; z) \notin \Omega,$$

whenever $r = q(\zeta)$, $s = m\zeta q'(\zeta)$, $\operatorname{Re}\frac{t}{s}+1 \geq m\operatorname{Re}\left[\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right]$, $z \in \mathrm{U}$, $\zeta \in \partial\mathrm{U}\setminus E(q)$ and $m \geq n$. This class will be denoted by $\Psi_n[\Omega, q]$.

We write $\Psi[\Omega, q] \equiv \Psi_1[\Omega, q]$. For the special case when $\Omega \neq \mathbb{C}$ is a simply connected domain and h is a conformal mapping of U onto Ω, we use the notation $\Psi_n[h, q] \equiv \Psi_n[\Omega, q]$.

Remark 1. *If $\psi : \mathbb{C}^2 \times \mathrm{U} \to \mathbb{C}$, then the above defined admissibility condition reduces to*

$$\psi(q(\zeta), m\zeta q'(\zeta); z) \notin \Omega,$$

when $z \in \mathrm{U}$, $\zeta \in \partial\mathrm{U} \setminus E(q)$ and $m \geq n$.

Lemma 6 (see [5, 9])**.** *Let h be univalent in U and $\psi : \mathbb{C}^3 \times \mathrm{U} \to \mathbb{C}$. Suppose that the differential equation*

$$\psi(q(z), zq'(z), z^2q''(z); z) = h(z)$$

has a solution q, with $q(0) = a$, and one of the following conditions is satisfied:

(*i*) *$q \in Q$ and $\psi \in \Psi[h, q]$,*

(*ii*) *q is univalent in U and $\psi \in \Psi[h, q_\rho]$, for some $\rho \in (0, 1)$, where $q_\rho(z) = q(\rho z)$, or*

(*iii*) *q is univalent in U and there exists $\rho_0 \in (0, 1)$ such that $\psi \in \Psi[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$, where $h_\rho(z) = h(\rho z)$ and $q_\rho(z) = q(\rho z)$.*

If $p(z) = a + a_1z + \ldots \in H(\mathrm{U})$ and $\psi(p(z), zp'(z), z^2p''(z); z) \in H(\mathrm{U})$, then

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \quad \textit{implies} \quad p(z) \prec q(z)$$

and q is the best dominant.

3. Main results

First we need to determine the subset $\mathcal{K} \subset H(\mathrm{U})$ such that the integral operator $A^h_{\alpha,\beta,\gamma,\delta}$ given by (2) in Section 1 will be well-defined. If we choose in Lemma 1 the correspondent functions $\Phi \equiv 1$ and $\phi \equiv h \in H[1, 1]$, with $h(z) \neq 0$ for all $z \in \mathrm{U}$, then we obtain the next Lemma:

Lemma 7. *Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\beta \neq 0$, $\alpha + \delta = \beta + \gamma$ and $\operatorname{Re}(\beta + \gamma) > 0$. For the function $h \in H[1, 1]$, with $h(z) \neq 0$ for all $z \in \mathrm{U}$, we define the set $\mathcal{K} \subset H(\mathrm{U})$ by*

$$\mathcal{K} = \mathcal{K}^h_{\alpha,\delta} = \left\{ f \in A : \alpha\frac{zf'(z)}{f(z)} + \frac{zh'(z)}{h(z)} + \delta \prec R_{\alpha+\delta}(z) \right\}.$$

Then $f \in \mathcal{K}^h_{\alpha,\delta}$ implies $F \in A$, $\frac{F(z)}{z} \neq 0$, $z \in \mathrm{U}$ and $\operatorname{Re}\left[\beta\frac{zF'(z)}{F(z)} + \gamma\right] > 0$, $z \in \mathrm{U}$, where $F(z) = A^h_{\alpha,\beta,\gamma,\delta}[f](z)$.

