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ON CERTAIN CHARACTER SUMS OVER SMOOTH NUMBERS

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Abstract. We give nontrivial bounds in various ranges for character sums of the form

$$\sum_{n\in\mathcal{S}(x,\,y)} \chi(R_1(n))\mathbf{e}_q(R_2(n)),$$

where χ is a nonprincipal multiplicative character modulo a prime q, R_1 and R_2 are rational functions modulo q, and $\mathcal{S}(x,y)$ is the set of positive integers $n \le x$ that are divisible only by primes $p \le y$. We also give sharper bounds in some special cases.

1. Introduction

It is a central problem in analytic number theory to estimate sums of the type

$$\sum_{n\in\mathcal{N}} F(n), \tag{1.1}$$

where $\mathcal{N}$ is a finite subset of integers and $F : \mathcal{N} \to \mathbb{C}$ is a periodic function of period q. Usually, the sparser the set $\mathcal{N}$ is, the harder the sums (1.1) become to control.

Let χ be a nonprincipal multiplicative character modulo a prime q. R_1, R_2 are rational functions modulo q. Assume

$$R_1 = f_1/g_1, \qquad R_2 = f_2/g_2,$$

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where f_1, g_1, f_2, g_2 are integral polynomials such that $\gcd(f_i, g_i) = 1$, f_1, g_1 are monic, and $\gcd(g_1(x)g_2(x), q) = 1$ for any integer x. As usual, we denote $\mathbf{e}_q(z) := \exp(2\pi iz/q)$.

In 1962, combining his generalization of the Weil estimates ([11]) with a variant of Vinogradov sieve, Perel'muter ([12]) succeeded in controlling character sums

$$\sum_{p\le N} \chi(R_1(p))\mathbf{e}_q(R_2(p)),$$

where p runs through consecutive primes, and R_1, R_2 satisfy certain nondegenerate condition.

In this paper, motivated by recent work of Shparlinski ([14]) on linear character sums over shifted smooth integers, we study character sums in general form

$$S = \sum_{n\in\mathcal{S}(x,\,y)} \chi(R_1(n))\mathbf{e}_q(R_2(n)), \tag{1.2}$$

where $\mathcal{S}(x, y)$ is the set of y-smooth numbers in $[1, x]$. Recall that a positive integer n is called to be y-smooth if $P(n) \le y$, where $P(n)$ is the largest prime divisor of n. We will follow the approach of Shparlinski ([14]) and give nontrivial bounds for (1.2) in different ranges. For convenience we denote $\Phi(n) := \chi(R_1(n))\mathbf{e}_q(R_2(n))$.

Using some results of Karatsuba (see [6, 7] or the survey [8]) on character sums over shifted primes and Bourgain ([1]) on exponential sums over primes, much better estimates can be obtained for some special cases of (1.2).

Throughout the paper the implied constants in the symbols '$\ll$' and 'O' depend only on $\deg f_i$, $\deg g_i$ and ε. The letters p and q always signify prime numbers, x, y real numbers, and n a positive integer.

In what follows, we always assume that R_1, R_2 satisfy the condition

$$\textit{if } R_2 = ax + b, \textit{ then } R_1 \ne x,\ \frac{1}{x},\ \textit{or a constant}. \tag{1.3}$$

2. Some lemmas

We need a bound of Perel'muter ([12]) for character sums over primes.

Lemma 2.1. *For any $\varepsilon > 0$ and $x \ge y \ge q^{1+\varepsilon}$, we have*

$$\sum_{y\le p\le x} \Phi(p) \ll xq^{-\delta},$$

where $\delta = \delta(\varepsilon) > 0$.

The following two lemmas can also be found in [12].

LEMMA 2.2. *If at least one of $\chi(R_1(x))$ and $R_2(x)$ is not constant, then we have*

$$\sum_{a=0}^{q-1}\Phi(a)\ll\sqrt{q}.$$

LEMMA 2.3. *Let K, L, X, Y be integers with $0<X<q$, $Y>0$; a an integer, $\gcd(a,q)=1$; integers l,l_1 run through the interval $(L,L+Y]$ independently. Then*

$$\sum_{k=K+1}^{K+X}\Phi(ak)\ll\sqrt{q}\log q$$

uniformly with respect to a, K, and X;

$$\sum_{k=1}^{q}\Phi(akl)\overline{\Phi(akl_1)}\ll\sqrt{q}$$

uniformly with respect to a, l and l_1 for all (l,l_1) with $\ll Y+\frac{Y^2}{q}$ possible exceptional pairs.

