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Geodesics and geodesic spheres in $\widetilde{\mathbf{SL}(\mathbf{2},\mathbb{R})}$ geometry*

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Abstract. In this paper geodesics and geodesic spheres in $\widetilde{SL(2,\mathbb{R})}$ geometry are considered. Exact solutions of ODE system that describes geodesics are obtained and discussed, geodesic spheres are determined and visualization of $\widetilde{SL(2,\mathbb{R})}$ geometry is given as well.
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Key words: $\widetilde{SL(2,\mathbb{R})}$ geometry, geodesics, geodesic sphere

1. Introduction

$\widetilde{SL(2,\mathbb{R})}$ geometry is one of the eight homogeneous Thurston 3-geometries

$$E^3, S^3, H^3, S^2\times\mathbb{R}, H^2\times\mathbb{R}, \widetilde{SL(2,\mathbb{R})}, Nil, Sol.$$

$\widetilde{SL(2,\mathbb{R})}$ is a universal covering group of $SL(2,\mathbb{R})$ that is a 3-dimensional Lie group of all 2×2 real matrices with determinant one. $\widetilde{SL(2,\mathbb{R})}$ is also a Lie group and it admits a Riemann metric invariant under right multiplication. The geometry of $\widetilde{SL(2,\mathbb{R})}$ arises naturally as geometry of a fibre line bundle over a hyperbolic base plane $\mathbb{H}^2$. This is similar to Nil geometry in a sense that Nil is a nontrivial fibre line bundle over the Euclidean plane and $\widetilde{SL(2,\mathbb{R})}$ is a twisted bundle over $\mathbb{H}^2$.

In $\widetilde{SL(2,\mathbb{R})}$, we can define the infinitesimal arc length square using the method of Lie algebras. However, by means of a projective spherical model of homogeneous Riemann 3-manifolds proposed by E. Molnar, the definition can be formulated in a more straightforward way. The advantage of this approach lies in the fact that we get a unified, geometrical model of these sorts of spaces.

Our aim is to calculate explicitly the geodesic curves in $\widetilde{SL(2,\mathbb{R})}$ and discuss their properties. The calculation is based upon the metric tensor, calculated by E.

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Molnar using his projective model (see [3]). It is not easy to calculate the geodesics because in the process of solving the problem we face a nonlinear system of ordinary differential equations of the second order with certain limits at the origin. We will also explain and determine the geodesic spheres of $\widetilde{SL(2,\mathbb{R})}$ geometry.

The paper is organized as follows. In Section 2 we give a description of the hyperboloid model of $\widetilde{SL(2,\mathbb{R})}$ geometry. Further, in Section 3, the geodesics of $\widetilde{SL(2,\mathbb{R})}$ space are explicitly calculated and discussed. Finally, in Section 4 the geodesic half-spheres in $SL(2,\mathbb{R})$ are given and illustrated for radii $R < \frac{\pi}{2}$ small enough.

2. Hyperboloid model of $\widetilde{SL(2,\mathbb{R})}$ geometry

In this section we describe in detail the hyperboloid model of $\widetilde{SL(2,\mathbb{R})}$ geometry, introduced by E. Molnar in [3].

The idea is to start with the collineation group which acts on projective 3-space $\mathcal{P}^3(R)$ and preserves a polarity i.e. a scalar product of signature $(--++)$. Let us imagine the one-sheeted hyperboloid solid

$$\mathcal{H}: \ -x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3 < 0$$

in the usual Euclidean coordinate simplex with the origin $E_0 = (1;0;0;0)$ and the ideal points of the axes $E_1^\infty(0;1;0;0)$, $E_2^\infty(0;0;1;0)$, $E_3^\infty(0;0;0;1)$. With an appropriate choice of a subgroup of the collineation group of $\mathcal{H}$ as an isometry group, the universal covering space $\widetilde{\mathcal{H}}$ of our hyperboloid $\mathcal{H}$ will give us the so-called hyperboloid model of $\widetilde{SL(2,\mathbb{R})}$ geometry.