Theorem 1. *Let $\alpha,\beta,\gamma,\delta\in\mathbb{C}$ with $\beta\neq 0$, $1<\beta+\gamma\leq 2$, $\alpha+\delta=\beta+\gamma$. Let $g\in\mathcal{K}^h_{\alpha,\delta}$, and for $\alpha\neq 1$ suppose in addition that $g(z)/z\neq 0$ for $z\in\mathrm{U}$. Suppose that*

$$\operatorname{Re}\left[1+\frac{z\varphi''(z)}{\varphi'(z)}\right]>\frac{1-(\beta+\gamma)}{2},\ z\in\mathrm{U},\tag{4}$$

where $\varphi(z)=zh(z)\left[\frac{g(z)}{z}\right]^{\alpha}$.

Let $f\in\mathcal{K}^h_{\alpha,\delta}$ such that $zh(z)\left[\frac{f(z)}{z}\right]^{\alpha}$ is univalent in U *and* $z\left[\frac{A^h_{\alpha,\beta,\gamma,\delta}[f](z)}{z}\right]^{\beta}\in Q$. *Then*

$$zh(z)\left[\frac{g(z)}{z}\right]^{\alpha}\prec zh(z)\left[\frac{f(z)}{z}\right]^{\alpha}\ \textit{implies}\ z\left[\frac{A^h_{\alpha,\beta,\gamma,\delta}[g](z)}{z}\right]^{\beta}\prec z\left[\frac{A^h_{\alpha,\beta,\gamma,\delta}[f](z)}{z}\right]^{\beta},$$

and the function $z\left[\frac{A^h_{\alpha,\beta,\gamma,\delta}[g](z)}{z}\right]^{\beta}$ *is the best subordinant.*

Proof. Denoting $G=A^h_{\alpha,\beta,\gamma,\delta}[g]$, $F=A^h_{\alpha,\beta,\gamma,\delta}[f]$, $\varphi(z)=zh(z)[g(z)/z]^{\alpha}$, $\psi(z)=zh(z)[f(z)/z]^{\alpha}$, $\Phi(z)=z[G(z)/z]^{\beta}$ and $\Psi(z)=z[F(z)/z]^{\beta}$, we need to prove that $\varphi(z)\prec\psi(z)$ implies $\Phi(z)\prec\Psi(z)$.

Because $g,f\in\mathcal{K}^h_{\alpha,\delta}$, then $\psi,\varphi\in A$ and by Lemma 1 we have $G(z)/z\neq 0$ and $F(z)/z\neq 0$, $z\in\mathrm{U}$, hence $\Phi,\Psi\in H(\mathrm{U})$ and moreover $\Phi,\Psi\in A$.

If we differentiate the relations $G(z)=A^h_{\alpha,\beta,\gamma,\delta}[g](z)$ and $\Phi(z)=z\left[\frac{G(z)}{z}\right]^{\beta}$ we have respectively

$$z^{\gamma}\left[\frac{G(z)}{z}\right]^{\beta}\left[\beta\frac{zG'(z)}{G(z)}+\gamma\right]=(\beta+\gamma)g^{\alpha}(z)h(z)z^{\delta-\beta},\tag{5}$$

$$\beta\frac{zG'(z)}{G(z)}=\beta-1+\frac{z\Phi'(z)}{\Phi(z)},\tag{6}$$

and replacing (6) in (5), together with the fact that $\alpha+\delta=\beta+\gamma$, we get

$$\varphi(z)=\left(1-\frac{1}{\beta+\gamma}\right)\Phi(z)+\frac{1}{\beta+\gamma}z\Phi'(z)=\chi(\Phi(z),z\Phi'(z)).\tag{7}$$

Letting

$$L(z;t)=\left(1-\frac{1}{\beta+\gamma}\right)\Phi(z)+\frac{t}{\beta+\gamma}z\Phi'(z),\tag{8}$$

then $L(z;1)=\varphi(z)$. If we denote $L(z;t)=a_1(t)z+\dots$, then

$$a_1(t)=\frac{\partial L(0;t)}{\partial z}=\left(1+\frac{t-1}{\beta+\gamma}\right)\Phi'(0)=1+\frac{t-1}{\beta+\gamma},$$

hence $\lim\limits_{t\to+\infty}|a_1(t)|=+\infty$, and from $\beta+\gamma>1$ we obtain $a_1(t)\neq 0$, $\forall t\geq 0$.

