SOLUTIONS OF A CLASS OF QUARTIC THUE INEQUALITIES

Bo He, Borka Jadrijević and Alain Togbé

ABa Teacher's College, China, University of Split, Croatia and Purdue University North Central, USA

ABSTRACT. Let c be a positive integer. In this paper, we use the method of Tzanakis to transform the quartic Thue inequality

$$|x^4 - 4x^3y - (2c-2)x^2y^2 + (4c+4)xy^3 - (2c-1)y^4| \leq \max\left\{\frac{c}{4}, 4\right\}$$

into systems of Pellian equations. Then we find all primitive solutions of this inequality using continued fractions.

1. INTRODUCTION

A Thue equation is a Diophantine equation of the form

$$F(x, y) = k,$$

where $F \in \mathbb{Z}[X, Y]$ is an irreducible binary form of degree $d \geq 3$ and k is a non-zero rational integer; the unknowns x and y being rational integers. The name is given in honor of the Norwegian mathematician A. Thue ([16]) who proved that it has only finitely many solutions. Upper bounds for the solutions have been given using A. Baker's theory on linear forms in logarithms of algebraic numbers (see [1]). In fact, since E. Thomas ([15]) has solved the first parameterized family of Thue equations of positive discriminant, several families of parameterized Thue equations have been studied, many authors are able to solve cubic, quartic, quintic, sextic Thue equations. In 2004,

Heuberger, Togbé, and Ziegler ([10]) solved the first octic family of Thue equations. In 2006, the third author solved a family of sextic Thue equations by means of Baker's method (see [17]).

In 1993, Tzanakis ([18]) considered Thue equations of the form $F(x, y) = k$, where F is a quartic form that corresponds to quartic field $\mathbb{K}$ which is the composite of two real quadratic fields. Tzanakis proved that solving the equation $F(x, y) = k$ consists in solving a system of Pellian equations. Therefore the method is called the Tzanakis method. The method was successfully applied by several authors not only for families of quartic Thue equations but also to families of quartic Thue inequalities. One can cite for examples [6–9, 11–13, 21]. One advantage of the method is that one can use the theory of continued fractions to determine small values of k for which the equation $F(x, y) = k$ has a solution. Worley ([20]), Dujella ([5]), Dujella-Ibrahimpašić ([6]) have extended or generalized the classical results of Legendre and Fatou concerning Diophantine approximations of irrational numbers.

Let $c \geq 1$ be an integer. In this paper, we consider the Thue inequality

$$(1.1) \quad |x^4 - 4x^3y - (2c-2)x^2y^2 + (4c+4)xy^3 - (2c-1)y^4| \leq \max\left\{\frac{c}{4}, 4\right\},$$

or, equivalently, the Thue equations of the form

$$(1.2) \qquad x^4 - 4x^3y - (2c-2)x^2y^2 + (4c+4)xy^3 - (2c-1)y^4 = m,$$

where $|m| \leq \max\left\{\frac{c}{4}, 4\right\}$. The aim of the paper is to prove the following theorem.

THEOREM 1.1. *Let c, m be integers such that $c \geq 1$ and $|m| \leq \max\left\{\frac{c}{4}, 4\right\}$ and such that the Thue equation* (1.2) *has a primitive solution. Then*

$$m \in \begin{cases} \{1, -1, 4, -4\}, & if \ \ c = 1, \\ \{1, -3, 4\}, & if \ \ c = 2, \\ \{1, 4\}, & if \ \ c \geq 3. \end{cases}$$

All primitive solutions are given as follows

1. *If $c = 2n^2 - 2, n \in \mathbb{N}, n > 1$,*

$$(x, y) = \begin{cases} (\pm 1, 0), \ \pm (n+1, n), \ \pm (n-1, n) & if \ \ m = 1, \\ \pm (1, 1), \ \pm (2n+1, 1), \ \pm (2n-1, -1) & if \ \ m = 4. \end{cases}$$

