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ON n-FOLD HYPERSPACES OF CONTINUA

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Abstract. We continue our study of n-fold hyperspaces and n-fold hyperspace suspensions.We present more properties of these hyperspaces.

1. Introduction

The notion of n-fold hyperspace suspension was introduced in [16]. This concept is a natural extension of the notion of hyperspace suspension introduced by Nadler [24].

Our purpose is to continue the study of the properties of the n-fold hyperspaces and n-fold hyperspace suspensions. For example:

In [6, Example 4.5] the authors present two continua X and Y such that X is indecomposable, Y is decomposable and the hyperspace of subcontinua of X is homeomorphic to the hyperspace of subcontinua of Y; we prove that this does not happen, even for n-fold hyperspaces, if X has the property of Kelley. With the same techinque, it may be shown that the hyperspace of nonempty closed subsets of X is not homeomorphic to the hyperspace of nonempty closed subsets of Y, for these continua X and Y. We characterize n-fold hyperspaces which are homogeneous. We prove that for $n \geq 2$ the n-fold hyperspace of a finite-dimensional continuum X is not homeomorphic to its topological suspension. If $n \geq 2$ and the n-fold hyperspace of a continuum X is homeomorphic to the topological suspension of a finite-dimensional continuum Z, then X is hereditarily decomposable and does not contain terminal subcontinua. Also, we show that the n-fold hyperspace of a continuum X is

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not homeomorphic to the n-fold symmetric product of X. If X is an indecomposable continuum, then the n-fold hyperspace of X is not homeomorphic to the n-fold hyperspace suspension of X.

Regarding n-fold hyperspace suspensions, we prove that if $n \geq 2$ then the n-fold hyperspace suspension of a finite-dimensional continuum X is not homeomorphic to its topological cone. If $n \geq 2$ and the n-fold hyperspace suspension of a continuum X is homeomorphic to the topological cone of a finite-dimensional continuum Z, then X is hereditarily decomposable and does not contain terminal subcontinua. If X is a finite-dimensional continuum and the 1-fold hyperspace suspension of X is homeomorphic to the 2-fold symmetric product of X, then X is homeomorphic to $[0,1]$. If X is a finite-dimensional continuum and n is an integer greater than two, then the 1-fold hyperspace suspension of X is not homeomorphic to n-fold symmetric product of X. If X is an absolute retract, then the n-fold hyperspace suspension of X is an absolute retract.

2. Definitions

If (Z,d) is a metric space, then given $A \subset Z$ and $\varepsilon > 0$, the open ball about A of radius ε is denoted by $\mathcal{V}^d_\varepsilon(A)$, the interior of A is denoted by $Int_Z(A)$.

A *map* means a continuous function. Let X and Z be metric spaces and let $\varepsilon > 0$ be given. A map $f\colon X \to Z$ is an ε-*map* if $\text{diam}(\big(f^{-1}(f(x))\big) < \varepsilon$ for each $x \in X$.

Given a metric space Z, $\text{Cone}(Z)$ denotes the topological *cone over* Z, and $\Sigma(Z)$ denotes the topological *suspension over* Z; also, v_1 and v_2 denote the vertexes of $\Sigma(Z)$.

A *continuum* is a nonempty compact, connected metric space. A *subcontinuum* is a continuum contained in a space Z. A continuum X is said to be *indecomposable* provided that it cannot be written as the union of two of its proper subcontinua. A continuum is *hereditarily indecomposable* if all of its subcontinua are indecomposable. A continuum is *decomposable* if it is not indecomposable. A continuum is *hereditarily decomposable* provided that each of its nondegenerate subcontinuum is decomposable.

A continuum X is *acyclic* if $\check{H}^1(X,\mathbb{Z}) = 0$; i.e., the first Čech cohomology group with integer coefficients is trivial. The continuum X has *property* (b) provided that each map $f\colon X \to \mathcal{S}^1$ is homotopic to a constant map, where $\mathcal{S}^1$ is the unit circle in the plane.

A subcontinuum A of a continuum X is *terminal*, if for any subcontinuum Y of X such that $Y \cap A \neq \emptyset$, we have that either $A \subset Y$ or $Y \subset A$.