From definition (8), a simple computation shows the equality

$$\operatorname{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] = \beta+\gamma-1+t\operatorname{Re}\left[1+\frac{z\Phi''(z)}{\Phi'(z)}\right].$$

Using the above relation together with the assumption $\beta+\gamma-1>0$, and according to Lemma 2, in order to prove that $L(z;t)$ is a subordination chain we need to prove that the next inequality holds:

$$\operatorname{Re}\left[1+\frac{z\Phi''(z)}{\Phi'(z)}\right]>0,\ z\in\mathrm{U}. \tag{9}$$

If we let $q(z)=1+\dfrac{z\Phi''(z)}{\Phi'(z)}$, by differentiating (7) we have

$$\varphi'(z)=\left(1-\frac{1}{\beta+\gamma}\right)\Phi'(z)+\frac{1}{\beta+\gamma}\left[\Phi'(z)+z\Phi''(z)\right],$$

and by computing the logarithmical derivative of the above equality we deduce that

$$q(z)+\frac{zq'(z)}{q(z)+\beta+\gamma-1}=1+\frac{z\varphi''(z)}{\varphi'(z)}\equiv H(z). \tag{10}$$

From (4) we have

$$\operatorname{Re}\left[H(z)+\beta+\gamma-1\right]>\frac{\beta+\gamma-1}{2}>0,\ z\in\mathrm{U},$$

and by using Lemma 4 we conclude that differential equation (10) has a solution $q\in H(\mathrm{U})$, with $q(0)=H(0)=1$.

Next, using Lemma 3 we will prove that under our assumption inequality (9) holds. If in Lemma 3 we replace the parameters β and γ by $\widetilde{\beta}=1$ and $\widetilde{\gamma}=\beta+\gamma-1$ respectively, then the conditions $\widetilde{\beta}=1>0$ and $\widetilde{\beta}+\widetilde{\gamma}=\beta+\gamma>0$ are satisfied.

The assumption $\beta+\gamma>1$ implies $\alpha_0=\max\left\{\dfrac{\widetilde{\beta}-\widetilde{\gamma}-1}{2\widetilde{\beta}};-\dfrac{\widetilde{\gamma}}{\widetilde{\beta}}\right\}=\dfrac{1-(\beta+\gamma)}{2}$.

Using Lemma 3 for the case $\alpha=\alpha_0=\dfrac{1-(\beta+\gamma)}{2}$, we obtain that the solution q of differential equation (10) satisfies

$$\begin{aligned}\operatorname{Re}q(z)&>\frac{\beta+\gamma}{{}_2F_2(1,\beta+\gamma+1,\beta+\gamma+1;1/2)}+1-(\beta+\gamma)=\\&=\frac{\beta+\gamma}{2}+1-(\beta+\gamma)=1-\frac{\beta+\gamma}{2}\geq 0,\ z\in\mathrm{U},\end{aligned}$$

whenever $\beta+\gamma\leq 2$. It follows that inequality (9) is satisfied, and according to Lemma 2 the function $L(z;t)$ is a subordination chain.

Using (9) and the fact that $\Phi\in A$, we have that Φ is convex (univalent) in U, i.e. the differential equation $\chi(\Phi(z),z\Phi'(z))=\varphi(z)$ has the univalent solution Φ.

From Lemma 5, we conclude that $\varphi(z)\prec\psi(z)$ implies $\Phi(z)\prec\Psi(z)$, and furthermore, since Φ is a univalent solution of the differential equation $\chi(\Phi(z),z\Phi'(z))=\varphi(z)$, hence it is the best subordinant of the given differential superordination. □