We also need the following upper bounds for weighted double sums.

LEMMA 2.4. *Let K, L, X, Y be integers with $X,Y>0$, a an integer, $\gcd(a,q)=1$. Then for any complex sequence (γ_l) supported on the interval $(L,L+Y]$ with $|\gamma_l|\le 1$, we have*

$$\sum_{K<k\le K+X}\left|\sum_{L<l\le L+Y}\gamma_l\Phi(akl)\right|\ll\sqrt{XY\Big(\frac{X}{q}+1\Big)\Big(\frac{Y}{\sqrt{q}}+1\Big)q}$$

uniformly with respect to a.

PROOF. By Cauchy inequality,

$$\begin{aligned}
&\left(\sum_{K<k\le K+X}\left|\sum_{L<l\le L+Y}\gamma_l\Phi(akl)\right|\right)^2\\
&\qquad\le X\sum_{K<k\le K+X}\left|\sum_{L<l\le L+Y}\gamma_l\Phi(akl)\right|^2\\
&\qquad\le X\left(\frac{X}{q}+1\right)\sum_{k=1}^{q}\left|\sum_{L<l\le L+Y}\gamma_l\Phi(akl)\right|^2\\
&\qquad\le X\left(\frac{X}{q}+1\right)\sum_{L<l,\,l_1\le L+Y}\left|\sum_{k=1}^{q}\Phi(akl)\overline{\Phi(akl_1)}\right|.
\end{aligned}$$

By Lemma 2.3, the inner sums are at most $\sqrt{q}$ in absolute value if $l \not\equiv l_1 \pmod q$ and are at most q otherwise (which happens for at most $Y(Y/q+1)$ pairs (l, l_1)). Therefore

$$\left(\sum_{K<k\le K+X}\left|\sum_{L<l\le L+Y}\gamma_l\Phi(akl)\right|\right)^2$$
$$\ll X\left(\frac{X}{q}+1\right)\left(Y(Y/q+1)q+Y^2\sqrt{q}\right) \ll XY\left(\frac{X}{q}+1\right)\left(\frac{Y}{\sqrt{q}}+1\right)q$$

as required. ∎

The following result, which is Lemma 10.1 of [15], helps us to relate the double sums of Lemma 2.4 to the sums over smooth numbers.

LEMMA 2.5. *Suppose that $2 \le y \le z \le n \le x$ and $n \in \mathcal{S}(x,y)$. Then there exists a unique triple (p,u,v) of integers with the properties:*

(i) $n = uv$;
(ii) $u \in \mathcal{S}(x/v, p)$;
(iii) $z < v \le zp$;
(iv) $p|v$;
(v) *If $r|v$ for a prime r, then $p \le r \le y$.*

3. Main results

THEOREM 3.1. *For any $\varepsilon > 0$ and $x \ge y \ge q^{1+\varepsilon}$, we have*

$$S \ll xq^{-\delta}\log x,$$

where $\delta = \delta(\varepsilon) > 0$.

PROOF. We have

$$S = \sum_{\substack{n\le x\\ P(n)\le y}}\Phi(n) = \sum_{n\le x}\Phi(n) - \sum_{\substack{n\le x\\ P(n)>y\\ \gcd(n,\,q)=1}}\Phi(n) + O(x/q).$$

For the first sum, by Lemma 2.3, we have

$$\sum_{n\le x}\Phi(n) \ll \sqrt{q}\log q.$$

For the second sum, since each integer n included in it can be uniquely represented as $n = kp$ with a prime $p > y$ and a positive integer k such that

$P(k) \le p$, then

$$\begin{aligned}\sum_{\substack{n\le x\\ P(n)>y\\ \gcd(n,q)=1}} \Phi(n) &= \sum_{\substack{p>y\\ p\ne q}} \sum_{\substack{k\le x/p\\ P(k)\le p\\ \gcd(k,q)=1}} \Phi(kp)\\ &= \sum_{\substack{k\le x/y\\ \gcd(k,q)=1}} \chi(k^{\deg R_1}) \sum_{\substack{L_k<p\le x/k\\ p\ne q}} \Phi_k(p)\end{aligned}$$

by taking $L_k = \max\{y, P(k)-1\}$, where $\deg R_1 = \deg f_1 - \deg g_1$ and the rational functions in Φ_k still satisfy the condition (1.3).