A subcontinnuum A is a *retract* of the continuum X provided that there exists a map $r\colon X \twoheadrightarrow A$ such that $r(a) = a$ for each $a \in A$, the map r is called a *retraction*. A continuum X is an *absolute retract* provided that whenever X is embedded as a subset X' of a space Z, X' is a retract of Z.

A *dendroid* is an arcwise connected continuum such that the intersection of any two of its subcontinua is connected. A *graph* is a continuum which can be written as the union of finitely many arcs any two of which are either disjoint or intersect in one or both of their endpoints. A *tree* is a graph without simple closed curves.

An *arc* is any space homeomorphic to $[0,1]$. The countable product of intervals, $\prod_{n=1}^{\infty}[0,1]$, with the product topology, is called the *Hilbert cube*. The symbol $\mathcal{Q}$ denotes the Hilbert cube.

A continuum X is *arc-like* (*circle-like*) if for each $\varepsilon > 0$, there exists a surjective ε-map $f\colon X \twoheadrightarrow [0,1]$ ($f\colon X \twoheadrightarrow \mathcal{S}^1$, respectively).

Given a continuum X, we consider the following *hyperspaces*:

$$2^X = \{A \subset X \mid A \text{ is nonempty and closed}\}$$

and

$$\mathcal{C}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ components}\},$$

where n is a positive integer. $\mathcal{C}_n(X)$ is called the *n-fold hyperspace* of X. These spaces are topologized with the Hausdorff metric defined as follows:

$$\mathcal{H}(A,B) = \inf\{\varepsilon > 0 \mid A \subset \mathcal{V}^d_\varepsilon(B) \text{ and } B \subset \mathcal{V}^d_\varepsilon(A)\},$$

$\mathcal{H}$ always denotes the Hausdorff metric on 2^X. When $n = 1$, we write $\mathcal{C}(X)$ instead of $\mathcal{C}_1(X)$.

The symbol $\mathcal{F}_n(X)$ denotes the *n-fold symmetric product* of X; that is:

$$\mathcal{F}_n(X) = \{A \in \mathcal{C}_n(X) \mid A \text{ has at most } n \text{ points}\}.$$

Note that, by definition, $\mathcal{F}_n(X) \subset \mathcal{C}_n(X)$. It is known that $\mathcal{C}_n(X)$ is an arcwise connected continuum (for $n = 1$, see [23, (1.12)]; for $n \geq 2$, see [14, 3.1]).

By the *n-fold hyperspace suspension* of a continuum X, which is denoted by $HS_n(X)$, we mean the quotient space:

$$HS_n(X) = \mathcal{C}_n(X)/\mathcal{F}_n(X)$$

with the quotient topology. The fact that $HS_n(X)$ is a continuum follows from [26, 3.10]. Note that $HS_1(X)$ corresponds to the hyperspace suspension $HS(X)$ defined by Nadler in [24].

Given a continuum X, $q^n_X\colon \mathcal{C}_n(X) \twoheadrightarrow HS_n(X)$ denotes the quotient map. Also, let F^n_X and T^n_X denote the points $q^n_X(\mathcal{F}_n(X))$ and $q^n_X(X)$, respectively.

Remark 2.1. Note that the sets $HS_n(X)\setminus\{F^n_X\}$ and $HS_n(X)\setminus\{T^n_X, F^n_X\}$ are homeomorphic to $\mathcal{C}_n(X)\setminus\mathcal{F}_n(X)$ and $\mathcal{C}_n(X)\setminus(\{X\}\cup\mathcal{F}_n(X))$, respectively, using the appropriate restriction of q^n_X.

A continuum X has the *property of Kelley* provided that for each $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x, x' \in X$, $d(x, x') < \delta$, and $x \in A \in \mathcal{C}(X)$, then there exists $B \in \mathcal{C}(X)$ such that $x' \in B$ and $\mathcal{H}(A, B) < \varepsilon$.

3. n-fold Hyperspaces

In [6, Example 4.5] the authors present two continua X and Y such that X is indecomposable, Y is decomposable and $\mathcal{C}(X)$ is homeomorphic to $\mathcal{C}(Y)$. The following theorem shows that this cannot happen when X has the property of Kelley.