We start with the one parameter group of matrices

$$\begin{pmatrix} \cos\varphi & \sin\varphi & 0 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \cos\varphi & -\sin\varphi \\ 0 & 0 & \sin\varphi & \cos\varphi \end{pmatrix}, \tag{1}$$

which acts on $\mathcal{P}^3(R)$ and leaves the polarity of signature $(--++)$ and the hyperboloid solid $\mathcal{H}$ invariant. By a right action of this group on the point $(x^0;x^1;x^2;x^3)$ we obtain its orbit

$$(x^0\cos\varphi - x^1\sin\varphi; x^0\sin\varphi + x^1\cos\varphi; x^2\cos\varphi + x^3\sin\varphi; -x^2\sin\varphi + x^3\cos\varphi), \tag{2}$$

which is the unique line (fibre) through the given point. We have pairwise skew fibre lines. Fibre (2) intersects base plane $E_0E_2E_3$ ($z^1 = 0$) at the point

$$Z = (x^0x^0 + x^1x^1;\ 0;\ x^0x^2 - x^1x^3;\ x^0x^3 + x^1x^2). \tag{3}$$

This action is called a fibre translation and φ is called a fibre coordinate (see Figure 1).

By usual inhomogeneous E^3 coordinates $x = \frac{x^1}{x^0}$, $y = \frac{x^2}{x^0}$, $z = \frac{x^3}{x^0}$, $\quad x^0 \neq 0$ fibre (2) is given by

$$(1, x, y, z) \mapsto \left(1, \frac{x + \tan\varphi}{1 - x \cdot \tan\varphi}, \frac{y + z \cdot \tan\varphi}{1 - x \cdot \tan\varphi}, \frac{z - y \cdot \tan\varphi}{1 - x \cdot \tan\varphi}\right),$$

where $\varphi \neq \frac{\pi}{2} + k\pi$. Particularly, the fibre through the base plane point $(0, y, z)$ is given by $(\tan\varphi, y + z \cdot \tan\varphi, z - y \cdot \tan\varphi)$ and through the origin by $(\tan\varphi, 0, 0)$.

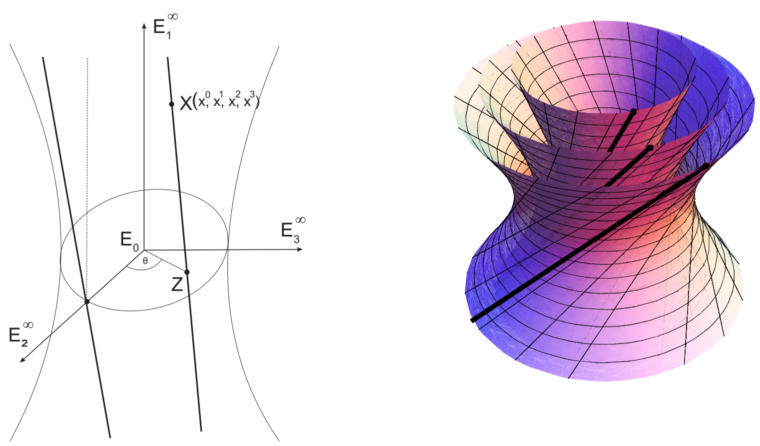

Figure 1. *Hyperboloid model of* $\widetilde{SL(2,\mathbb{R})}$

The subgroup of collineations that acts transitively on the points of $\tilde{\mathcal{H}}$ and maps the origin $E_0(1;0;0;0)$ onto $X(x^0;x^1;x^2;x^3)$ is represented by the matrix

$$\mathbf{T} : (t_i^j) := \begin{pmatrix} x^0 & x^1 & x^2 & x^3 \\ -x^1 & x^0 & x^3 & -x^2 \\ x^2 & x^3 & x^0 & x^1 \\ x^3 & -x^2 & -x^1 & x^0 \end{pmatrix}, \tag{4}$$

whose inverse up to a positive determinant factor Q is

$$\mathbf{T^{-1}} : (t_i^j)^{-1} = \frac{1}{Q} \cdot \begin{pmatrix} x^0 & -x^1 & -x^2 & -x^3 \\ x^1 & x^0 & -x^3 & x^2 \\ -x^2 & -x^3 & x^0 & -x^1 \\ -x^3 & x^2 & x^1 & x^0 \end{pmatrix}. \tag{5}$$

