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ON SOME FUNCTIONAL EQUATIONS ON STANDARD OPERATOR ALGEBRAS

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Abstract. The main purpose of this paper is to prove the following result. Let X be a real or complex Banach space, let $L(X)$ be the algebra of all bounded linear operators on X, let $A(X) \subseteq L(X)$ be a standard operator algebra, and let $T : A(X) \to L(X)$ be an additive mapping satisfying the relation $T(A^{2n+1}) = \sum_{i=1}^{2n+1} (-1)^{i+1} A^{i-1} T(A) A^{2n+1-i}$, for all $A \in A(X)$ and some fixed integer $n \geq 1$. In this case T is of the form $T(A) = AB+BA$, for all $A \in A(X)$ and some fixed $B \in L(X)$. In particular, T is continuous.

Throughout, R will represent an associative ring. Given an integer $n > 1$, a ring R is said to be n-torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. An additive mapping $x \mapsto x^*$ on a ring R is called an involution if $(xy)^* = y^* x^*$ and $x^{**} = x$ hold for all pairs $x, y \in R$. A ring equipped with an involution is called a ring with involution or *-ring. Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime in case $aRa = (0)$ implies $a = 0$. Let A be an algebra over the real or complex field and let B be a subalgebra of A. A linear mapping $D : B \to A$ is called a linear derivation in case $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in B$. In case we have a ring R an additive mapping $D : R \to R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. A derivation D is inner in case there exists $a \in R$, such that $D(x) = ax - xa$ holds for all $x \in R$. Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein [7] asserts that any

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Jordan derivation on a 2-torsion free prime ring is a derivation. Cusack [4] generalized Herstein's result to 2-torsion free semiprime rings. It should be mentioned that Beidar, Brešar, Chebotar and Martindale have considerably generalized Herstein's theorem (see [2, Theorem 4.4]. For explanation of the symmetric Martindale ring of quotients of a semiprime ring R, which will be denoted by $Q_S(R)$, we refer to [1]. Let X be a real or complex Banach space, and let $L(X)$ and $F(X)$ denote the algebra of all bounded linear operators on X, and the ideal of all finite rank operators in $L(X)$, respectively. An algebra $A(X) \subseteq L(X)$ is said to be standard in case $F(X) \subseteq A(X)$. Let us point out that any standard algebra is prime, which is a consequence of Hahn-Banach theorem. A projection $P \in L(H)$, where H is a complex Banach space, is called bicircular in case all mappings of the form $e^{i\alpha}P + e^{i\beta}(I - P)$, where I denotes the identity operator, are isometric for all pairs of real numbers α, β.

Vukman, Kosi-Ulbl and Eremita [14] have proved the following result.

Theorem 1 ([14, Theorem 2.1]). *Let R be a 2-torsion free semiprime ring. Suppose that $T : R \to R$ is an additive mapping satisfying the relation*

$$T(xyx) = T(x)yx - xT(y)x + xyT(x) \tag{1}$$

for all pairs $x, y \in R$. In this case T is of the form

$$2T(x) = qx + xq,$$

for all $x \in R$ and some fixed $q \in Q_s(R)$.

Putting in the relation (1) $y = x$ we obtain

$$T(x^3) = T(x)x^2 - xT(x)x + x^2T(x), \qquad x \in R. \tag{2}$$

Fošner and Vukman [6] have recently proved the following result.

Theorem 2 ([6, Theorem 3.2]). *Let R be a 2-torsion free prime ring. Suppose that $T : R \to R$ is an additive mapping satisfying the relation (2) for all $x \in R$. In this case T is of the form*

$$4T(x) = qx + xq$$

for all $x \in R$ and some fixed $q \in Q_s(R)$.

From the relation (2) one obtains by induction the following generalization

$$T(x^{2n+1}) = \sum_{i=1}^{2n+1} (-1)^{i+1} x^{i-1} T(x) x^{2n+1-i}, \qquad x \in R, \tag{3}$$

where $n \geq 1$ is some fixed integer. In this paper we consider the relation (3) in standard operator algebras.

THEOREM 3. *Let X be a real or complex Banach space and let $A(X)$ be a standard operator algebra on X. Suppose $T : A(X) \to L(X)$ is an additive mapping satisfying the relation*

$$T(A^{2n+1}) = \sum_{i=1}^{2n+1} (-1)^{i+1} A^{i-1} T(A) A^{2n+1-i},$$

for all $A \in A(X)$ and some fixed integer $n \geq 1$. In this case T is of the form

$$T(A) = AB + BA,$$

for all $A \in A(X)$ and some fixed $B \in L(X)$. In particular, T is continuous.