Theorem 3.1. *Let X and Y be continua, where X is indecomposable with the property of Kelley, and let n be a positive integer. If $\mathcal{C}_n(X)$ is homeomorphic to $\mathcal{C}_n(Y)$, then Y is indecomposable.*

Proof. Let $h\colon \mathcal{C}_n(X) \twoheadrightarrow \mathcal{C}_n(Y)$ be a homeomorphism. Since X is indecomposable and has the property of Kelley, X is the only point at which $\mathcal{C}_n(X)$ is locally connected ([15, 3.7]). Thus, $h(X)$ is the only point at which $\mathcal{C}_n(Y)$ is locally connected. Since $\mathcal{C}_n(Y)$ is always locally connected at Y ([21, 2.3]), we have that $h(X) = Y$.

Now, since X is indecomposable, $\mathcal{C}_n(X) \setminus \{X\}$ is not arcwise connected ([14, 6.3]). Hence, $\mathcal{C}_n(Y) \setminus \{Y\}$ is not arcwise connected. Therefore, Y is indecomposable ([14, 6.3]). □

Using [23, (1.139)], [23, (1.136)] and [23, (11.4)] instead of [15, 3.7], [21, 2.3] and [14, 6.3], respectively, in the proof of Theorem 3.1, we obtain:

Theorem 3.2. *Let X and Y be continua, where X is indecomposable. If 2^X is homeomorphic to 2^Y, then Y is indecomposable.*

Remark 3.3. Note that in Theorem 3.2, X is not required to have the property of Kelley. Let us also observe that even though the hyperspaces of subcontinua of the continua X and Y of [6, Example 4.5] are homeomorphic, by Theorem 3.2, 2^X is not homeomorphic to 2^Y.

The following theorem shows that the converse of [14, 7.1] is true:

Theorem 3.4. *Let X be a continuum, and let n be a positive integer. If $\mathcal{C}_n(X)$ is homeomorphic to the Hilbert cube $\mathcal{Q}$, then X is locally connected and does not contain free arcs.*

Proof. Since $\mathcal{Q}$ is locally connected, $\mathcal{C}_n(X)$ is locally connected. Hence, X is locally connected ([14, 3.2]). Suppose X contains a free arc α. Let $a \in Int_X(\alpha)$. Then $\{a\}$ has arbitrary small neighborhoods in $\mathcal{C}_n(X)$ homeomorphic to $\mathcal{C}_n([0,1])$. Thus, since $\mathcal{C}_n(X)$ is homogeneous ([22, 6.1.6]), $\dim(\mathcal{C}_n(X)) = 2n$ ([15, 5.3]). A contradiction to the fact that $\dim(\mathcal{Q}) = \infty$. Therefore, X does not contain free arcs. □

Using the technique of the proof of [14, 3.4], the following lemma is easy to prove.

LEMMA 3.5. *If X is a graph topologically different from an arc and a simple closed curve, and n is a positive integer, then* $\dim(\mathcal{C}_n(X)) \geq 2n+1$.

LEMMA 3.6. *Let n be a positive integer. Then neither $\mathcal{C}_n([0,1])$ nor $\mathcal{C}_n(\mathcal{S}^1)$ is homogeneous.*

PROOF. The lemma follows from the facts that there are points of $\mathcal{C}_n([0,1])$ and of $\mathcal{C}_n(\mathcal{S}^1)$ which have open $2n$-cell neighborhoods in $\mathcal{C}_n([0,1])$ and $\mathcal{C}_n(\mathcal{S}^1)$ ([19, 4.2 and 4.3]), respectively, and points, like any element of $\mathcal{C}_n(X)$ with less that n components, which do not have that property, apply the Brouwer Invariance of Domain Theorem ([9, Theorem VI 9, p. 95]). □

The following theorem extends ([23, (17.2)]) to n-fold hyperspaces.