Theorem 2. *Let $\alpha,\beta,\gamma,\delta\in\mathbb{C}$ with $\beta\neq 0$, $0<\beta+\gamma\leq 2$, $\alpha+\delta=\beta+\gamma$. Let $f,g\in\mathcal{K}^h_{\alpha,\delta}$, and for $\alpha\neq 1$ suppose in addition that $f(z)/z\neq 0$, $g(z)/z\neq 0$ for $z\in\mathrm{U}$. If*

$$\mathrm{Re}\left[1+\frac{z\varphi''(z)}{\varphi'(z)}\right]>\alpha_0=\max\left\{\frac{1-(\beta+\gamma)}{2};1-(\beta+\gamma)\right\},\ z\in\mathrm{U},\tag{11}$$

where $\varphi(z)=zh(z)\left[\frac{g(z)}{z}\right]^\alpha$, then $zh(z)\left[\frac{f(z)}{z}\right]^\alpha\prec zh(z)\left[\frac{g(z)}{z}\right]^\alpha$ implies

$$z\left[\frac{A^h_{\alpha,\beta,\gamma,\delta}[f](z)}{z}\right]^\beta\prec z\left[\frac{A^h_{\alpha,\beta,\gamma,\delta}[g](z)}{z}\right]^\beta,$$

and the function $z\left[\frac{A^h_{\alpha,\beta,\gamma,\delta}[g](z)}{z}\right]^\beta$ is the best dominant of the given subordination.

Proof. Like in the proof of Theorem 1, if we denote $F=A^h_{\alpha,\beta,\gamma,\delta}[f]$, $G=A^h_{\alpha,\beta,\gamma,\delta}[g]$, $\psi(z)=zh(z)[f(z)/z]^\alpha$, $\varphi(z)=zh(z)[g(z)/z]^\alpha$, $\Psi(z)=z[F(z)/z]^\beta$ and $\Phi(z)=z[G(z)/z]^\beta$, then we need to prove that $\psi(z)\prec\varphi(z)$ implies $\Psi(z)\prec\Phi(z)$.

Since $f,g\in\mathcal{K}^h_{\alpha,\delta}$, it follows that $\psi,\varphi\in A$ and by Lemma 1 we have $F(z)/z\neq 0$ and $G(z)/z\neq 0$, $z\in\mathrm{U}$, hence $\Psi,\Phi\in H(\mathrm{U})$ and moreover $\Psi,\Phi\in A$.

Differentiating the relations $G(z)=A^h_{\alpha,\beta,\gamma,\delta}[g](z)$ and $\Phi(z)=z\left[\frac{G(z)}{z}\right]^\beta$, we obtain respectively

$$z^\gamma\left[\frac{G(z)}{z}\right]^\beta\left[\beta\frac{zG'(z)}{G(z)}+\gamma\right]=(\beta+\gamma)g^\alpha(z)h(z)z^{\delta-\beta},\tag{12}$$

$$\beta\frac{zG'(z)}{G(z)}+\gamma=\beta+\gamma-1+\frac{z\Phi'(z)}{\Phi(z)},\tag{13}$$

and replacing in (13) in (12), together with the assumption $\alpha+\delta=\beta+\gamma$, we deduce that

$$\varphi(z)=\left(1-\frac{1}{\beta+\gamma}\right)\Phi(z)+\frac{1}{\beta+\gamma}z\Phi'(z).\tag{14}$$

If we let

$$L(z;t)=\left(1-\frac{1}{\beta+\gamma}\right)\Phi(z)+\frac{1+t}{\beta+\gamma}z\Phi'(z),\tag{15}$$

then $L(z;0)=\varphi(z)$. Denoting $L(z;t)=a_1(t)z+\ldots$, then

$$a_1(t)=\frac{\partial L(0;t)}{\partial z}=\left(1+\frac{t}{\beta+\gamma}\right)\Phi'(0)=1+\frac{t}{\beta+\gamma},$$

hence $\lim\limits_{t\to+\infty}|a_1(t)|=+\infty$, and because $\beta+\gamma>0$ we obtain $a_1(t)\neq 0$, $\forall t\geq 0$.