2. *If $c \neq 2n^2 - 2, n \in \mathbb{N}, n > 1$,*

$$(x, y) = \begin{cases} (\pm 1, 0), & if \ \ m = 1, \\ \pm (1, 1), & if \ \ m = 4. \end{cases}$$

Moreover, there are additional primitive solutions

$$(x, y) = \begin{cases} (0, \pm 1),\ \pm(2, 1), & if \ \ m = -1, \\ \pm(3, 1), & if \ \ m = -4, \end{cases} \quad for \quad c = 1,$$

and

$$(x, y) = \pm\,(2, 1)\,, \ \ \pm\,(0, 1) \ \ if \ \ m = -3, \quad for \quad c = 2.$$

From the above theorem, one can easily deduce the following consequence.

COROLLARY 1.2. *Let $c \geq 1$ be an integer. Then, for all $c \geq 1$, the Thue inequality* (1.1) *has primitive solutions of the form $(x, y) = (\pm 1, 0), \pm(1, 1)$. These solutions are the only primitive solutions if $c \neq 2n^2 - 2, n \in \mathbb{N}, n > 1$ and $c \neq 1, 2$. The additional primitive solutions are given by:*

   i) $(x, y) \ = \ \pm\,(n + 1, n)\,, \ \pm\,(n - 1, n)\,, \ \pm\,(2n + 1, 1)\,, \ \pm\,(2n - 1, -1) \ \ for$
     $c = 2n^2 - 2, n \in \mathbb{N}, n > 1;$
  ii) $(x, y) = \pm\,(0, 1)\,, \ \pm\,(2, 1) \ for \ c = 2;$
 iii) $(x, y) = (0, \pm 1),\ \pm(2, 1),\ \pm(3, 1) \ for \ c = 1.$

First we will use Tzanakis method to transform the Thue equation (1.2) into the following system of Pellian equations

$$(1.3) \qquad\qquad\qquad cU^2 - (c + 4)\, V^2 = -4m,$$

$$(1.4) \qquad\qquad\qquad U^2 - 2\,(c + 4)\, Z^2 = m.$$

This is done in the next section. Second, we recall some results due to Worley, Dujella, and Dujella-Jadrijević on continued fractions in Section 3. We use these results in order to find all possible values of m for which system (1.3) and (1.4) has a solution. In the last two sections we solve equation (1.2) for all values of m that are obtained in Section 3, using results of Cohn ([3, 4]) and Walsh ([19]).

2. AN APPLICATION OF THE METHOD OF TZANAKIS

In this section, we recall the method of Tzanakis and use it to transform equation (1.2) into a system of Pellian equations. So let us consider the Thue equation

$$(2.1) \qquad\qquad\qquad\qquad f(x, y) = m,$$

where

$$(2.2) \quad f(x, y) = a_0 x^4 + 4a_1 x^3 y + 6a_2 x^2 y^2 + 4a_3 x y^3 + a_4 y^4 \in \mathbb{Z}[x, y], \ \ a_0 > 0.$$

We assign to this equation the cubic equation

$$(2.3) \qquad\qquad\qquad\qquad 4\rho^3 - g_2\rho - g_3 = 0,$$

where

$$g_2 = a_0 a_4 - 4a_1 a_3 + 3a_2^2 \in \frac{1}{12}\mathbb{Z}, \quad g_3 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} \in \frac{1}{432}\mathbb{Z}.$$

In 1993, Tzanakis [18] proved that a necessary condition to apply his method is that equation (2.3) has three rational roots ρ_1, ρ_2, ρ_3 that satisfy

$$(2.4) \qquad \frac{a_1^2}{a_0} - a_2 \geq \max{(\rho_1, \rho_2, \rho_3)}.$$

Let $H(x,y)$ and $G(x,y)$ be the quartic and sextic covariants of $f(x,y)$ respectively, i.e.

$$H(x,y) = -\frac{1}{144}\begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix} \in \frac{1}{48}\mathbb{Z}[x,y],$$

$$G(x,y) = -\frac{1}{8}\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} \end{vmatrix} \in \frac{1}{96}\mathbb{Z}[x,y].$$