Remark 1. *A bijection between* $\mathcal{H}$ *and* $SL(2,\mathbb{R})$, *which maps point* $(x^0;x^1;x^2;x^3)$ *to matrix* $\begin{pmatrix} d & b \\ c & a \end{pmatrix}$ *is provided by the following coordinate transformations*

$$a = x^0 + x^3, \quad b = x^1 + x^2, \quad c = -x^1 + x^2, \quad d = x^0 - x^3.$$

This will be an isomorphism between translations (4) *and* $\begin{pmatrix} d & b \\ c & a \end{pmatrix}$ *with the usual multiplication operations, respectively. Moreover, the request* $bc - ad < 0$*, by using the mentioned coordinate transformations, corresponds to our hyperboloid solid*

$$-x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3 < 0.$$

Similarly to fibre (2) *that we obtained by acting of group* (1) *on the point* $(x^0; x^1; x^2; x^3)$ *in* $\tilde{\mathcal{H}}$*, a fibre in* $\widetilde{SL(2,\mathbb{R})}$ *is obtained by acting of group* $\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$*, on the "point"* $\begin{pmatrix} d & b \\ c & a \end{pmatrix} \in SL(2,\mathbb{R})$ *(see [3] also for other respects).*

Let us introduce new coordinates

$$\begin{aligned} x^0 &= \cosh r \cos\varphi \\ x^1 &= \cosh r \sin\varphi \\ x^2 &= \sinh r \cos(\vartheta - \varphi) \\ x^3 &= \sinh r \sin(\vartheta - \varphi)) \end{aligned} \tag{6}$$

as hyperboloid coordinates for $\tilde{\mathcal{H}}$, where (r, ϑ) are polar coordinates of the hyperbolic base plane and φ is just the fibre coordinate (by (2) and (3)). Notice that

$$-x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3 = -\cosh^2 r + \sinh^2 r = -1 < 0.$$

Now, we can assign an invariant infinitesimal arc length square by the standard method called pull back into the origin. Under action of (5) on the differentials $(dx^0; dx^1; dx^2; dx^3)$, by using (6) we obtain the following result

$$(ds)^2 = (dr)^2 + \cosh^2 r \sinh^2 r (d\vartheta)^2 + \big((d\varphi) + \sinh^2 r (d\vartheta)\big)^2. \tag{7}$$

Therefore, the symmetric metric tensor field g is given by

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sinh^2 r(\cosh^2 r + \sinh^2 r) & \sinh^2 r \\ 0 & \sinh^2 r & 1 \end{pmatrix}. \tag{8}$$

Remark 2. *Note that inhomogeneous coordinates corresponding to (6), that are important for a later visualization of geodesics and geodesic spheres in* E^3*, are given by*

$$\begin{aligned} x &= \frac{x^1}{x^0} = \tan\varphi, \\ y &= \frac{x^2}{x^0} = \tanh r \cdot \frac{\cos(\vartheta - \varphi)}{\cos\varphi}, \\ z &= \frac{x^3}{x^0} = \tanh r \cdot \frac{\sin(\vartheta - \varphi)}{\cos\varphi}. \end{aligned} \tag{9}$$

3. Geodesics in $\widetilde{SL(2,\mathbb{R})}$

The local existence, uniqueness and smoothness of a geodesics through any point $p \in M$ with initial velocity vector $v \in T_pM$ follow from the classical ODE theory on a smooth Riemann manifold. Given any two points in a complete Riemann manifold, standard limiting arguments show that there is a smooth curve of minimal length between these points. Any such curve is a geodesic.

Geodesics in *Sol* and *Nil* geometry are considered in [2], [5] and [6].

In local coordinates (u^1, u^2, u^3) around an arbitrary point $p \in \widetilde{SL(2,\mathbb{R})}$ one has a natural local basis $\{\partial_1, \partial_2, \partial_3\}$, where $\partial_i = \frac{\partial}{\partial u^i}$. The Levi-Civita connection ∇ is defined by $\nabla_{\partial_i}\partial_j := \Gamma^k_{ij}\partial_k$, and the Cristoffel symbols Γ^k_{ij} are given by

$$\Gamma^k_{ij} = \frac{1}{2} g^{km}\Big(\partial_i g_{mj} + \partial_j g_{im} - \partial_m g_{ij}\Big), \tag{10}$$

where the Einstein-Schouten index convention is used and (g^{ij}) denotes the inverse matrix of (g_{ij}).