From (15) we may easily deduce the equality

$$\mathrm{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right]=\mathrm{Re}\left[\beta+\gamma+\frac{z\Phi''(z)}{\Phi'(z)}\right]+t\,\mathrm{Re}\left[1+\frac{z\Phi''(z)}{\Phi'(z)}\right].$$

Using the above relation and according to Lemma 2, in order to prove that $L(z;t)$ is a subordination chain we need to show that the next two inequalities hold:

$$\operatorname{Re}\left[1+\frac{z\Phi''(z)}{\Phi'(z)}\right]>0,\ z\in \mathrm{U} \tag{16}$$

and

$$\operatorname{Re}\left[\beta+\gamma+\frac{z\Phi''(z)}{\Phi'(z)}\right]>0,\ z\in \mathrm{U}. \tag{17}$$

If we let $q(z)=1+\dfrac{z\Phi''(z)}{\Phi'(z)}$, by differentiating (14) we have

$$\varphi'(z)=\left(1-\frac{1}{\beta+\gamma}\right)\Phi'(z)+\frac{1}{\beta+\gamma}\left[\Phi'(z)+z\Phi''(z)\right],$$

and from the logarithmical derivative of the above equality we deduce

$$q(z)+\frac{zq'(z)}{q(z)+\beta+\gamma-1}=1+\frac{z\varphi''(z)}{\varphi'(z)}\equiv H(z). \tag{18}$$

From (11) we have

$$\operatorname{Re}[H(z)+\beta+\gamma-1]>\alpha_0+\beta+\gamma-1\geq 0,\ z\in \mathrm{U}$$

and by using Lemma 4 we conclude that differential equation (18) has a solution $q\in H(\mathrm{U})$, with $q(0)=H(0)=1$.

Now we will use Lemma 3 to prove that under our assumption the inequalities (16) and (17) hold. If we replace parameters β by $\widetilde{\beta}=1$ and γ by $\widetilde{\gamma}=\beta+\gamma-1$ in Lemma 3, the conditions $\widetilde{\beta}=1>0$ and $\widetilde{\beta}+\widetilde{\gamma}=\beta+\gamma>0$ are satisfied.

Because

$$\alpha_0=\max\left\{\frac{1-(\beta+\gamma)}{2};1-(\beta+\gamma)\right\}=\begin{cases}1-(\beta+\gamma), & \text{if } \beta+\gamma\leq 1,\\[2mm] \dfrac{1-(\beta+\gamma)}{2}, & \text{if } \beta+\gamma\geq 1,\end{cases}$$

we need to discuss the following two cases.

In the first case, if $\beta+\gamma\leq 1$, by using Lemma 3 for $\alpha=\alpha_0=1-(\beta+\gamma)$ we obtain that the solution q of differential equation (18) satisfies

$$\begin{aligned}\operatorname{Re} q(z) &> \frac{\beta+\gamma}{{}_2F_1(1,2(\beta+\gamma),\beta+\gamma+1;1/2)}+1-(\beta+\gamma)\\ &=\frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\beta+\gamma+1/2\right)}{\Gamma(\beta+\gamma)}+1-(\beta+\gamma)>0,\ z\in \mathrm{U},\end{aligned}$$

hence (16) holds. From this inequality we also deduce that

$$\operatorname{Re}\left[\beta+\gamma+\frac{z\Phi''(z)}{\Phi'(z)}\right]=\operatorname{Re} q(z)+\beta+\gamma-1>\frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\beta+\gamma+1/2\right)}{\Gamma(\beta+\gamma)}>0,\ z\in \mathrm{U},$$

hence (17) holds.

In the second case, if $\beta+\gamma\geq 1$, by using Lemma 3 for $\alpha=\alpha_0=\dfrac{1-(\beta+\gamma)}{2}$ we obtain that the solution q of differential equation (18) satisfies

$$\begin{aligned}\operatorname{Re} q(z) &> \frac{\beta+\gamma}{{}_2F_2(1,\beta+\gamma+1,\beta+\gamma+1;1/2)}+1-(\beta+\gamma)\\ &=\frac{\beta+\gamma}{2}+1-(\beta+\gamma)=1-\frac{\beta+\gamma}{2}\geq 0,\ z\in \mathrm{U},\end{aligned}$$

if $\beta+\gamma\leq 2$, hence (16) holds. From this inequality we also deduce that

$$\operatorname{Re}\left[\beta+\gamma+\frac{z\Phi''(z)}{\Phi'(z)}\right]=\operatorname{Re} q(z)+\beta+\gamma-1>\frac{\beta+\gamma}{2}\geq\frac{1}{2}>0,\ z\in \mathrm{U},$$

hence (17) holds.