Let us point out that in the theorem above we obtain as a result the continuity of T under purely algebraic assumptions concerning the mapping T. Therefore, the above result might be of some interest from the automatic continuity point of view. In the proof of Theorem 3 we shall use Theorem 2.

PROOF OF THEOREM 3. We have the relation

$$T(A^{2n+1}) = \sum_{i=1}^{2n+1} (-1)^{i+1} A^{i-1} T(A) A^{2n+1-i}, \quad A \in A(X). \tag{4}$$

Let A be from $F(X)$ and let $P \in F(X)$ be a projection with $AP = PA = A$. Putting $A + mP$ $(m \in \mathbb{N})$ for A in the relation (4) and comparing the coefficients of m^{2n}, we obtain

$$\begin{aligned}(2n+1)\,T\,(A) \\ &= (T\,(A)\,P + 2nT(P)A) + (PT\,(A) + 2nAT(P)) \\ &\quad + (-PT\,(A)\,P + PT\,(A)\,P - \cdots + PT\,(A)\,P - PT\,(A)\,P) \\ &\quad + (-1 + 2 - \cdots + (2n-2) - (2n-1))\,(AT(P)P + PT(P)A)\,.\end{aligned}$$

The above equation reduces to

$$\begin{aligned}(2n+1)\,T\,(A) = T\,(A)\,P + PT\,(A) - PT\,(A)\,P \\ + n\,(2T(P)A + 2AT(P) - AT(P)P - PT(P)A)\,.\end{aligned} \tag{5}$$

Multiplying the above relation from both sides by P we obtain

$$2PT(A)P = AT(P)P + PT(P)A, \tag{6}$$

which reduces (5) to

$$\begin{aligned}(2n+1)\,T(A) = T(A)P + PT(A) - PT\,(A)\,P \\ + 2n\,(T(P)A + AT(P) - PT\,(A)\,P)\,.\end{aligned} \tag{7}$$

Left multiplication of the above relation by P gives

$$PT(A) = PT(P)A + AT(P) - PT\,(A)\,P. \tag{8}$$

Similarly, we obtain

$$T(A)P = AT(P)P + T(P)A - PT\,(A)\,P. \tag{9}$$

Adding up (8) and (9), and applying (6), we obtain

$$T(A)P + PT(A) = AT(P) + T(P)A.$$

Inserting this in (7), we get

$$T(A) = T(P)A + AT(P) - PT(A)P. \tag{10}$$

From (10) and (6) we obtain

$$2T(A) = A(2T(P) - T(P)P) + (2T(P) - PT(P))A.$$

Hence $2T(A) = AQ + RA$ with $Q = 2T(P) - T(P)P$ and $R = 2T(P) - PT(P)$; note that $AQA = ARA$. Direct calculation yields

$$T(A^3) = T(A)A^2 - AT(A)A + A^2T(A). \tag{11}$$

From the relation (10) one can conclude that T maps $F(X)$ into itself. Therefore we have an additive mapping $T : F(X) \to F(X)$ satisfying the relation (11) for all $A \in F(X)$. Since $F(X)$ is prime one can apply Theorem 2, which means that T is of the form

$$4T(A) = AC + CA,$$

for all $A \in F(X)$ and some $C \in Q_s(F(X))$. Since $Q_s(F(X)) = L(X)$ (this is the direct consequence of [1, Theorem 4.3.8] and [8, p.78, Example 5]), one can conclude that T is of the form

$$T(A) = AB + BA, \tag{12}$$

for all $A \in F(X)$ and some $B \in L(X)$. It remains to prove that the relation (12) holds on $A(X)$ as well. Let us introduce $T_1 : A(X) \to L(X)$ by $T_1(A) = AB + BA$ and consider $T_0 = T - T_1$. The mapping T_0 is, obviously, linear and satisfies the relation (4). Besides, T_0 vanishes on $F(X)$. It is our aim to prove that T_0 vanishes on $A(X)$ as well. Let $A \in A(X)$, let P be an one-dimensional projection and