THEOREM 3.7. *If X is a continuum and n is a positive integer, then the following are equivalent:*

(1) *$\mathcal{C}_n(X)$ is homogeneous;*
(2) *X is locally connected and does not contain free arcs;*
(3) *$\mathcal{C}_n(X)$ is homeomorphic to the Hilbert cube $\mathcal{Q}$.*

PROOF. Suppose $\mathcal{C}_n(X)$ is homogeneous. Since $\mathcal{C}_n(X)$ is homogeneous and locally connected at X ([21, 2.3]), $\mathcal{C}_n(X)$ is locally connected. Hence, X is locally connected ([14, 3.2]). Suppose X contains a free arc. With the same argument as the one given in the proof of Theorem 3.4, we conclude that $\dim(\mathcal{C}_n(X)) = 2n$. Thus, X is a graph ([15, 5.1]). Then, by Lemma 3.5, X must be an arc or a simple closed curve. But, by Lemma 3.6, neither $\mathcal{C}_n([0,1])$ nor $\mathcal{C}_n(\mathcal{S}^1)$ is homogeneous, a contradiction. Therefore, X does not contain a free arc.

Now, if X is a locally connected continuum without free arcs, then $\mathcal{C}_n(X)$ is homeomorphic to $\mathcal{Q}$ ([14, 7.1]).

Finally, since $\mathcal{Q}$ is homogeneous ([22, 6.1.6]), if $\mathcal{C}_n(X)$ is homeomorphic to $\mathcal{Q}$, then $\mathcal{C}_n(X)$ is homogeneous. □

Now we turn our attention to the comparison of n-fold hyperspaces and suspensions.

THEOREM 3.8. *If X is a finite-dimensional continuum, then $\mathcal{C}_n(X)$ is not homeomorphic to $\Sigma(X)$ for any integer n greater than one.*

PROOF. Suppose $\mathcal{C}_n(X)$ is homeomorphic to $\Sigma(X)$. Since X is finite-dimensional, $\Sigma(X)$ is finite-dimensional; in fact, $\dim(\Sigma(X)) = \dim(X)+1$ ([9, p. 34]). Since $\mathcal{C}_n(X)$ is finite-dimensional, $\mathcal{C}(X)$ is finite-dimensional. Hence, $\dim(X) = 1$ ([12, 2.1]). Thus, $2 = \dim(\Sigma(X)) = \dim(\mathcal{C}_n(X))$. Therefore, since $\mathcal{C}_n(X)$ contains n-cells ([19, 3.4]), we have that $n = 2$. We consider two cases.

Case (1). X contains a proper decomposable subcontinuum.

Note that, by [17, 4.1.11], $\dim(\mathcal{C}_2(X)) \geq 3$, a contradiction to the fact that $\dim(\mathcal{C}_2(X)) = 2$.

Case (2). Every proper subcontinuum of X is indecomposable.

In this case, X is hereditarily indecomposable ([19, 3.1]). It is well known that $\Sigma(X)$ is not arcwise disconnected by any of its points. On the other hand, since X is hereditarily indecomposable, for each $A \in \mathcal{C}(X) \setminus \mathcal{F}_1(X)$, $\mathcal{C}_2(X) \setminus \{A\}$ is not arcwise connected ([14, 6.9]), a contradiction.

Therefore, $\mathcal{C}_n(X)$ is not homeomorphic to $\Sigma(X)$. □

Let us recall that R. Schori proved that $\mathcal{C}_2([0,1])$ is homeomorphic to $[0,1]^4$ (a proof of this fact may be found in [17, 6.8.11]). Note that $[0,1]^4$ is homeomorphic to $\Sigma([0,1]^3)$. In connection with this we have the following:

Theorem 3.9. *Let X be a continuum and let $n \geq 2$ be an integer. If Z is a finite-dimensional continuum such that $\Sigma(Z)$ is homeomorphic to $\mathcal{C}_n(X)$, then X is hereditarily decomposable, and X does not contain nondegenerate proper terminal subcontinua. Also, Z is arcwise connected.*

Proof. Let $h\colon \mathcal{C}_n(X) \twoheadrightarrow \Sigma(Z)$ be a homeomorphism. Suppose X contains a nondegenerate indecomposable subcontinuum A. Since Z is finite dimensional, with a similar argument to the one given in the proof of Theorem 3.8, we have that $\mathcal{C}(X)$ is finite-dimensional. Then $\mathcal{C}_n(X) \setminus \{A\}$ is not arcwise connected ([19, 3.4]). Hence, $\Sigma(Z) \setminus \{h(A)\}$ is not arcwise connected. A contradiction to the fact that $\Sigma(Z)$ is not arcwise disconnected by any of its points. Therefore, X is hereditarily decomposable.

