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On closure axioms for a matroid using Galois connections*

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Abstract. We present two systems of closure axioms for a matroid with the assistance of Galois connections. These axioms give a mathematical foundation for the connections between matroids, Galois connections and concept lattices. We deal with some relationship between matroids and geometric lattices by the above axioms. We also discuss some applications between matroids and concept lattices with the above two closure axioms for a matroid.

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1. Introduction

It is well known that there are many equivalent methods to define a matroid, and many notions within matroid theory have a variety of equivalent formulations (see [10, 15, 16]). The rich equivalent definitions cause matroids to have widespread applications (see [10, 12, 16]). We note the following facts about matroids:

(α) One of the most important results in matroid theory is the relation between matroids and geometric lattices (see [15, Ch.3]). This relation is based on the class of closed sets of a matroid. It is certainly useful to connect matroids with lattices.

(β) [10, 15, 16] present many properties about matroids with their closure operators or their families of closed sets.

(γ) We know that a closure operator determines its closed sets uniquely, and vice versa.

(δ) A system of axioms is essential for a theory and can help the theory to be more widely applied.

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$(\alpha) - (\delta)$ motivate us to look for new closure axioms for a matroid besides the definitions such as [15, pp.8-9].

Galois connections occur in many areas of mathematics. They are not only related to the classical Galois theory, but also to the idea of order (see [13]). It is also widely applied in Data Mining (see [3]), Formal Concept Analysis (see [4,8]) and so on (see [5]). This paper will provide a series of closure axioms of a matroid using Galois connections.

The structure of this paper is as follows. It first recalls some preliminary knowledge of Galois connections, concept lattices and matroids in Section 2. In Section 3 a series of closure axioms of matroids are provided. These axioms lead to some other properties about matroids using Galois connections in this section also. At last, Section 4 presents some applications of those axioms given in Section 3.

2. Preliminaries

The section summarizes the known facts about Galois connections, concept lattices and matroids that are needed later on. Although some of the definitions appearing in this section do not require the sets involved to be finite, we make a standing assumption that all the sets under consideration are finite.

Let X be a set. $\mathcal{P}(X)$ is the collection of all subsets of X. Let Y be a poset. $\uparrow x = \{y \in Y | x \leq y\}$ and $\downarrow x = \{y \in Y | y \leq x\}$ for any $x \in Y$.

Definition 1 (see [4], pp.155–157)**.**

(1) Let O and P be sets. Let R be a relation between O and P; in symbols, $R \subseteq O \times P$. For any subset X of O, let

$$K(X) := \{y \in P | \forall x \in X, (x, y) \in R\}.$$

For any subset Y of P, let

$$L(Y) := \{x \in O | \forall y \in Y, (x, y) \in R\}.$$

This pair (K, L) of mappings is a Galois connection between O and P called the Galois connection induced by R.

(2) A mapping $J : \mathcal{P}(O) \to \mathcal{P}(O)$ is a closure operator on O if it satisfies the following conditions for all subsets X and Y of O:

(s1) $X \subseteq J(X)$.

(s2) $X \subseteq Y \Rightarrow J(X) \subseteq J(Y)$.

(s3) $JJ(X) = J(X)$.

(3) (see [4], p.146 & [8]) An element $X \in \mathcal{P}(O)$ is called closed if $J(X) = X$.

Though the "Galois connection" in Definition 1 is not the original description as [4], Definition 1 is also a Galois connection according to the original description in [4]. In addition, [8, p.13] show that Definition 1 is the same as the Galois connection in [4, 8] between O and P. Hence, we use Definition 1 as our definition of a Galois connection in this paper.

Lemma 1 (see [4], p.160)**.** *Let O and P be sets and let $K : \mathcal{P}(O) \to \mathcal{P}(P)$ and $L : \mathcal{P}(P) \to \mathcal{P}(O)$ be maps which form a Galois connection. Then the map LK is a closure operator on O.*

For the knowledge of lattice theory we refer to [2, 4]. Next we recall some needed notation of concept lattice theory. More information may be found in [4] and [8].