Since $y \ge q^{1+\varepsilon}$, we can use Lemma 2.1 to estimate the sum over p, getting

$$\sum_{\substack{k\le x/y\\ \gcd(k,q)=1}} \chi(k^{\deg R_1}) \sum_{\substack{L_k<p\le x/k\\ p\ne q}} \Phi_k(p) \ll xq^{-\delta} \sum_{k\le x/y} \frac{1}{k} \ll xq^{-\delta}\log x\,,$$

which completes the proof. □

THEOREM 3.2. *We have*

$$S \le (xy^{1/2}q^{-1/4} + q^{3/2})x^{o(1)}.$$

PROOF. We follow the approach of Shparlinski ([14]). Let z be a fixed real number such that $2 \le y \le z \le x$. Then

$$S = \sum_{\substack{n\in\mathcal{S}(x,y)\\ n>z}} \Phi(n) + O(z) = \sum_{p\le y} U_\Phi(p,x,y,z) + O(z), \tag{3.1}$$

where

$$U_\Phi(p,x,y,z) = \sum_{v\in\mathcal{Q}(p,y,z)} \sum_{u\in\mathcal{S}(x/v,p)} \Phi(uv)$$

and

$$\mathcal{Q}(p,y,z) = \{v : z < v \le zp,\ p|v,\ \text{and if a prime } r|v,\ \text{then } p \le r \le y\}.$$

Writing $v = pw$, we have

$$|U_\Phi(p,x,y,z)| \le \sum_{z/p<w\le z} \left| \sum_{u\in\mathcal{S}(x/wp,p)} \Phi(pwu) \right|.$$

We can assume that

$$y < q < \frac{(\frac{x}{2})^{2/3}}{(\log x)^{8/3}} \tag{3.2}$$

since otherwise the result is trivial. Thus $p \ne q$ for any primes $p \le y$ for which we need to estimate $U_\Phi(p,x,y,z)$.

We now partition the summation range of w into level sets and bound $U_\Phi(p,x,y,z)$ by the double sums of Lemma 2.4.

Let us fix some real number Δ in the range

$$y/z \ll \Delta < 1/2 \tag{3.3}$$

and let

$$\mathcal{M}(p,z) = \left\{ \frac{z}{2p}(1+\Delta)^j : 0 \le j \le N_p \right\},$$

where

$$N_p = \left\lfloor \frac{\log(2p)}{\log(1+\Delta)} \right\rfloor \ll \Delta^{-1} \log p \ll \Delta^{-1} \log x.$$

We also have

$$\sum_{A \in \mathcal{M}(p,z)} A^{1/2} \ll \Delta^{-1} z^{1/2} \qquad \text{and} \qquad \sum_{A \in \mathcal{M}(p,z)} A^{-1/2} \ll \Delta^{-1} (p/z)^{1/2}.$$

Since for $A < w \le A(1+\Delta)$ we have

$$0 \le \#\mathcal{S}(x/Ap, p) - \#\mathcal{S}(x/wp, p) \le \frac{x}{Ap} - \frac{x}{wp} \le \frac{\Delta x}{Ap}$$

and $\Delta A \gg 1$ for all $A \in \mathcal{M}(p,z)$ by the assumption (3.3), therefore

$$\begin{aligned}
&|U_\Phi(p,x,y,z)| \\
&\qquad \le \sum_{A \in \mathcal{M}(p,z)} \sum_{A < w \le A(1+\Delta)} \left| \sum_{u \in \mathcal{S}(x/wp,\,p)} \Phi(pwu) \right| \\
&\qquad \le \sum_{A \in \mathcal{M}(p,z)} \sum_{A < w \le A(1+\Delta)} \left(\left| \sum_{u \in \mathcal{S}(x/Ap,\,p)} \Phi(pwu) \right| + O(\Delta x/Ap) \right) \\
&\qquad \le \sum_{A \in \mathcal{M}(p,z)} \left(\sum_{A < w \le A(1+\Delta)} \left| \sum_{u \in \mathcal{S}(x/Ap,\,p)} \Phi(pwu) \right| + O(\Delta^2 x/p) \right) \\
&\qquad = \sum_{A \in \mathcal{M}(p,z)} W(p,x,z,A) + O\left(\sum_{A \in \mathcal{M}(p,z)} \frac{\Delta^2 x}{p} \right),
\end{aligned}$$

where

$$W(p,x,z,A) = \sum_{A < w \le A(1+\Delta)} \left| \sum_{u \in \mathcal{S}(x/Ap,\,p)} \Phi(pwu) \right|.$$