Now, suppose X contains a nondegenerate proper terminal subcontinuum B. Then $\mathcal{C}_n(X) \setminus \{B\}$ is not arcwise connected ([14, 6.4]). Hence, like in the previous paragraph, we obtain a contradiction. Therefore, X does not contain nondegenerate proper terminal subcontinua.

To show that Z is arcwise connected, it is enough to prove that $\Sigma(Z) \setminus \{v_1, v_2\}$ is arcwise connected.

Let $A_j = h^{-1}(v_j)$, $j \in \{1,2\}$. We prove that $\mathcal{C}_n(X) \setminus \{A_1, A_2\}$ is arcwise connected. Since X does not contain nondegenerate proper terminal subcontinua, $\mathcal{C}_n(X) \setminus \{A_j\}$, $j \in \{1,2\}$, is arcwise connected ([14, 6.2 and 6.4]). We consider three cases.

Case (1). $A_1, A_2 \in \mathcal{C}(X)$.

Observe that, by [14, 6.4], $\mathcal{C}(X) \setminus \{A_j\}$ is arcwise connected, $j \in \{1,2\}$. Hence, $\mathcal{C}(X) \setminus \{A_1, A_2\}$ is arcwise connected ([23, (9.2)]). Therefore, $\mathcal{C}_n(X) \setminus \{A_1, A_2\}$ is arcwise connected.

Case (2). $A_1, A_2 \in \mathcal{C}_n(X) \setminus \mathcal{C}(X)$.

Given $B \in \mathcal{C}_n(X) \setminus \{A_1, A_2\}$, it is easy to construct an arc from B to X in $\mathcal{C}_n(X) \setminus \{A_1, A_2\}$. Therefore, $\mathcal{C}_n(X) \setminus \{A_1, A_1\}$ is arcwise connected.

CASE (3). $A_1 \in \mathcal{C}_n(X) \setminus \mathcal{C}(X)$ and $A_2 \in \mathcal{C}(X)$.

Suppose $A_2 \neq X$. Let $B \in \mathcal{C}_n(X) \setminus \{A_1, A_2\}$. Since X is decomposable and does not contain nondegenerate proper terminal subcontinua, $\mathcal{C}(X)\setminus\{A_2\}$ is arcwise connected ([14, 6.3 and 6.4]). Now, it is easy to construct an arc from B to X. Hence, in this case, $\mathcal{C}_n(X) \setminus \{A_1, A_2\}$ is arcwise connected.

Assume that $A_2 = X$. Let $B_0, B_1 \in \mathcal{C}_n(X) \setminus \{A_1, X\}$. Since X is decomposable, $\mathcal{C}_n(X) \setminus \{X\}$ is arcwise connected ([14, 6.3]). Thus, there exists an arc $\alpha\colon [0,1] \to \mathcal{C}_n(X) \setminus \{X\}$ such that $\alpha(0) = B_0$ and $\alpha(1) = B_1$. Suppose that $A_1 \in \alpha([0,1])$. Let k be the number of components of A_1. Note that $k \geq 2$. Then there exists a k-cell, $\mathcal{K}$, in $\mathcal{C}_n(X) \setminus \mathcal{C}(X)$ such that $A_1 \in \mathcal{K}$ (see the proof of [14, 3.4]). Now it is easy to find an arc $\beta\colon [0,1] \to \mathcal{C}_n(X)\setminus\{A_1, X\}$ such that $\beta(0) = B_0$ and $\beta(1) = B_1$. Therefore, $\mathcal{C}_n(X) \setminus \{A_1, X\}$ is arcwise connected.