Then we have $4H^3 - g_2 H f^2 - g_3 f^3 = G^2$. Let us put

$$H = \frac{1}{48}H_0, G = \frac{1}{96}G_0, \rho_i = \frac{1}{12}r_i, i = 1, 2, 3,$$

then

$$H_0, G_0 \in \mathbb{Z}[x,y], r_i \in \mathbb{Z}, i = 1, 2, 3,$$

and

$$(H_0 - 4r_1 f)(H_0 - 4r_2 f)(H_0 - 4r_3 f) = 3G_0^2.$$

There exist positive square-free integers k_1, k_2, k_3 and quadratic $G_1, G_2, G_3 \in \mathbb{Z}[x,y]$ such that

$$H_0 - 4r_i f = k_i G_i^2, \quad i = 1, 2, 3$$

and $k_1 k_2 k_3 (G_1 G_2 G_3)^2 = 3G_0^2$. If $(x,y) \in \mathbb{Z} \times \mathbb{Z}$ is a solution of (2.1), then we obtain

$$(2.5) \qquad k_2 G_2^2 - k_1 G_1^2 = 4(r_1 - r_2)m,$$

$$(2.6) \qquad k_3 G_3^2 - k_1 G_1^2 = 4(r_1 - r_3)m.$$

Now we apply the reduction method to equation (1.2). In this particular case, we obtain

$$a_0 = 1, \quad a_1 = -1, \quad a_2 = -\frac{c-1}{3}, \quad a_3 = c+1, \quad a_4 = -2c+1,$$

$$g_2 = \frac{c^2 + 4c + 16}{3}, \quad g_3 = \frac{c^3 + 6c^2 - 24c - 64}{27}.$$

One can see that equation (2.3) has three rational roots

$$\rho_1 = -\frac{1}{6}c + \frac{2}{3}, \quad \rho_2 = -\frac{1}{6}c - \frac{4}{3}, \quad \rho_3 = \frac{1}{3}c + \frac{2}{3}.$$

Since

$$\frac{a_1^2}{a_0} - a_2 = \frac{1}{3}c + \frac{2}{3} \geq \max\{\rho_1, \rho_2, \rho_3\},$$

so the reduction method can be applied to equation (1.2). We get

$$
\begin{aligned}
H_0(x, y) - 4r_1 F(x, y) &= 24c(x^2 - 2xy + 3y^2)^2, \\
H_0(x, y) - 4r_2 F(x, y) &= 24(c + 4)(x^2 - 2xy - y^2)^2, \\
H_0(x, y) - 4r_3 F(x, y) &= 48c(c + 4)(xy - y^2)^2.
\end{aligned}
$$

We take

$$k_1 = 6c, \ k_2 = 6(c + 4), \ k_3 = 3c(c + 4),$$

and

$$G_1 = 2(x^2 - 2xy + 3y^2), \ G_2 = 2(x^2 - 2xy - y^2), \ G_3 = 4y(x - y).$$

Let

(2.7) $\quad U = x^2 - 2xy + 3y^2, \quad V = |x^2 - 2xy - y^2|, \quad Z = |y(x - y)|.$

Then, by the method of Tzanakis, solving equation (1.2) reduces to solving the system of Pellian equations (1.3) and (1.4), where $|m| \leq \max\{\frac{c}{4}, 4\}$.

Assuming that x and y are relatively prime, from (2.7) we see that $\gcd(U, V, Z) = 1$. Since $U^2 - V^2 = 8Z^2$, we see $\gcd(U, V) = 1$ or 2.

3. CONTINUED FRACTIONS

In this section, we will consider the connections between solutions of the equations (1.3) and continued fraction expansion of the corresponding quadratic irrational.

The simple continued fraction expansion of a quadratic irrational $\alpha = \frac{a+\sqrt{d}}{b}$ is periodic. This expansion can be obtained using the following algorithm. Multiplying the numerator and the denominator by b, if necessary, we may assume that $b | (d - a^2)$. Let $s_0 = a$, $t_0 = b$ and

(3.1) $\quad a_n = \left\lfloor \frac{s_n + \sqrt{d}}{t_n} \right\rfloor, \quad s_{n+1} = a_n t_n - s_n, \quad t_{n+1} = \frac{d - s_{n+1}^2}{t_n}, \quad$ for $n \geq 0$.