Applying Lemma 2.4, we obtain

$$\begin{aligned} W(p,x,z,A) &\le (A\Delta)^{1/2}\left(\frac{x}{Ap}\right)^{1/2}\left(\frac{A\Delta}{q}+1\right)^{1/2}\left(\frac{x}{Ap\sqrt{q}}+1\right)^{1/2} q^{1/2} \\ &\le \left(\frac{\Delta x q}{p}\right)^{1/2}\left(\frac{\Delta x}{pq^{3/2}}+\frac{A\Delta}{q}+\frac{x}{Ap\sqrt{q}}+1\right)^{1/2} \\ &\le \frac{\Delta x}{pq^{1/4}}+\frac{\Delta A^{1/2}x^{1/2}}{p^{1/2}}+\frac{\Delta^{1/2}xq^{1/4}}{A^{1/2}p}+\frac{\Delta^{1/2}x^{1/2}q^{1/2}}{p^{1/2}}. \end{aligned}$$

Consequently,

$$U_\Phi(p,x,y,z) \ll \sum\nolimits_1+\sum\nolimits_2+\sum\nolimits_3+\sum\nolimits_4+\sum\nolimits_5,$$

where

$$\begin{aligned} \sum\nolimits_1 &= \frac{\Delta x}{pq^{1/4}}\sum_{A\in\mathcal{M}(p,\,z)} 1 \ll \frac{x\log x}{pq^{1/4}}, \\ \sum\nolimits_2 &= \frac{\Delta x^{1/2}}{p^{1/2}}\sum_{A\in\mathcal{M}(p,\,z)} A^{1/2} \ll \frac{x^{1/2}z^{1/2}}{p^{1/2}}, \\ \sum\nolimits_3 &= \frac{\Delta^{1/2}xq^{1/4}}{p}\sum_{A\in\mathcal{M}(p,\,z)} A^{-1/2} \ll \frac{xq^{1/4}}{\Delta^{1/2}z^{1/2}p^{1/2}}, \\ \sum\nolimits_4 &= \frac{\Delta^{1/2}x^{1/2}q^{1/2}}{p^{1/2}}\sum_{A\in\mathcal{M}(p,\,z)} 1 \ll \frac{x^{1/2}q^{1/2}\log x}{\Delta^{1/2}p^{1/2}}, \\ \sum\nolimits_5 &= \frac{\Delta^2 x}{p}\sum_{A\in\mathcal{M}(p,\,z)} 1 \ll \frac{\Delta x\log x}{p}. \end{aligned}$$

We choose

$$z = \left(\frac{xq^{1/2}}{\Delta}\right)^{1/2}$$

to balance the bounds on $\sum_2$ and $\sum_3$ as $O(x^{3/4}q^{1/8}\Delta^{-1/4}p^{-1/2})$. Then we choose

$$\Delta = x^{-1}q^{3/2}\log^4 x$$

to balance the bounds on $\sum_3$ and $\sum_4$ as $O(xq^{-1/4}p^{-1/2}\log^{-1} x)$. With this choice we see that $\sum_5 \ll q^{3/2}p^{-1}\log^5 x$.

We have $z = \left(\frac{xq^{1/2}}{\Delta}\right)^{1/2} = xq^{-1/2}\log^{-2} x$. Therefore the inequalities (3.2) imply that (3.3) as well as the condition $x \ge z \ge y$ is satisfied for the above choice of z and Δ.

Therefore

$$U_\Phi(p,x,y,z) \ll \frac{x\log x}{pq^{1/4}}+\frac{x}{q^{1/4}p^{1/2}\log x}+\frac{q^{3/2}\log^5 x}{p} \ll \frac{x\log x}{q^{1/4}p^{1/2}}+\frac{q^{3/2}\log^5 x}{p}.$$

Summing this up over all $p \le y$ and using (3.1), we obtain:

$$S \ll \sum_{p \le y} \left(\frac{x \log x}{q^{1/4} p^{1/2}} + \frac{q^{3/2} \log^5 x}{p} \right) + x q^{-1/2} \log^{-2} x$$
$$\le x^{1+o(1)} y^{1/2} q^{-1/4} + q^{3/2} x^{o(1)} + x q^{-1/2} \log^{-2} x.$$