Therefore, Z is arcwise connected. □

REMARK 3.10. The statement of [19, 4.1] is incorrect. In order to apply [23, (v), p. 312], we need to know that the hyperspace of subcontinua of the continuum X is finite-dimensional. This is not guaranteed by the hypothesis stated. A correct statement is: "*Let X be a continuum and let n be an integer greater than one. If $\mathcal{C}_n(X)$ is homeomorphic to the product of two nondegenerate and finite-dimensional continua, then X is hereditarily decomposable and X has no nondegenerate proper terminal continua*". With this statement, the proof given for [19, 4.1] is correct.

Next, we compare n-fold hyperspaces with products of continua.

THEOREM 3.11. *Let X be an acyclic continuum. If X is homeomorphic to the product of two nondegenerate continua Y and Z, then Y and Z are acyclic.*

PROOF. Since X is acyclic, X has property (b) ([3, 8.1]). Also, since the projection maps π_Y and π_Z are monotone, Y and Z both have property (b) ([11, Theorem 2, p. 434]). Therefore, Y and Z are both acyclic ([3, 8.1]). □

LEMMA 3.12. *Let X be a dendroid and let n be an integer greater than one. Then $\mathcal{C}_n(X)$ is finite-dimensional if and only if X is a graph.*

PROOF. If $\mathcal{C}_n(X)$ is finite-dimensional, then $\mathcal{C}(X)$ is finite-dimensional. Hence, X is a graph ([25, (2.6)]). If X is a graph, then $\mathcal{C}_n(X)$ is finite-dimensional ([15, 5.1]). □

Professor Sam Nadler proved that if $\dim(\mathcal{C}(X)) < \infty$ and if $\mathcal{C}(X)$ is homeomorphic to $X \times Z$, then X must be an arc ([25, (2.7)]). We present a partial extension to this result to n-fold hyperspaces:

Theorem 3.13. *Let X and Z be finite-dimensional continua and let n be an integer greater than one. If $\mathcal{C}_n(X)$ is homeomorphic to $X \times Z$, then X is a tree.*

Proof. Since $\mathcal{C}_n(X)$ is arcwise connected ([14, 3.1]), X and Z are arcwise connected. Since $\mathcal{C}_n(X)$ has property (b) ([14, 4.8]), $\mathcal{C}_n(X)$ is acyclic ([3, 8.1]). Hence, X and Z are acyclic, by Theorem 3.11. Also, X is hereditarily decomposable (see [19, 4.1] and Remark 3.10). Since X is finite-dimensional, X is a dendroid ([25, (1.2)]). Since $X \times Z$ is finite-dimensional, $\mathcal{C}_n(X)$ is finite-dimensional. Thus, X is a graph, by Lemma 3.12. Therefore, since X is acyclic, X is a tree. □

The following theorem is an extension of [13, Theorem 12].

Theorem 3.14. *If X is a finite-dimensional continuum, then $\mathcal{C}_n(X)$ is not homeomorphic to $\mathcal{F}_n(X)$ for any positive integer n.*

Proof. Let n be a positive integer. Suppose $\mathcal{C}_n(X)$ is homeomorphic to $\mathcal{F}_n(X)$. Since X is finite dimensional, $\dim(\mathcal{F}_n(X)) = \dim(X^n) < \infty$ ([27, 22.12]). Hence, $\dim(\mathcal{C}_n(X)) < \infty$ and $\dim(X) = 1$ ([12, 2.1]).

Since $\dim(\mathcal{C}(X)) \geq 2$ ([5, Theorem 1]) and $\dim(X) = 1$, $\mathcal{C}(X)$ is not homeomorphic to $\mathcal{F}_1(X)$ ($\mathcal{F}_1(X)$ is homeomorphic to X).

Suppose that $n \geq 2$. Note that, by [9, Theorem III 4, p. 33], $\dim(X^n) \leq n$. On the other hand, since $\dim(X) = 1$, by [8, (a), p. 197], $\dim(X^n) \geq n$. Therefore, $\dim(X^n) = n$. Thus, $\dim(\mathcal{C}_n(X)) = n$. Hence, X is hereditarily indecomposable ([17, 6.1.11]). Since $\mathcal{C}_n(X)$ is arcwise connected, $\mathcal{F}_n(X)$ is arcwise connected. Thus, X is arcwise connected ([2, 2.2]). A contradiction to the fact that X is hereditarily indecomposable.