Let us write $u^1 = r, u^2 = \vartheta, u^3 = \varphi$. Now by formula (10) we obtain Cristoffel symbols Γ^k_{ij}, as follows

$$\begin{aligned}
\Gamma^1_{ij} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}(1-2\cosh 2r)\sinh 2r & -\cosh r \sinh r \\ 0 & -\cosh r\sinh r & 0 \end{pmatrix}, \\
\Gamma^2_{ij} &= \begin{pmatrix} 0 & \coth r + 2\tanh r & \frac{1}{\cosh r \sinh r} \\ \coth r + 2\tanh r & 0 & 0 \\ \frac{1}{\cosh r\sinh r} & 0 & 0 \end{pmatrix}, \\
\Gamma^3_{ij} &= \begin{pmatrix} 0 & -2\sinh^2 r\tanh r & -\tanh r \\ -2\sinh^2 r \tanh r & 0 & 0 \\ -\tanh r & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{11}$$

Further, geodesics are given by the well-known system of differential equations

$$\ddot{u}^k + \dot{u}^i \dot{u}^j \Gamma^k_{ij} = 0. \tag{12}$$

After having substituted coefficients of Levi-Civita connection given by (11) into equation (12) and by assuming first $r > 0$, we obtain the following nonlinear system of the second order ordinary differential equations

$$\ddot{r} = \sinh(2r)\ \dot{\vartheta}\ \dot{\varphi} + \frac{1}{2}\big(\sinh(4r) - \sinh(2r)\big)\dot{\vartheta}\dot{\vartheta}, \tag{13}$$

$$\ddot{\vartheta} = -\frac{2\dot{r}}{\sinh(2r)}\big[(3\cosh(2r) - 1)\dot{\vartheta} + 2\dot{\varphi}\big], \tag{14}$$

$$\ddot{\varphi} = 2\dot{r}\tanh r\big(2\sinh^2 r\ \dot{\vartheta} + \dot{\varphi}\big). \tag{15}$$

By homogeneity of $\widetilde{SL(2,\mathbb{R})}$, we can extend the solution to limit $r \to 0$, due to the given assumption, as follows later on.

From (14) we get

$$\dot{\varphi} = -\frac{\ddot{\vartheta}\sinh(2r)}{4\dot{r}} - \frac{1}{2}\big(3\cosh(2r) - 1\big)\dot{\vartheta}, \tag{16}$$

and after inserting (16) into (13) we have

$$\frac{2\ddot{r}}{\sinh(2r)} = -\frac{\dot{\vartheta}\ddot{\vartheta}\sinh(2r)}{2\dot{r}} - \cosh(2r)\dot{\vartheta}\dot{\vartheta}. \tag{17}$$

Multiplying (17) by $2\sinh(2r)\dot{r}$ we get a differential

$$\frac{1}{2}\frac{d}{dt}\left(4\dot{r}\dot{r} + \sinh^2(2r)\dot{\vartheta}\dot{\vartheta}\right) = 0 \tag{18}$$

and hence

$$4(\dot{r})^2 + \sinh^2(2r)(\dot{\vartheta})^2 = 4C^2, \tag{19}$$

where C is the constant of integration, depending on initial conditions to be discussed later on.

Therefore we obtain

$$\dot{\vartheta} = \pm\frac{2\sqrt{C^2 - (\dot{r})^2}}{\sinh(2r)}. \tag{20}$$

As a consequence of (13) and (14), the sign will be $(-)$ due to the geometric interpretation of a fibre translation, but we will discuss this later.