The third term never dominates, which completes the proof. □

4. Special cases

In this section, we give sharper bounds in some special cases of (1.2). All of them are consequences of the corresponding sums over primes.

i) For the linear case

$$\sum_{n \in \mathcal{S}(x,y)} \chi(n+a), \qquad \gcd(a,q) = 1,$$

nontrivial bounds have been established by Shparlinski ([14, Theorems 4, 5]).

ii) For the quadratic case

$$\sum_{n \in \mathcal{S}(x,y)} \chi((n+a)(n+b)), \qquad a \not\equiv b \pmod q,$$

we can deduce the following theorems from the estimates of Karatsuba ([7, 8]) on character sums over shifted primes and the double sums estimates of Shparlinski ([14, Lemma 2]).

Theorem 4.1. *There exists an absolute constant $c > 0$ such that for any $\varepsilon > 0$ and $x \ge y \ge q^{3/4+\varepsilon}$ we have*

$$\sum_{n \in \mathcal{S}(x,y)} \chi((n+a)(n+b)) \ll x q^{-c\varepsilon^2} \log x.$$

Theorem 4.2. *We have*

$$\sum_{n \in \mathcal{S}(x,y)} \chi((n+a)(n+b)) \le \begin{cases} (x y^{1/2} q^{-1/2} + q^2) x^{o(1)}, & \text{if } ab \equiv 0 \pmod q; \\ (x y^{1/2} q^{-1/4} + q^{3/2}) x^{o(1)}, & \text{if } ab \not\equiv 0 \pmod q. \end{cases}$$

iii) For $R_2(x) = \frac{f_2(x)}{g_2(x)}$ is not constant or linear, Fouvry and Michel ([3, Theorem 1.1]) established the following estimates

$$\sum_{p \le x} \mathbf{e}_q(R_2(p)) \ll x^{25/32} q^{3/16+\varepsilon} \tag{4.1}$$

with any $\varepsilon > 0$, $1 \le x \le q$, and the implied constant depends at most on $\deg f_2$, $\deg g_2$ and ε. Obviously (4.1) is nontrivial for the range $q^{6/7+\varepsilon} \le x \le q$, thus we can bound the corresponding sums over smooth integers as follows.

THEOREM 4.3. *For $R_2(x)$ is not constant or linear, there exists an absolute constant $c > 0$ such that for any $\varepsilon > 0$ and $x \geq y \geq q^{6/7+\varepsilon}$ we have*

$$\sum_{n \in \mathcal{S}(x,y)} \mathbf{e}_q(R_2(n)) \ll xq^{-\delta} \log x$$

for some $\delta = \delta(\varepsilon) > 0$.

iv) Since Bourgain ([1, Theorem A.9]) improved (4.1) for the cases $R_2(x) = ax^{-1} + bx$ or $x^k + ux$, k, $u \in \mathbb{Z}$, $k < 0$, we can immediately improve Theorem 4.3 for such R_2.

THEOREM 4.4. *For $R_2(x) = ax^{-1} + bx$ or $x^k + ux$, k, $u \in \mathbb{Z}$, $k < 0$, there exists an absolute constant $c > 0$ such that for any $\varepsilon > 0$ and $x \geq y \geq q^{1/2+\varepsilon}$ we have*

$$\sum_{n \in \mathcal{S}(x,y)} \mathbf{e}_q(R_2(n)) \ll xq^{-\delta} \log x$$

for some $\delta = \delta(\varepsilon) > 0$.

5. Remarks

We note that Fouvry and Tenenbaum ([4]), Maier ([10]), de la Bretèche and Tenenbaum ([2]) have studied exponential sums with multiplicative coefficients over smooth integers. Particularly, they give nontrivial estimates for sums

$$\sum_{n \in \mathcal{S}(x,y)} \chi(n) \exp(2\pi i n \vartheta) = \sum_{n \in \mathcal{S}(x,y)} \chi(n) \mathbf{e}_q(an)$$

for various ranges of q, x and y, where $\vartheta = a/q$, $\gcd(a, q) = 1$. However, we remark that these results are not included in the present paper, since we are bounding the sums (1.2) under the condition (1.3).

Since the method of Shparlinski ([14]) can also be applied to similar character sums (for example, see Rakhmonov ([13])) modulo a composite number, we remark that one can obtain much better bounds for some special moduli. For example, Kopaneva ([9]) recently proved some strong results on sums of characters modulo a power of a fixed prime numbers.

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