Hence we conclude that, if $0<\beta+\gamma\leq 2$, inequalities (16) and (17) are satisfied, then according to Lemma 2 the function $L(z;t)$ is a subordination chain. Moreover, inequality (16) and the fact that $\Phi\in A$ show that Φ is convex (univalent) in U.

Next we will show that $\Psi(z)\prec\Phi(z)$. Without loss of generality, we can assume that φ and Φ are analytic and univalent in $\overline{\mathrm{U}}$ and $\Phi'(\zeta)\neq 0$ for $|\zeta|=1$. If not, then we could replace φ with $\varphi_\rho(z)=\varphi(\rho z)$ and Φ with $\Phi_\rho(z)=\Phi(\rho z)$, where $\rho\in(0,1)$. These new functions will have the desired properties and we would prove our result using part *(iii)* of Lemma 6.

With our assumption, we will use part *(i)* of Lemma 6. If we denote by $\chi(\Phi(z), z\Phi'(z))=\varphi(z)$, we need to show that $\chi\in\Psi[\varphi,\Phi]$, i.e. χ is an admissible function. Because

$$\chi(\Phi(\zeta),m\zeta\Phi'(\zeta))=\left(1-\frac{1}{\beta+\gamma}\right)\Phi(\zeta)+\frac{1+t}{\beta+\gamma}\zeta\Phi'(\zeta)=L(\zeta;t),$$

where $m=1+t$, $t\geq 0$, since $L(z;t)$ is a subordination chain and $\varphi(z)=L(z;0)$, it follows that

$$\chi(\Phi(\zeta),m\zeta\Phi'(\zeta))\notin\varphi(\mathrm{U}).$$

Then, according to Remark 1, we have $\chi\in\Psi[\varphi,\Phi]$, and using Lemma 6 we obtain that $\Psi(z)\prec\Phi(z)$ and, moreover, Φ is the best dominant. □

If we combine this result together with Theorem 1, then we obtain the following differential sandwich-type theorem.

Theorem 3. *Let $\alpha,\beta,\gamma,\delta\in\mathbb{C}$ with $\beta\neq 0$, $1<\beta+\gamma\leq 2$, $\alpha+\delta=\beta+\gamma$. Let $g_1,g_2\in\mathcal{K}^h_{\alpha,\delta}$, and for $\alpha\neq 1$ suppose in addition that $g_k(z)/z\neq 0$ for $z\in$ U and $k=1,2$. Suppose that the next two conditions are satisfied*

$$\operatorname{Re}\left[1+\frac{z\varphi_k''(z)}{\varphi_k'(z)}\right]>\frac{1-(\beta+\gamma)}{2},\ z\in \mathrm{U},\ \textit{for}\ k=1,2,\tag{19}$$

where $\varphi_k(z)=zh(z)\left[\dfrac{g_k(z)}{z}\right]^\alpha$ *and* $k=1,2$.

Let $f \in \mathcal{K}^h_{\alpha,\delta}$ *such that* $zh(z)\left[\frac{f(z)}{z}\right]^{\alpha}$ *is univalent in* U *and* $z\left[\frac{A^h_{\alpha,\beta,\gamma,\delta}[f](z)}{z}\right]^{\beta} \in Q$. *Then*

$$zh(z)\left[\frac{g_1(z)}{z}\right]^{\alpha} \prec zh(z)\left[\frac{f(z)}{z}\right]^{\alpha} \prec zh(z)\left[\frac{g_2(z)}{z}\right]^{\alpha}$$

implies

$$z\left[\frac{A^h_{\alpha,\beta,\gamma,\delta}[g_1](z)}{z}\right]^{\beta} \prec z\left[\frac{A^h_{\alpha,\beta,\gamma,\delta}[f](z)}{z}\right]^{\beta} \prec z\left[\frac{A^h_{\alpha,\beta,\gamma,\delta}[g_2](z)}{z}\right]^{\beta}.$$

Moreover, the functions $z\left[\frac{A^h_{\alpha,\beta,\gamma,\delta}[g_1](z)}{z}\right]^{\beta}$ *and* $z\left[\frac{A^h_{\alpha,\beta,\gamma,\delta}[g_2](z)}{z}\right]^{\beta}$ *are the best subordinant and the best dominant, respectively.*

Remark 2. *Note that this theorem generalizes the previous one [3, Theorem 3.2], that may be obtained for the case* $\alpha = \beta$ *and* $h \equiv 1$.