Let (O, P, R) be a context, $Gal(O, P, R)$ the concept lattice corresponding to (O, P, R), where O and P are sets and $R \subseteq O \times P$. For $A \subseteq O$ and $B \subseteq P$, $A' = \{m \in P | \forall g \in A, (g, m) \in R\}, B' = \{g \in O | \forall m \in B, (g, m) \in R\}$. Let $(\mathcal{B}_O := \{A \subseteq O | A'' = A\}, \subseteq)$ be the extent lattice for (O, P, R). Combining [4, p.159, 7.2.6 & 6,8] with [4, pp.160-162, p.68 & 8] shows us the following facts:

(v1) a Galois connection (K, L) between O and P is one to one corresponding to a context (O, P, R), and further one to one to the class $Gal(O, P, R)$ of concepts.

(v2) $\mathcal{B}_O = \mathcal{B}(LK)$, where $\mathcal{B}(LK) = \{X \subseteq O | LK(X) = X\}$. Additionally, $(\mathcal{B}_O, \subseteq)$ is isomorphic to $Gal(O, P, R)$.

We will also use the usual matroid theoretical notation in [15]. We give matroid properties we shall use here.

Lemma 2. *It holds:*

(1) (see [15, pp.8-9]) A function σ on O is the closure operator of a matroid M on a finite set O if and only if σ satisfies (s1)-(s3) and the following condition:

(s4) for $Y \subseteq O$ and $y, z \in O$, if $y \notin \sigma(Y)$ but $y \in \sigma(Y \cup \{z\})$, then $z \in \sigma(Y \cup \{y\})$.

(2) (see [15, p.54]) The correspondence between a geometric lattice $\mathfrak{L}$ and the matroid $M(\mathfrak{L})$ on the set of atoms of $\mathfrak{L}$ is a bijection between the set of finite geometric lattices and the set of simple matroids.

Based on Lemma 2, in this paper we denote a matroid M on O with a closure operator σ as (O, σ). From Lemmas 1 and 2 we see that a closure operator on a matroid is certainly a closure operator in the usual sense, but a closure operator need not satisfy the additional condition (s4) to make it a matroid closure operator. This motivates the search for additional conditions which make a closure operator a matroid closure operator.

3. Closure axioms

In this section we present two additional conditions (s5) and (s6), which allow us to give iff characterizations of closure operators for matroids. These conditions use Galois connections and concept lattices and give a mathematical foundation to connect these concepts to matroids. We also discuss some relationships between matroids and Galois connections which are needed for the applications in the next section.

Theorem 1. *Let $\sigma : \mathcal{P}(O) \rightarrow \mathcal{P}(O)$ be a map and $\mathcal{F} = \{X \subseteq O | X = \sigma(X)\}$. Then the following statements are equivalent.*

(1) σ is the closure operator of a matroid M on O,

(2) σ satisfies (s4), and

(s5) there exists a Galois connection (K, L) on $\mathcal{F}$ such that up to isomorphism between $(\mathcal{F}, \subseteq)$ and $(\mathcal{B}_{\mathcal{F}}, \subseteq)$, the following is true

$$X = \sigma(X) = \bigcap_{b \in \bigcap_{a \in X} \uparrow a} \downarrow b \text{ for any } X \subseteq O, \text{ where } \mathcal{B}_{\mathcal{F}} = \{X \subseteq \mathcal{F} | LK(X) = X\}$$

Proof. (1)⇒(2): Suppose $M = (O, \sigma)$ is a matroid. Then $\mathcal{F}$ is the family of all closed sets of M. Also, Lemma 2 implies that $(\mathcal{F}, \subseteq)$ is a geometric lattice and σ satisfies (s4).