From derivative of (20) we get

$$\ddot{\vartheta} = -\frac{2\dot{r}\ddot{r}}{\sinh(2r)\left(\pm\sqrt{C^2 - (\dot{r})^2}\right)} \mp 2\sqrt{C^2 - (\dot{r})^2}\,\frac{2\dot{r}\cosh(2r)}{\sinh^2(2r)}. \tag{21}$$

Further, by inserting (20) and (21), equation (16) has the following form

$$\dot{\varphi} = \frac{\ddot{r}}{2\left(\pm\sqrt{C^2 - (\dot{r})^2}\right)} - (2\cosh(2r) - 1)\frac{\pm\sqrt{C^2 - (\dot{r})^2}}{\sinh(2r)}. \tag{22}$$

Now we put (20) and (22) in (15) and get

$$\ddot{\varphi} - \tanh(r)\frac{\dot{r}\ddot{r}}{\left(\pm\sqrt{C^2 - (\dot{r})^2}\right)} + \frac{\pm\sqrt{C^2 - (\dot{r})^2}}{\cosh^2(r)}\;\dot{r} = 0. \tag{23}$$

From this equation it follows

$$\dot{\varphi} + \tanh(r)\left(\pm\sqrt{C^2 - (\dot{r})^2}\right) = D, \tag{24}$$

where D is a new constant of integration.

By equalizing $\dot{\varphi}$ from (22) and (24) we have

$$\frac{\ddot{r}}{2\left(\pm\sqrt{C^2 - (\dot{r})^2}\right)} - (2\cosh(2r) - 1)\frac{\pm\sqrt{C^2 - (\dot{r})^2}}{\sinh(2r)} = D - \tanh(r)\Big(\pm\sqrt{C^2 - (\dot{r})^2}\Big).$$

By reordering and multiplying by $-2\dot{r}\sinh(2r)$ we get

$$\frac{\dot{r}\ddot{r}}{\pm\sqrt{C^2-(\dot{r})^2}}\sinh(2r)+2\dot{r}D\sinh(2r)+2\dot{r}\cosh(2r)\left(\pm\sqrt{C^2-(\dot{r})^2}\right)=0,$$

which is again a differential and implies

$$\pm\sqrt{C^2-(\dot{r})^2}\sinh(2r)+D\cosh(2r)=E. \tag{25}$$

In consistence with homogeneity we may consider $\lim_{t\to 0} r(t)=0$. This implies $D=E$, and relation (25) then obtains the following form

$$\pm\sqrt{C^2-(\dot{r})^2}=-D\tanh r. \tag{26}$$

Now from (26), (20) and (24) we have respectively

$$\dot{r}=\pm\sqrt{C^2-D^2\tanh^2 r}, \tag{27}$$

$$\dot{\vartheta}=\frac{-D}{\cosh^2 r}, \tag{28}$$

$$\dot{\varphi}=D(1+\tanh^2 r)=2D+\dot{\vartheta}. \tag{29}$$

Here we see the consistence with $r\to 0$

$$\dot{r}(0)=C,\quad \dot{\vartheta}(0)=-D,\quad \dot{\varphi}(0)=D. \tag{30}$$

At the same time we can assume $r(0)=0$, $\vartheta(0)=0$, $\varphi(0)=0$, as initial conditions.

Further we consider the arc length

$$s=\int_0^t d\tau\sqrt{(\dot{r})^2+\cosh^2(r)\sinh^2(r)(\dot{\vartheta})^2+(\dot{\varphi}+\sinh^2(r)\dot{\vartheta})^2}, \tag{31}$$

that by (27), (28) and (29) gives

$$s=\int_0^t d\tau\sqrt{C^2+D^2}, \tag{32}$$

normalized with $C^2+D^2=1$ i.e. $C=\dot{r}(0)=\cos\alpha$, $D=\dot{\varphi}(0)=\sin\alpha$ and $\dot{\vartheta}(0)=-D=-\sin\alpha$ can be assumed.

Now, we have to consider three different cases: $D=C>0$, $D>C\geq 0$ and $C>D\geq 0$, with respect to the former equations as well.

(i) Case $D=C>0$, or equivalently $\alpha=\frac{\pi}{4}$.

In this case we obtain $Dt=\int_0^{r(t)}\cosh\rho\, d\rho=\sinh r(t)$, and hence

$$r(t)=\operatorname{arsinh}(Dt). \tag{33}$$

From (28) and (29), with initial conditions $\varphi(0)=0$ and $\vartheta(0)=0$, we obtain

$$\vartheta(t)=-\arctan(Dt), \tag{34}$$

$$\varphi(t)=2Dt-\arctan(Dt).$$

Particularly, $C = D$ implies $\alpha = \frac{\pi}{4}$ and hence $D = \frac{\sqrt{2}}{2}$.