For the case $\alpha = \beta = 1$ *and* $h \equiv 1$, *the result was obtained in [10, Corollary 6.1], where the authors assumed that* $\operatorname{Re}\gamma \geq 0$ *and* g_1, g_2 *are convex functions.*

Since the conditions that the functions $zh(z)\left[\frac{f(z)}{z}\right]^{\alpha}$ and $z\left[\frac{A^h_{\alpha,\beta,\gamma,\delta}[f](z)}{z}\right]^{\beta}$ need to be univalent in U are difficult to be checked, we will replace these assumptions by other simple sufficient conditions on f, g_1 and g_2 which implies the univalence of the above functions.

Corollary 1. *Let* $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ *with* $\beta \neq 0$, $1 < \beta + \gamma \leq 2$, $\alpha + \delta = \beta + \gamma$. *Let* $f, g_1, g_2 \in \mathcal{K}^h_{\alpha,\delta}$, *and for* $\alpha \neq 1$ *suppose in addition that* $f(z)/z \neq 0$, $g_k(z)/z \neq 0$ *for* $z \in$ U *and* $k = 1, 2$. *Suppose that the next three conditions are satisfied*

$$\operatorname{Re}\left[1 + \frac{z\varphi_k''(z)}{\varphi_k'(z)}\right] > \frac{1 - (\beta + \gamma)}{2},\ z \in \mathrm{U},\ \textit{for}\ k = 1, 2, 3, \tag{20}$$

where $\varphi_k(z) = zh(z)\left[\frac{g_k(z)}{z}\right]^{\alpha}$, $k = 1, 2$ *and* $\varphi_3(z) = zh(z)\left[\frac{f(z)}{z}\right]^{\alpha}$. *Then*

$$zh(z)\left[\frac{g_1(z)}{z}\right]^{\alpha} \prec zh(z)\left[\frac{f(z)}{z}\right]^{\alpha} \prec zh(z)\left[\frac{g_2(z)}{z}\right]^{\alpha}$$

implies

$$z\left[\frac{A^h_{\alpha,\beta,\gamma,\delta}[g_1](z)}{z}\right]^{\beta} \prec z\left[\frac{A^h_{\alpha,\beta,\gamma,\delta}[f](z)}{z}\right]^{\beta} \prec z\left[\frac{A^h_{\alpha,\beta,\gamma,\delta}[g_2](z)}{z}\right]^{\beta}.$$

Moreover, the functions $z\left[\frac{A^h_{\alpha,\beta,\gamma,\delta}[g_1](z)}{z}\right]^{\beta}$ *and* $z\left[\frac{A^h_{\alpha,\beta,\gamma,\delta}[g_2](z)}{z}\right]^{\beta}$ *are the best subordinant and the best dominant, respectively.*

Proof. In order to use straight Theorem 3, we need to show that inequality (20) for $k=3$ implies the univalence of the functions

$$\varphi_3(z)=zh(z)\left[\frac{f(z)}{z}\right]^{\alpha} \text{ and } \Phi(z)=z\left[\frac{A^{\phi,\varphi}_{\alpha,\beta,\gamma,\delta}[f](z)}{z}\right]^{\beta}.$$

The condition (20) for $k=3$ means that

$$\varphi_3\in K\left(\frac{1-(\beta+\gamma)}{2}\right)\subseteq K\left(-\frac{1}{2}\right)$$

and from [4] it follows that φ_3 is a close-to-convex function, hence it is univalent. If we denote by $F=A^h_{\alpha,\beta,\gamma,\delta}[f]$ and $\psi(z)=zh(z)[f(z)/z]^{\alpha}$, then $\Psi(z)=z[F(z)/z]^{\beta}$, and using a proof similar to that of Theorem 1 and Theorem 2 we conclude that Ψ is a convex function, hence it is univalent in U. □

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