If $(s_j, t_j) = (s_k, t_k)$ for $j < k$, then

$$\alpha = [a_0, \ldots, a_{j-1}, \overline{a_j, \ldots, a_{k-1}}].$$

Applying this algorithm to quadratic irrational $\sqrt{(c+4)/c}$, we find

$$\sqrt{\frac{c+4}{c}} = \left[1; \overline{\frac{c}{2}, 2}\right],$$
$$(s_0, t_0) = (0, c), (s_1, t_1) = (c, 4),$$
$$(s_2, t_2) = (c, c), (s_3, t_3) = (c, 4),$$

if c is an even positive integer, and

$$\sqrt{\frac{c+4}{c}} = \left[1, \overline{\frac{c-1}{2}, 1, 2c+2, 1, \frac{c-1}{2}, 2}\right],$$
$$(s_0, t_0) = (0, c), (s_1, t_1) = (c, 4),$$
$$(s_2, t_2) = (c-2, 2c-1), (s_3, t_3) = (c+1, 1),$$
$$(s_4, t_4) = (c+1, 2c-1), (s_5, t_5) = (c-2, 4),$$
$$(s_6, t_6) = (c, c), (s_7, t_7) = (c, 4),$$

if $c > 1$ is an odd positive integer. If $c = 1$, we have

$$\sqrt{\frac{c+4}{c}} = \sqrt{5} = \left[2, \overline{4}\right],$$
$$(s_0, t_0) = (0, 1), (s_1, t_1) = (2, 1), (s_2, t_2) = (2, 1).$$

Let (U, V, Z) be integer solution of the system of equations (1.3) and (1.4). One can observe that if one of the integers U, V, Z is equal to zero, then equations (1.3) and (1.4) imply that the two other integers are not equal to zero. Moreover, one can see that $U = 0$ or $V = 0$ is impossible, which follows immediately from $U^2 - V^2 = 8Z^2$.

Assume now that (U, V, Z) is a nonnegative solution of the system (1.3) and (1.4). Then $\frac{U}{V}$ is a good rational approximation of $\sqrt{\frac{c+4}{c}}$. We find that

$$\sqrt{\frac{c+4}{c}} + \frac{U}{V} > \sqrt{\frac{c+4}{c}}$$

which implies

$$\left|\sqrt{\frac{c+4}{c}} - \frac{U}{V}\right| = \left|\frac{c+4}{c} - \frac{U^2}{V^2}\right| \cdot \left|\sqrt{\frac{c+4}{c}} + \frac{V}{U}\right|^{-1}$$

$$< \frac{4|m|}{cV^2} \cdot \sqrt{\frac{c}{c+4}} \leq \begin{cases} \frac{4 \cdot \frac{c}{4}}{cV^2} \cdot 1 = \frac{1}{V^2}, & \text{if } c \geq 16, \\ \frac{16}{\sqrt{c(c+4)}V^2}, & \text{if } c \leq 15. \end{cases}$$

This implies that

$$\left|\sqrt{\frac{c+4}{c}} - \frac{U}{V}\right| < \frac{M}{V^2},$$

where $M = 1$ if $c \geq 15$, $M = 2$ if $7 \leq c \leq 14$, $M = 3$ if $4 \leq c \leq 6$, $M = 4$ if $c = 3$, $M = 5$ if $c = 2$ and $M = 8$ if $c = 1$.

Let p_k/q_k denote the kth convergent of α. The following result of Worley ([20]) and Dujella ([5]) extends classical results of Legendre and Fatou concerning Diophantine approximations of the form $\left|\alpha - \frac{a}{b}\right| < \frac{1}{2b^2}$ and $\left|\alpha - \frac{a}{b}\right| < \frac{1}{b^2}$.