(ii) Case $C > D \geq 0$, or equivalently $\tan\alpha < 1$.

From (27) we have

$$t = \int_{r(0)}^{r(t)} \frac{d\rho}{\sqrt{C^2 - D^2\tanh^2\rho}} = \int_0^{r(t)} \frac{\cosh\rho \, d\rho}{\sqrt{(C^2 - D^2)\sinh^2\rho + C^2}}, \tag{35}$$

and by substitution $u = \sqrt{C^2 - D^2}\,\sinh\rho$, after integration, we obtain

$$t = \frac{1}{\sqrt{C^2 - D^2}}\ \operatorname{arsinh}\frac{u}{C}$$

and hence

$$r(t) = \operatorname{arsinh}\left(\frac{C}{\sqrt{C^2 - D^2}}\ \sinh(\sqrt{C^2 - D^2}\ t)\right) \tag{36}$$

According to (28), we have

$$\dot{\vartheta} = \frac{-D(C^2 - D^2)}{C^2\cosh^2(\sqrt{C^2 - D^2}\ t) - D^2} = \frac{\frac{-D(C^2 - D^2)}{\cosh^2(\sqrt{C^2 - D^2}\ t)}}{(C^2 - D^2) + D^2\tanh^2(\sqrt{C^2 - D^2}\ t)},$$

and hence by using substitution $u = D\tanh(\sqrt{C^2 - D^2}\ t)$, after integration, we get

$$\vartheta(t) = -\arctan\left(\frac{D}{\sqrt{C^2 - D^2}}\ \tanh\left(\sqrt{C^2 - D^2}\ t\right)\right). \tag{37}$$

Finally, from (29) we have $\varphi(t) = 2D\ t + \vartheta(t)$ and hence

$$\varphi(t) = 2D\ t - \arctan\left(\frac{D}{\sqrt{C^2 - D^2}}\ \tanh\left(\sqrt{C^2 - D^2}\ t\right)\right). \tag{38}$$

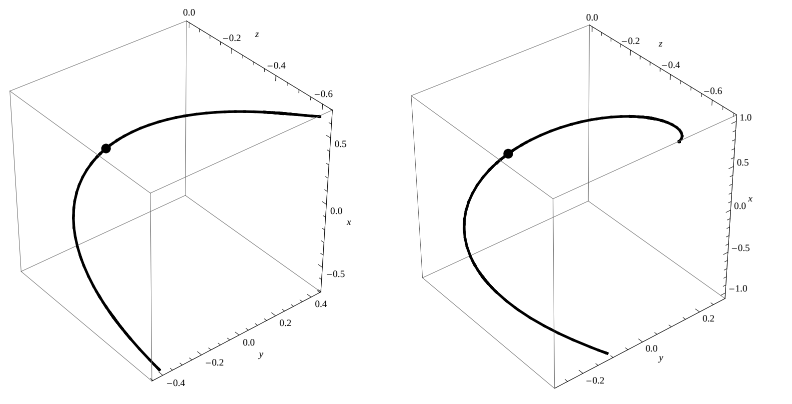

Figure 2. *Geodesics in* $\widetilde{SL(2,\mathbb{R})}$ - *Case* $\alpha = \frac{\pi}{6}$ *and* $\alpha = \frac{\pi}{4}$

Figure 2 shows geodesics through the origin for $C=\frac{\sqrt{3}}{2}$, $D=\frac{1}{2}$ and $C=D=\frac{\sqrt{2}}{2}$, and parameter $t\in[-1,1]$, respectively.
(iii) Case $D>C\geq 0$, or equivalently $\tan\alpha>1$.