By the fundamental theorem of concept lattices (see [8, p.20 & 4, p.72]), $(\mathcal{F}, \subseteq)$ is isomorphic to $Gal(\mathcal{F}, \mathcal{F}, \subseteq)$. Additionally, the Galois connection (K, L) generated by $(\mathcal{F}, \mathcal{F}, \subseteq)$ is defined as

$$K(Z) = \bigcap_{x \in Z} \uparrow x \text{ and } L(Y) = \bigcap_{y \in Y} \downarrow y \text{ for any } Z \subseteq \mathcal{F} \text{ and } Y \subseteq \mathcal{F}.$$

So $(\mathcal{F}, \subseteq)$ is isomorphic to $(\mathcal{B}_{\mathcal{F}}, \subseteq)$ by virtue of (v2) and the above idea. Namely, up to lattice isomorphism, we have $(\mathcal{F}, \subseteq) = (\mathcal{B}_{\mathcal{F}}, \subseteq)$. In light of the discussion in [4, pp.73-75 & 8], we obtain that

$$Y \in \mathcal{B}_{\mathcal{F}} \Leftrightarrow Y = \bigcap_{b \in \bigcap_{a \in Y} \uparrow a} \downarrow b.$$

That is, under isomorphism between $(\mathcal{F}, \subseteq)$ and $(\mathcal{B}_{\mathcal{F}}, \subseteq)$, for every $X \subseteq O$,

$$X = \sigma(X) = \bigcap_{b \in \bigcap_{a \in X} \uparrow a} \downarrow b$$

holds.

(2)⇒(1): By (s5) and (v1), we can let $(O_{\mathcal{F}}, P_{\mathcal{F}}, R_{\mathcal{F}})$ be the context generated from (K, L). (s5) implies that up to isomorphism, $(\mathcal{F}, \subseteq)$ is the extent lattice $(\mathcal{B}_{O_{\mathcal{F}}}, \subseteq)$ of $(O_{\mathcal{F}}, P_{\mathcal{F}}, R_{\mathcal{F}})$(i.e. $(\mathcal{B}_{\mathcal{F}}, \subseteq)$). Equivalently, up to isomorphism we have $\mathcal{F} = \mathcal{B}_{O_{\mathcal{F}}} = \mathcal{B}_{\mathcal{F}}$. By virtue of Lemma 1, the fact that (K, L) is a Galois connection means that LK is a closure operator, and so LK satisfies (s1)-(s3). Additionally, since LK is the closure operator on $\mathcal{F}$ and $\mathcal{F} = \mathcal{B}_{\mathcal{F}} = \mathcal{B}_{O_{\mathcal{F}}}$ up to isomorphism, we get $\sigma = LK$ under isomorphism. So σ satisfies (s1)-(s3). Furthermore, because σ satisfies (s4) and Lemma 2, we get that σ is the closure operator of some matroid on O. □

Theorem 1 arises from concept lattice theory. The proof is not a direct one. Theorem 2 will be proved in a more direct way.

Theorem 2. *Let $\sigma : \mathcal{P}(O) \to \mathcal{P}(O)$ be a map. Then the following statements are equivalent.*

(1) σ is the closure operator of a matroid on O,

(2) σ satisfies (s4), and

(s6) There is a Galois connection (K, L) which satisfies $\sigma = LK$.

Proof. (1)⇒(2): Lemma 2 implies that we only need to prove that σ satisfies (s6). Let $R \subseteq O \times \mathcal{P}(O)$ be defined as $(x, Y) \in R \Leftrightarrow \sigma(Y \cup x) = \sigma(Y)$ for any $x \in O$ and $Y \subseteq O$. Then obviously, $(O, \mathcal{P}(O), R)$ is a context. Let (K, L) be the Galois connection corresponding to $(O, \mathcal{P}(O), R)$. Then

$$K(X) = \{Y \subseteq O : \forall x \in X, (x, Y) \in R\} = \{Y \subseteq O : \forall x \in X, \sigma(Y \cup x) = \sigma(Y)\},$$

and so,

$$\begin{aligned} L(K(X)) &= \{x \in O : \forall Y \in K(X), (x, Y) \in R\} \\ &= \{x \in O : \forall Y \in K(X), \sigma(Y \cup x) = \sigma(Y)\}. \end{aligned}$$

Our analysis for the proof of $\sigma = LK$ is as follows.