THEOREM 3.1 (Worley [20], Dujella [5]). *Let α be a real number and a and b coprime nonzero integers, satisfying the inequality*

$$\left|\alpha - \frac{a}{b}\right| < \frac{M}{b^2},$$

where M is a positive real number. Then $(a, b) = (rp_{k+1} \pm up_k, rq_{k+1} \pm uq_k)$, for some $k \geq -1$ and nonnegative integers r and u such that $ru < 2M$.

Explicit version of Theorem 3.1 for $M = 2$, was given by Worley ([20, Corollary, p. 206]). Recently, Dujella and Ibrahimpašić ([6, Propositions 2.1 and 2.2]) extended Worley's work and gave explicit and sharp versions of Theorem 3.1 for $M = 3, 4, ..., 12$.

We would like to apply Theorem 3.1 in order to determine all values of m with $|m| \leq \max\left\{\frac{c}{4}, 4\right\}$ for which equation (1.3) has solutions. We need the following lemma (see Dujella and Jadrijević [8, Lemma 1]).

LEMMA 3.2. *Let $\alpha\beta$ be a positive integer which is not a perfect square, and let p_k/q_k denotes the kth convergent of continued fraction expansion of $\sqrt{\frac{\alpha}{\beta}}$. Let the sequences (s_k) and (t_k) be defined by (3.1) for the quadratic irrational $\frac{\sqrt{\alpha\beta}}{\beta}$. Then*

$$(3.2) \quad \alpha(rq_{k+1}+uq_k)^2 - \beta(rp_{k+1}+up_k)^2 = (-1)^k(u^2 t_{k+1}+2rus_{k+2}-r^2 t_{k+2}).$$

Since the period length of the continued fraction expansion of $\sqrt{\frac{c+4}{c}}$ is: equal to 2 if c is even, equal to 6 if $c > 1$ is odd and equal to 1 if $c = 1$, according to Lemma 3.2, we have to consider only the fractions $(rp_{k+1} + up_k)/(rq_{k+1} + uq_k)$ for: $k = 0, 1$ if c is even and $c = 1$ and $k = 0, 1, ..., 5$ if $c > 1$ is odd.

By checking all possibilities, it is now easy to prove the following result.

PROPOSITION 3.3. *Let c and m be integers such that $c \geq 1$ and $|m| \leq \max\left\{\frac{c}{4}, 4\right\}$ and such that equation (1.3) has a solution in integers U and V*

with $\gcd(U, V) = 1$ *or* 2. *Then*

$$
m \in \begin{cases}
\{1, 4, -\frac{c}{4}\}, & if \ c \equiv 0 \pmod 4, \ c \geq 16, \\
\{1, 4\}, & if \ c \not\equiv 0 \pmod 4, c \geq 13 \ and \ c = 10, 9, 6, 5, \\
\{1, -3, 4\}, & if \ c = 12, 2, \\
\{1, -2, 4\}, & if \ c = 11, \\
\{1, -2, 3, 4\}, & if \ c = 8, \\
\{1, 3, 4\}, & if \ c = 7, \\
\{1, -1, 2, -2, 4, -4\}, & if \ c = 4, \\
\{1, 2, -2, -3, 4, -4\}, & if \ c = 3, \\
\{1, -1, 4, -4\}, & if \ c = 1.
\end{cases}
$$

Furthermore, all solutions of this equation in relatively prime positive integers are given by $(U, V) = (p_{2k}, q_{2k})$ *if* $m = 1$ *and* $c \equiv 0 \pmod 4$, $c \geq 16$; $(U, V) = (p_{2k+1}, q_{2k+1})$ *if* $m = -\frac{c}{4}$ *and* $c \equiv 0 \pmod 4$; $(U, V) = (p_{6k}, q_{6k})$, $(p_{6k+2} + p_{6k+2}, q_{6k+2} + q_{6k+2})$, (p_{6k+4}, q_{6k+2}) *if* $m = 1$ *and* $c \equiv 1, 3 \pmod 4$, $c \geq 15$. *While all solutions of this equation in integers with* $\gcd(U, V) = 2$ *are given by* $(U, V) = (2p_{2k}, 2q_{2k})$ *if* $m = 4$ *and* $c \equiv 0 \pmod 4$, $c \geq 16$; $(U, V) = (2p_{6k}, 2q_{6k})$, $(2p_{6k+2} + 2p_{6k+2}, 2q_{6k+2} + 2q_{6k+2})$, $(2p_{6k+4}, 2q_{6k+2})$ *if* $m = 4$ *and* $c \equiv 1, 3 \pmod 4$, $c \geq 15$.
Here p_k/q_k *denotes the* k^{th} *convergent of continued fraction expansion of* $\sqrt{\frac{c+4}{c}}$ *and* $(p_{-1}, q_{-1}) = (1, 0)$.