$$S = A + PAP - (AP + PA).$$

We have $T_0(S) = T_0(A)$ and $SP = PS = 0$. We have

$$T_0(A^{2n+1}) = \sum_{i=1}^{2n+1} (-1)^{i+1} A^{i-1} T_0(A) A^{2n+1-i},$$

for all $A \in A(X)$. Applying the above relation we obtain

$$\begin{aligned}
&\sum_{i=1}^{2n+1}(-1)^{i+1}S^{i-1}T_0(S)S^{2n+1-i}\\
&\quad= T_0(S^{2n+1}) = T_0(S^{2n+1}+P) = T_0((S+P)^{2n+1})\\
&\quad= \sum_{i=1}^{2n+1}(-1)^{i+1}(S+P)^{i-1}T_0(S+P)(S+P)^{2n+1-i}\\
&\quad= T_0(A)\left(S^{2n}+P\right) + \sum_{i=2}^{2n}(-1)^{i+1}\left(S^{i-1}+P\right)T_0(A)\left(S^{2n+1-i}+P\right)\\
&\qquad+\left(S^{2n}+P\right)T_0(A)\\
&\quad= T_0(A)S^{2n} + T_0(A)P + \sum_{i=2}^{2n}(-1)^{i+1}S^{i-1}T_0(A)S^{2n+1-i}\\
&\qquad+\sum_{i=2}^{2n}(-1)^{i+1}PT_0(A)S^{2n+1-i} + \sum_{i=2}^{2n}(-1)^{i+1}S^{i-1}T_0(A)P\\
&\qquad+S^{2n}T_0(A) + PT_0(A) - PT_0(A)P\\
&\quad= \sum_{i=1}^{2n+1}(-1)^{i+1}S^{i-1}T_0(A)S^{2n+1-i} + \sum_{i=2}^{2n}(-1)^{i+1}PT_0(A)S^{2n+1-i}\\
&\quad+\sum_{i=2}^{2n}(-1)^{i+1}S^{i-1}T_0(A)P + T_0(A)P + PT_0(A) - PT_0(A)P.
\end{aligned}$$

We have therefore

$$\begin{aligned}
\sum_{i=2}^{2n}(-1)^{i+1}PT_0(A)S^{2n+1-i} + \sum_{i=2}^{2n}(-1)^{i+1}S^{i-1}T_0(A)P&\\
+ T_0(A)P + PT_0(A) - PT_0(A)P = 0.&
\end{aligned} \tag{13}$$

Multiplying the above relation from both sides by P we obtain

$$PT_0(A)P = 0,$$

which reduces the relation (13) to

$$\sum_{i=2}^{2n}(-1)^{i+1}PT_0(A)S^{2n+1-i}+\sum_{i=2}^{2n}(-1)^{i+1}S^{i-1}T_0(A)P+T_0(A)P+PT_0(A) = 0.$$

Right multiplication of the above relation by P gives

$$\sum_{i=1}^{2n}(-1)^{i+1}S^{i-1}T_0(A)P = 0. \tag{14}$$

Putting in the above relation $-A$ for A (note that in this case S becomes $-S$), and comparing the relation so obtained with the relation (14), we obtain

$$\sum_{i=1}^{n} S^{2(i-1)} T_0(A) P = 0. \tag{15}$$

Inserting mA ($m \in \mathbb{N}$) instead of A we get (since S is then replaced by mS)

$$\sum_{i=1}^{n} S^{2(i-1)} T_0(A) P m^{2i-1} = 0.$$

The coefficient of m is equal to $T_0(A)P$. We have therefore $T_0(A)P = 0$. Since P is an arbitrary one-dimensional projection, it follows that $T_0(A) = 0$, for any $A \in A(X)$, which completes the proof of the theorem. ∎

In [9] one can find the following result.

THEOREM 4 ([9, Theorem 1]). *Let X be a real or complex Banach space and let $A(X)$ be a standard operator algebra on X. Suppose $D : A(X) \to L(X)$ is a linear mapping satisfying the relation*

$$D(A^n) = \sum_{i=1}^{n} A^{i-1} D(A) A^{n-i},$$

for all $A \in A(X)$ and some integer $n > 1$. In this case D is of the form

$$D(A) = AB - BA,$$

for all $A \in A(X)$ and some $B \in L(X)$, which means that D is a linear derivation. In particular, D is continuous.

The history of the above result goes back to the classical result of Chernoff [3] (see also [12,13,16]) which states that in case there exists a linear derivation D, which maps a standard operator algebra $A(X)$ into $L(X)$, where X is a real or complex Banach space, then D is of the form $D(A) = AB - BA$, for all $A \in A(X)$ and some $B \in L(X)$. Theorem 4 generalizes the result we have just mentioned above. Let us point out that in Theorem 3 we assumed that T is an additive mapping, while in Theorem 4 we have stronger assumption that D is linear. In general Chernoff's result and therefore also Theorem 4 cannot be proved by assuming that D is additive as shown by Šemrl in [12].

In the proof of our next result we apply Theorem 3 and Theorem 4.