(T1) for any $x \in X$ and $Y \in K(X)$, it follows that $\sigma(Y \cup x) = \sigma(Y)$. This indicates $x \in \sigma(Y)$, and further, $X \subseteq \sigma(Y)$. So $\sigma(X) \subseteq \sigma(\sigma(Y)) = \sigma(Y)$. Besides, for any $x \in X$, it is easy to see $\sigma(X \cup x) = \sigma(X)$ and $\sigma(\sigma(X) \cup x)$ $= \sigma(\sigma(X)) = \sigma(X)$. This means $X \in K(X)$ and $\sigma(X) \in K(X)$.

(T2) for any $x \in L(K(X))$, we have $\sigma(X \cup x) = \sigma(X)$. By the properties of matroids, we know $\sigma(\sigma(X) \cup x) = \sigma(\sigma(X)) = \sigma(X)$ for any $x \in \sigma(X)$. Hence $L(K(X)) \subseteq \sigma(X)$.

(T3) for any $z \in \sigma(X)$ and any $Y \in K(X)$, the fact that $\sigma(\sigma(X) \cup z) = \sigma(X) \subseteq \sigma(Y)$ brings about $z \in \sigma(Y)$, and so $\sigma(Y \cup z) = \sigma(Y)$. Namely, $\sigma(X) \subseteq L(K(X))$.

Summing up (T1)-(T3), it follows that $(LK)(X) = \sigma(X)$ for any $X \subseteq O$, i.e. $LK = \sigma$.

(2)⇒(1): Let (K, L) be the Galois connection that appeared in (s6). Then Lemma 1 shows LK to be a closure operator, and it follows that σ satisfies (s1)-(s3). This with what is known and $\sigma = LK$ will show that σ is the closure operator of a matroid on O. □

If a Galois connection (K, L) can produce a matroid M with LK as its closure operator, we say that M *corresponds with* (K, L).

From Theorem 1 and Theorem 2, we see that different Galois connections could generate the same matroid. In other words, a matroid could correspond with different Galois connections. To overcome this non-uniqueness, we give a definition as follows: let (K_1, L_1) and (K_2, L_2) be two Galois connections between O and P_j, respectively $(j = 1, 2)$. If $X = (L_1K_1)(X) \Leftrightarrow X = (L_2K_2)(X)$ $(X \subseteq O)$, we say that (K_1, L_1) is *isomorphic* to (K_2, L_2), in notation, $(K_1, L_1) \cong (K_2, L_2)$. That is to say, two Galois connections are isomorphic if and only if they have the same collections of closed sets. Hence there is the following result:

Theorem 3. *Up to isomorphism, a matroid is uniquely determined by its corresponding Galois connections.*

As is well-known, for a Galois connection (K, L), LK does not necessarily satisfy (s4). Recalling Lemma 2, we get that the converse of Theorem 3 is not true.

[15, p.329] tells us the definition of automorphism of M, and [15, p.9] shows us the definition of isomorphism between two matroids. It is easy to know that an automorphism of M is an isomorphism on M. We can prove that π is an automorphism of M if and only if $X = \sigma(X) \Leftrightarrow \pi X = \sigma(\pi X)$ though it will not be proved here. That is to say, our definition of isomorphism between Galois connection above is significant. In the future, we will discuss the properties of isomorphic Galois connections.

4. Applications

This section uses the results in Section 3 to study some properties of matroids and concept lattices.

We first offer some properties connecting matroids and geometric lattices using the closure axioms for a matroid provided above. Though these properties are obvious, they are useful for the applications to follow.

Theorem 4. *It holds:*

(1) If σ is the closure operator of a matroid on O, then there is a Galois connection (K, L) such that $(\{Y \subseteq O | LK(Y) = Y\}, \subseteq)$ is a geometric lattice.

(2) If there is a Galois connection (K, L) such that $(\{Y \subseteq O | LK(Y) = Y\}, \subseteq)$ is a geometric lattice, then up to isomorphism, there is a matroid on O with LK as its closure operator.