Similarly to the previous case, we start with equation

$$t=\int_{r(0)}^{r(\tau)}\frac{d\rho}{\sqrt{C^2-D^2\tanh^2\rho}}=\int_{r(0)}^{r(\tau)}\frac{\cosh\rho\;d\rho}{\sqrt{C^2-(D^2-C^2)\sinh^2\rho}},$$

and by using substitution $u=\sqrt{D^2-C^2}\,\sinh\rho$, after integration, we obtain

$$t=\frac{1}{\sqrt{D^2-C^2}}\;\arcsin\frac{u}{C}$$

and hence

$$r(t)=\operatorname{arsinh}\left(\frac{C}{\sqrt{D^2-C^2}}\;\sin(\sqrt{D^2-C^2}\;t)\right). \tag{39}$$

From (28) we get

$$\dot{\vartheta}=\frac{-D(D^2-C^2)}{D^2-C^2\cos^2(\sqrt{D^2-C^2}\;t)}=\frac{\frac{-D(D^2-C^2)}{\cos^2(\sqrt{D^2-C^2}\;t)}}{(D^2-C^2)+D^2\tan^2(\sqrt{D^2-C^2}\;t)},$$

and hence, by using substitution $u=D\tan(\sqrt{D^2-C^2}\;t)$, after integration, we obtain

$$\vartheta(t)=-\arctan\left(\frac{D}{\sqrt{D^2-C^2}}\;\tan\left(\sqrt{D^2-C^2}\;t\right)\right). \tag{40}$$

Similarly to the former case $\varphi(t)=2D\;t+\vartheta(t)$ and hence

$$\varphi(t)=2D\;t-\arctan\left(\frac{D}{\sqrt{D^2-C^2}}\;\tan\left(\sqrt{D^2-C^2}\;t\right)\right). \tag{41}$$

Figure 3 shows geodesic through the origin for $C=\frac{1}{2}$, $D=\frac{\sqrt{3}}{2}$ and parameter $t\in[-1,1]$.

Remark 3. *One can easily observe special cases $\alpha=0$,*

$$\begin{array}{ll} r(s)=s, & x(s)=0\\ \vartheta(s)=0, & y(s)=\tanh s\\ \varphi(s)=0, & z(s)=0,\end{array}$$

and $\alpha=\frac{\pi}{2}$,

$$\begin{array}{ll} r(s)=0, & x(s)=\tan s\\ \vartheta(s)=-s, & y(s)=0\\ \varphi(s)=s, & z(s)=0.\end{array}$$

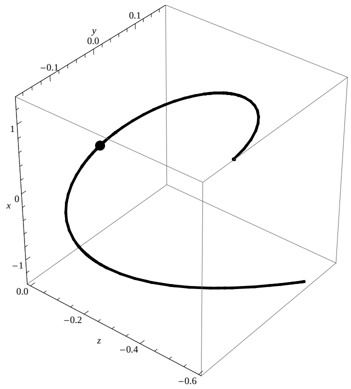

Figure 3. *Geodesic in* $\widetilde{SL(2,\mathbb{R})}$ - *Case* $\alpha = \frac{\pi}{3}$

Case	Geodesic line (hyperboloid coordinates)
$0 \le D = \sin\alpha < C = \cos\alpha$ $0 \le \alpha < \frac{\pi}{4}$ $t = s$ (H^2-like direction)	$r_\alpha(s) = \operatorname{arsinh}\left(\frac{\cos\alpha}{\sqrt{\cos 2\alpha}} \sinh(\sqrt{\cos 2\alpha}\ s)\right)$ $\vartheta_\alpha(s) = -\arctan\left(\frac{\sin\alpha}{\sqrt{\cos 2\alpha}} \tanh(\sqrt{\cos 2\alpha}\ s)\right)$ $\varphi_\alpha(s) = 2\sin\alpha\ s + \vartheta_\alpha(s)$
$D = C = \frac{\sqrt{2}}{2}$ $\alpha = \frac{\pi}{4}$ $t = s$ (separating light direction)	$r(s) = \operatorname{arsinh}\left(\frac{\sqrt{2}}{2} s\right)$ $\vartheta(s) = -\arctan\left(\frac{\sqrt{2}}{2} s\right)$ $\varphi(s) = \sqrt{2}\ s + \vartheta(s)$
$0 \le C = \cos\alpha < D = \sin\alpha$ $\frac{\pi}{4} < \alpha \le \frac{\pi}{2}$ $t = s$ (fibre-like direction)	$r_\alpha(s) = \operatorname{arsinh}\left(\frac{\cos\alpha}{\sqrt{-\cos 2\alpha}} \sin(\sqrt{-\cos 2\alpha}\ s)\right)$ $\vartheta_\alpha(s) = -\arctan\left(\frac{\sin\alpha}{\sqrt{-\cos 2\alpha}} \tan(\sqrt{-\cos 2\alpha}\ s)\right)$ $\varphi_\alpha(s) = 2\sin\alpha\ s + \vartheta_\alpha(s)$