COROLLARY 5. *Let X be a real or complex Banach space and let $A(X)$ be a standard operator algebra on X. Suppose $D, G : A(X) \to L(X)$ are linear* *mappings satisfying the relations*

$$\begin{aligned}(16)\quad D(A^{2n+1}) &= D(A)A^{2n} + AG(A)A^{2n-1} + A^2D(A)A^{2n-2} + \cdots \\ &\quad + A^{2n-1}G(A)A + A^{2n}D(A),\\ (17)\quad G(A^{2n+1}) &= G(A)A^{2n} + AD(A)A^{2n-1} + A^2G(A)A^{2n-2} + \cdots \\ &\quad + A^{2n-1}D(A)A + A^{2n}G(A),\end{aligned}$$

for all $A \in A(X)$ and some fixed integer $n \geq 1$. In this case D and G are of the form

$$D(A) = AB - CA, \qquad G(A) = AC - BA,$$

for all $A \in A(X)$ and some B and C from $L(X)$. In particular, D and G are continuous.

Proof. Adding up (16) with (17) we obtain

$$F(A^{2n+1}) = \sum_{i=1}^{2n+1} A^{i-1}F(A)A^{2n+1-i}, \tag{18}$$

for all $A \in A(X)$, where F stands for $D + G$. Subtracting (17) from (16) we obtain

$$H(A^{2n+1}) = \sum_{i=1}^{2n+1} (-1)^{i+1}A^{i-1}H(A)A^{2n+1-i}, \tag{19}$$

for all $A \in A(X)$, where H denotes $D - G$. Now, applying Theorem 3 and Theorem 4, we obtain

$$F(A) = D(A) + G(A) = AB - BA, \tag{20}$$

and

$$H(A) = D(A) - G(A) = AC + CA, \tag{21}$$

for all $A \in A(X)$ and some fixed B and C from $L(X)$. From (20) and (21) we obtain $2D(A) = A\,(B + C) + (C - B)\,A$, $2G(A) = A\,(B - C) - (B + C)\,A$. Replacing $\frac{1}{2}\,(B + C)$ by B and $\frac{1}{2}\,(B - C)$ by C we obtain $D(A) = AB - CA$, $G(A) = AC - BA$ for all $A \in A(X)$, which completes the proof of the corollary. □

Stachó and Zalar ([10, 11]) investigated bicircular projections on C^*-algebra $L(H)$, the algebra of all bounded linear operators on a complex Hilbert space H. According to Proposition 3.4 in [10] every bicircular projection $P : L(H) \to L(H)$ satisfies the relation

$$P(xyx) = P(x)yx - xP(y^*)^*x + xyP(x) \tag{22}$$

for all pairs $x, y \in L(H)$. Fošner and Ilišević [5] investigated the above functional equation in $2-$torsion free semiprime *-rings. They expressed the solution of the equation (22) in terms of derivations and so-called double centralizers. Vukman showed that applying more direct approach makes it possible to prove a more general result ([15]). Fošner and Vukman ([6]) investigated the following system of functional equations on 2-torsion free prime *-rings

$$P(x^3) = P(x)x^2 + xQ(x^*)^*x + x^2P(x), \quad x \in R, \tag{23}$$

$$Q(x^3) = Q(x)x^2 + xP(x^*)^*x + x^2Q(x), \quad x \in R. \tag{24}$$

The observations above lead to our next result.

COROLLARY 6. *Let H be a real or complex Hilbert space and let $A(H)$ be a standard operator algebra on H which is closed under the adjoint operation. Suppose $P, Q : A(H) \to L(H)$ are linear mappings satisfying the relations*

$$\begin{aligned} P(A^{2n+1}) &= P(A)A^{2n} + AQ(A^*)^*A^{2n-1} + A^2P(A)A^{2n-2} + \cdots \\ &\quad + A^{2n-1}Q(A^*)^*A + A^{2n}P(A), \end{aligned} \tag{25}$$

$$\begin{aligned} Q(A^{2n+1}) &= Q(A)A^{2n} + AP(A^*)^*A^{2n-1} + A^2Q(A)A^{2n-2} + \cdots \\ &\quad + A^{2n-1}P(A^*)^*A + A^{2n}Q(A), \end{aligned} \tag{26}$$

for all $A \in A(H)$ and some integer $n \geq 1$. In this case P and Q are of the form

$$P(A) = AB - CA, \qquad Q(A) = -AB^* + C^*A,$$

for all $A \in A(H)$ and some fixed $B, C \in L(H)$. In particular, P and Q are continuous.

PROOF. Put $D(A) = P(A)$ and $G(A) = Q(A^*)^*$ for all $A \in A(H)$ and apply Corollary 5. There exist $B, C \in L(H)$ such that

$$P(A) = AB - CA, \qquad Q(A^*)^* = AC - BA$$

for all $A \in A(H)$, that is

$$P(A) = AB - CA, \qquad Q(A) = -AB^* + C^*A$$

for all $A \in A(H)$. The proof of the corollary is complete. ∎

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