Now we will discuss the solvability in integers U, V, Z, with $\gcd(U, V) = 1$ or 2, of the system of equations (1.3) and (1.4) where m is one of the admissible values from Proposition 3.3.

Assume now that for $m = -\frac{c}{4}$ and $c \equiv 0 \pmod 4$, $c \geq 16$, the system (1.3) and (1.4) has a solution (U, V, Z). Then, by Proposition 3.3, we have $U = p_{2k+1}$ for an integer $k \geq 0$.

We will need the recursive relations for the convergents of $\sqrt{\frac{c+4}{c}}$, $c \equiv 0 \pmod 4$, with odd and even subscripts. Let $c = 4d$, $d \geq 1$. Then from

(3.3) $p_0 = 1, \ p_1 = 2d + 1,$

(3.4) $p_{2k} = 2p_{2k-1} + p_{2k-2},$

(3.5) $p_{2k+1} = 2dp_{2k} + p_{2k-1},$

it follows easily

$$p_{2k+1} = 2(2d + 1)p_{2k-1} - p_{2k-3}.$$

The following lemma can be proved easily by induction.

LEMMA 3.4. *Let the sequences* (p_m) *be defined by* (3.3), (3.4), *and* (3.5). *Then for all* $l \geq 0$ *we have*

(3.6) $$p_{4l+3} \equiv 1 \pmod{8d+8},$$

(3.7) $$p_{4l+1} \equiv 2d+1 \pmod{8d+8}.$$

From equation (1.4) we have

(3.8) $$U^2 + d \equiv 0 \pmod{8d+8}.$$

On the other hand, if $c = 4d$ and $d \geq 4$, from Lemma 3.4 and $U = p_{2k+1}$, we have

$$U^2 + d \equiv 1 + d \not\equiv 0 \pmod{8d+8} \quad \text{if} \quad U = p_{4l+3}$$

or

$$U^2 + d \equiv (2d+1)^2 + d \equiv (d+1)(4d+1) \not\equiv 0 \pmod{8d+8} \quad \text{if} \quad U = p_{4l+1}.$$

Hence, we have proved that for $m = -\frac{c}{4}$ and $c \equiv 0 \pmod 4$, $c \geq 16$ the system (1.3) and (1.4) has no solution. We will consider the remaining cases in the next two sections.

4. CASES $m = 1, 4$

Let $c \geq 1$ and $m = 1, 4$. Let $x = X + Y$ and $y = Y$. Then equation (1.2) gives

(4.1) $$\left(2Y^2\right)^2 - (c+2)\left(2Y^2\right)X^2 + X^4 = m.$$

Furthermore, by (2.7), we have $(U, V, Z) = (X^2 + 2Y^2, |X^2 - 2Y^2|, |XY|)$.

We will apply the results of Cohn [3, Equation 1 and Equation 5] and Walsh [19, Theorem 2].

THEOREM 4.1 (Cohn [3]). *Let* K *be an odd positive integer.*

1. *The only solutions to* $x^2 - Kxy^2 + y^4 = 1$ *in non-negative integers* (x, y) *are* $(K, 1), (0, 1), (1, 0)$, *unless either* K *is a perfect square, in which case there is also the solution* $(1, \sqrt{K})$;
2. *The only solution in non-negative integers* x, y *to the equation* $x^2 - Kxy^2 + y^4 = 4$ *is* $(x, y) = (2, 0)$.