Proof. (1): By Theorem 2, there is a Galois connection (K, L) satisfying $\sigma = LK$. By Lemma 2, $(\{\sigma(X) | X \subseteq O\}, \subseteq)$ is a geometric lattice, and so is

$$(\{Y \subseteq O | LK(Y) = Y\}, \subseteq).$$

(2): By the assumption, we may assume that (O, P, R) is the context decided by (K, L). By (v2) , $(\{Y \subseteq O | LK(Y) = Y\}, \subseteq)$ is isomorphic to $Gal(O, P, R)$. The given condition shows that $(\{Y \subseteq O | LK(Y) = Y\}, \subseteq)$ is geometric. Thus Lemma 2 implies that up to isomorphism, there is a matroid on O with LK as its closure operator. $\square$

Let $M = (O, \sigma)$ be a matroid with $\mathcal{F}$ as its family of all closed sets. [15] shows that there are many results depending on $\mathcal{F}$ such as [15, Ch.3, Ch.17, etc.]. This suggests that it is important to be able to find $\mathcal{F}$. Unfortunately, there are seldom algorithms to find out $\mathcal{F}$. Though [7] gives an approach to find out all closed sets for closure operators, it is an algorithm for general closure operators, not specially for the closure operators of matroids. Additionally, it is pointed out in [7] that in concrete problems, additional structural properties come into play which are not taken into consideration by this general algorithm. The two points above together hint that the algorithm in [7] is not easy to realize directly to find out all the closed sets for matroids. Here, according to our closure axioms, we will provide an idea for drawing $\mathcal{F}$. In addition, we recall from [7] that it is difficult to compare the algorithm in [7] with others, which as far as we know always handle special problems. So, it is also difficult for us to compare our algorithm in this paper with that in [7].

4.1. The sketch for drawing $\mathcal{F}$

For the context $(O, \mathcal{P}(O), R)$ given in Theorem 2, we have $(\mathcal{B}_O, \subseteq) = (\mathcal{F}, \subseteq)$. Reviewing [4, 8], there are many algorithms for generating $(\mathcal{B}_O, \subseteq)$ such as [4, p.76 & 8]. Currently there are many good modified algorithms for generating $\mathcal{B}_O$ such as these in [1] and [11], which are faster than those in [4] and [8]. In fact, all the algorithms for generating $\mathcal{B}_O$ could be used here.

That is to say, we obtain many algorithms to draw $\mathcal{F}$. For a concrete matroid, we will choose a more suitable algorithm for searching out $\mathcal{B}_O$, say, $\mathcal{F}$.

How about the converse part? Namely, could we use matroid algorithms to produce its corresponding Galois connection (K, L) or concept lattice $Gal(O, P, R)$ or $\mathcal{B}_O$? It is well known that there are some algorithms for searching out the construction of a matroid M such as [14] and the greedy algorithm (see [15, pp.357–360]) even though all these algorithms rely on the family $\mathcal{I}$ of all independent sets of M. In light of the definition of a matroid (see [15, p.6]) and Lemma 2, it happens that $\mathcal{F}$ is created when $\mathcal{I}$ is produced. This view combined with Theorem 4 together infers that we could construct the concept lattices and the extent lattices for some contexts or for some Galois connections by applying the algorithms for generating matroids. We believe that these algorithms to concrete concept lattices or extent lattices must be different from those we have known. We plan to implement these algorithms in the future.

For a Galois connection (K, L), or equivalently, a context (O, P, R), can we utilize our closure axioms here to give a way to decide whether there is a matroid on O corresponding with (K, L)? The answer is yes based on Theorem 4. We review Theorem 1 and Theorem 4 and describe the process as follows:

(ζ) First we need to obtain $(\mathcal{B}_O, \subseteq)$. We do not need a new algorithm to do so, since there are many existing algorithms to do this; see [1, 4, 8, 11] and so on.

We would better generate $(\mathcal{B}_O, \subseteq)$ by using an algorithm to construct $(\mathcal{B}_O, \subseteq)$ for (O, P, R), and meanwhile, the diagram graph(or line diagram, or Hasse diagram) of $(\mathcal{B}_O, \subseteq)$ is constructed. Then it is very easy to know the geometric property of $(\mathcal{B}_O, \subseteq)$ from its diagram graph. These algorithms may be found in [4, 6, 8, 9].

(η) If the decision is yes for the geometric property of $(\mathcal{B}_O, \subseteq)$, there is a matroid corresponding with (K, L). Otherwise, no matroid corresponds with (K, L).

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