Table 1. *Table of geodesics restricted to* $SL(2,\mathbb{R})$, $s \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

4. Geodesic spheres in $\widetilde{SL(2,\mathbb{R})}$ geometry

After having investigated geodesic curves, we can consider geodesic spheres. Geodesic spheres in *Sol* model geometry are visualized in [1]. For *Nil* geodesics, problems with geodesic Nil spheres and balls, and for analogous translation spheres and balls, we refer to [4], [5], [7] and [8], respectively.

In $\widetilde{SL(2,\mathbb{R})}$ geometry geodesic spheres of radius R are given by following equations

$$\begin{aligned} X(R,\phi,\alpha) &= x\ (s=R,\alpha), \\ Y(R,\phi,\alpha) &= y\ (s=R,\alpha)\cos\phi - z\ (s=R,\alpha)\sin\phi, \\ Z(R,\phi,\alpha) &= y\ (s=R,\alpha)\sin\phi + z\ (s=R,\alpha)\cos\phi, \end{aligned} \tag{42}$$

where x, y, z are Euclidean coordinates of geodesics given in Table 1, that are transformed according to formulas (9). Here $\phi \in (-\pi,\pi]$ denotes the longitude and $\alpha \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right]$ the altitude coordinate.

For $R \geq \frac{\pi}{2}$ we consider the projective extension and the universal covering space $\widetilde{SL(2,\mathbb{R})} = \tilde{\mathcal{H}}$ by (1) (see [3]) for the fibre coordinate $\varphi \in \mathbb{R}$ by extra conventions. That is not visual any more!

In Figure 4 geodesic half-spheres in $SL(2,\mathbb{R})$ are shown. Dark parts correspond to geodesics determined by $0 \leq \alpha < \frac{\pi}{4}$, light parts correspond to geodesics determined by $\frac{\pi}{4} < \alpha \leq \frac{\pi}{2}$ and black curves between these parts correspond to $\alpha = \frac{\pi}{4}$.

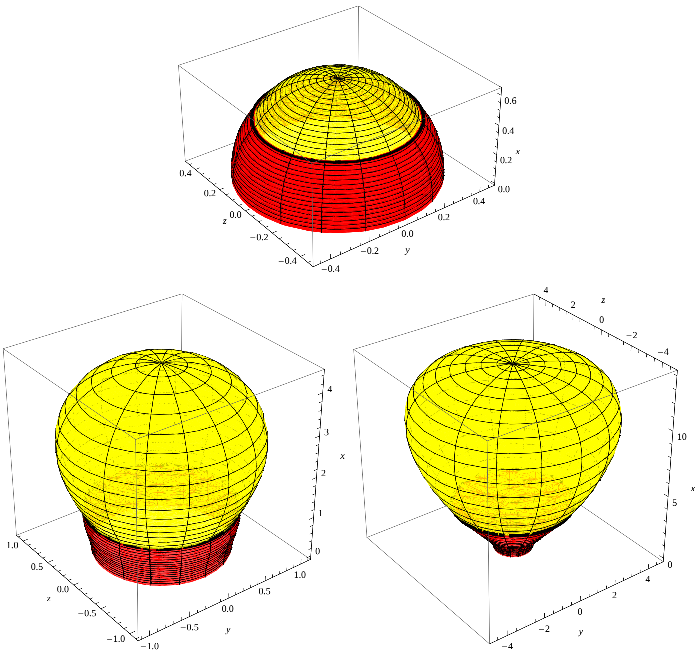

Figure 4. *Geodesic half-spheres in* $SL(2,\mathbb{R})$ *of radius* 0.5, 1 *and* 1.5, *respectively*

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