THEOREM 4.2 (Walsh [19]). *Let* K *be an even positive integer.*

1. *The only solutions to* $x^2 - Kxy^2 + y^4 = 1$ *in non-negative integers* (x, y) *are* $(K, 1), (0, 1), (1, 0)$, *unless either* K *is a perfect square, in which case there are also the solutions* $(1, \sqrt{K}), (K^2 - 1, \sqrt{K})$, *or* $K = 338$ *in which there are the solutions* $(x, y) = (114243, 6214), (13051348805, 6214)$
2. *The only solution in non-negative integers* x, y *to the equation* $x^2 - Kxy^2 + y^4 = 4$ *is* $(x, y) = (2, 0)$, *unless* $K = 2v^2$ *for some integer* v, *in which case there are also the solutions* $(2, \sqrt{2K}), (2K^2 - 2, \sqrt{2K})$.

Let us begin with $m = 1$ by applying the first part of the above theorems to equation (4.1).

CASE 1: For all $c \geq 1$, we obtain that possible solutions are given by $(2Y^2, \pm X) = (c + 2, 1), (0, 1), (1, 0)$.

• The first solution gives us $c = 2n^2 - 2, X = \pm 1, Y = \pm n$ for some positive integer n. This implies $(U, V, Z) = (2n^2 + 1, 2n^2 - 1, n)$ and the solutions of equation (1.2) are $(x, y) = \pm(n + 1, n), \pm(n - 1, n)$.

• The second solution gives us $X = \pm 1, Y = 0$. This implies $(U, V, Z) = (1, 1, 0)$, and the solutions are $(x, y) = (\pm 1, 0)$.

• It is obvious to see that there is no solution corresponding to $(2Y^2, \pm X) = (1, 0)$.

CASE 2: If $c + 2 = t^2$, for some positive integer t, then possible additional solutions are given by $(2Y^2, \pm X) = (1, t)$ for all $c \geq 1$ and $(2Y^2, \pm X) = (t^4 - 1, t)$ when c is an even integer.

• Using Cohn's result [4], one can see that the equation $2Y^2 = t^4 - 1$ has only the integer solutions $(Y, t) = (0, \pm 1)$. This implies $c = -1$ which gives us a contradiction.

• The second eventuality $(2Y^2, \pm X) = (1, t)$ is also impossible.

CASE 3: If $c + 2 = 338$, i.e., $c = 340$, then there is no additional solution because $2Y^2 \notin \{114243, 13051348805\}$.

Now we only apply the second part of Theorem 4.1 and Theorem 4.2 to deal with $m = 4$.

CASE 4: For all $c \geq 1$ possible solutions are given by $(2Y^2, \pm X) = (2, 0)$. This implies $(X, Y) = (0, \pm 1)$ and $(U, V, Z) = (2, 2, 0)$. Therefrom we obtain the solutions $(x, y) = \pm(1, 1)$.

CASE 5: If $K = 2n^2$, i.e. $c = 2n^2 - 2$ with $n > 1$, then we have two additional possibilities.

• First, we take $(2Y^2, \pm X) = (2, 2n)$. Therefore, we get $Y = \pm 1$. It follows that $(U, V, Z) = (2n^2 + 2, 2n^2 - 2, 2n)$, and gives the solutions $(x, y) = \pm(2n + 1, 1), \pm(2n - 1)$.

• Second, we take $(2Y^2, \pm X) = (2(2n^2)^2 - 2, 2n)$ and we get $Y^2 = (2n^2)^2 - 1$. This is impossible.

Therefore, all solutions are found when $m = 1, 4$.

REMARK 4.3. It is easy to see that when $m = 4$ and $X = 2X_1$, then from (4.1), we get

$$(2X_1^2)^2 - (c + 2)(2X_1^2)Y^2 + Y^4 = 1.$$

Conversely, one can see that any solution of equation (4.1) of the form $(X, Y) = (a, b)$ for $m = 1$ gives a solution of the form $(X, Y) = (2b, a)$ for $m = 4$.

5. Cases $m \neq 1, 4$

In this section, we study the remaining cases when $m \neq 1, 4$. By (1.3) and (1.4), we get

$$(5.1) \qquad V^2 - 2cZ^2 = m.$$

So if $c = 2$ and $m = -3$ then from equation (5.1) we obtain

$$V^2 - 4Z^2 = (V - 2Z)(V + 2Z) = -3.$$

This implies $V = \pm 1, Z = \pm 1$. Then, from (2.7), we obtain $(x, y) = \pm(2, 1)$ or $(x, y) = \pm(0, 1)$.

Now let us rewrite equation (4.1) to

$$(5.2) \qquad A^2 - \left(\delta^2 - 4\right) B^2 = m,$$

where $\delta = c + 2 \geq 3$, $A = X^2 - (c + 2) Y^2 = (x - y)^2 - (c + 2) y^2$, $B = Y^2 = y^2$. Since the period of continued fraction expansion of $\sqrt{\delta^2 - 4}$ is even if $\delta \geq 4$, there is no solution if $m = -1, -4$ and $c \geq 2$. If $\delta = 3$, i.e $c = 1$, the period is odd which implies that equation (5.2) has solutions with $m = -1$ and $m = -4$.

It remains to consider the cases:

- $c = 1$ and $m = -1$ or $m = -4$;
- $c = 3$ and $m = 2$ or $m = -2$ or $m = -3$;
- $c = 7$ and $m = -3$;
- $c = 8$ and $m = -2$ or $m = 3$;
- $c = 11$ and $m = -2$;
- $c = 12$ and $m = -3$.

We solve the remaining cases as follows.

CASE $c = 1$. When $m = -1$, equation (1.2) was solved by Tzanakis [18, p. 275]. For $m = -4$, we used *Thuesolver* of KASH ([14]). So we get

$$(x, y) = \begin{cases} (0, \pm 1), \pm(2, 1), & \text{if } m = -1, \\ \pm(3, 1), & \text{if } m = -4. \end{cases}$$

CASE $c = 3$. Equation (5.2) becomes $A^2 - 21B^2 = m$. If $m = 2$, it implies $A^2 \equiv 2 \pmod 3$. This is impossible. If $m = -2$, we obtain $A^2 \equiv -2 \equiv 5 \pmod 7$. So there is no solution. If $m = 3$, then $A^2 \equiv 3 \pmod 7$, but 3 is not a square modulo 7. Therefore, we have no solution (A, B) when $c = 3$.

CASE $c = 7, m = -3$. In this case, equation (5.2) has the form $A^2 - 77B^2 = -3$. By taking modulo 11, we obtain $A^2 \equiv -3 \pmod{11}$. Using the Legendre symbol, we get $(-3/11) = (8/11) = (2/11) = -1$. There is no solution (A, B).

CASE $c = 8$. Equation (5.2) becomes $A^2 - 96B^2 = m$. If $m = -2$, we get $A^2 \equiv -2 \equiv 2 \pmod 4$ which is impossible. If $m = 3$, similarly, $A^2 \equiv 3 \pmod 4$ gives us a contradiction.

CASE $c = 11, m = -2$. Then we have $A^2 - 165B^2 = -2$. We get that $A^2 \equiv -2 \pmod 5$. This is impossible.

CASE $c = 12, m = -3$. From equation (5.2), we deduce $A^2 - 192B^2 = -3$. Then $A^2 \equiv -3 \equiv 5 \pmod 8$ which is also impossible.

This completes the proof of Theorem 1.1.

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Bo He
Department of Mathematics
ABa Teacher's College
Wenchuan, Sichuan, 623000
P. R. China
E-mail: `bhe@live.cn`

B. Jadrijević
Faculty of Science
Department of Mathematics
University of Split
Teslina 12, 21000 Split
Croatia
E-mail: `borka@pmfst.hr`

A. Togbé
Mathematics Department
Purdue University North Central
1401 S, U.S. 421, Westville IN 46391
USA
E-mail: `atogbe@pnc.edu`