Therefore, $\mathcal{C}_n(X)$ is not homeomorphic to $\mathcal{F}_n(X)$. □

Observe that if X is a smooth fan, then $\mathcal{C}(X)$ is homeomorphic to $HS(X)$ ([7, 3.2]). If $n \in \{1, 2\}$, then $\mathcal{C}_n([0,1])$ is homeomorphic to $HS_n([0,1])$ (for $n = 1$, it is clear; for $n = 2$, it follows from [17, 6.8.11] and [20, 4.6]). Also, $\mathcal{C}_n(\mathcal{Q})$ is homeomorphic to $HS_n(\mathcal{Q})$ for each positive integer n, by [14, 7.1] and [16, 5.4]. This situation does not happen when the continuum is indecomposable:

Theorem 3.15. *If X is an indecomposable continuum, then $\mathcal{C}_n(X)$ is not homeomorphic to $HS_n(X)$ for any positive integer n.*

Proof. Note that if X is an indecomposable continuum, then $\mathcal{C}_n(X) \setminus \{X\}$ is not arcwise connected ([14, 6.3]). Meanwhile, it is easy to see that no point arcwise disconnects $HS_n(X)$. □

Theorem 3.16. *If X is either an arc-like or a circle-like continuum and n is a positive integer, then $\dim(\mathcal{C}_n(X)) \leq 2n$.*

Proof. Let n be a positive integer. Let X be an arc-like continuum and let $\varepsilon > 0$ be given. Then there exists an ε-map $f\colon X \twoheadrightarrow [0,1]$. Hence, the induced map $\mathcal{C}_n(f)\colon \mathcal{C}_n(X) \twoheadrightarrow \mathcal{C}_n([0,1])$ is an ε-map with respect to the Hausdorff metric ([1, Lemma 36]). Thus, since $\dim(\mathcal{C}_n([0,1])) = 2n$ ([15, 5.3]), we have that $\dim(\mathcal{C}_n(X)) \leq 2n$ ([27, 15.5]).

The proof for the case when X is circle-like is similar. We need to use [15, 5.6] instead of [15, 5.3]. □

4. n-fold Hyperspace Suspensions

The proof of the following lemma is similar to the one given for Lemma 3.6.

Lemma 4.1. *If n is an integer greater than one, then neither $HS_n([0,1])$ nor $HS_n(\mathcal{S}^1)$ is homogeneous.*

Remark 4.2. With respect to Lemma 4.1, observe that for $\mathcal{S}^1$ the result is not true when $n = 1$, since $HS(\mathcal{S}^1)$ is a 2-sphere, which is homogeneous. The lemma is true for $HS([0,1])$, since this is homeomorphic to a 2-cell.

Theorem 4.3. *Let X be a continuum and let n be a positive integer greater than one. If $HS_n(X)$ is homogeneous, then X is a locally connected continuum without free arcs.*

Proof. Let n be an integer greater than one. Since $\mathcal{C}_n(X)$ is locally connected at X ([21, 2.3]), $HS_n(X)$ is locally connected at T^n_X. Hence, since $HS_n(X)$ is homogeneous, $HS_n(X)$ is locally connected. Thus, X is locally connected ([16, 5.2]). Suppose X contains a free arc. Then, since $HS_n(X)$ is homogeneous, $\dim(HS_n(X)) = 2n$ ([20, 4.1]). Since X is locally connected and $\dim(HS_n(X)) = 2n$, X is a graph, by [16, 3.6] and [15, 5.1]. Since $2n = \dim(HS_n(X)) = \dim(\mathcal{C}_n(X))$ (the second equality follows from [16, 3.6]), by Lemma 3.5, X is an arc or a simple closed curve. A contradiction, because, by Lemma 4.1, neither $HS_n([0,1])$ nor $HS_n(\mathcal{S}^1)$ is homogeneous. Therefore, X does not contain free arcs. □

Remark 4.4. Regarding Theorem 4.3, we note that the case $n = 1$ was already proved in [7, 5.6]. Also, it is not clear that the converse of Theorem 4.3 is true, since there exists a locally connected continuum X, without free arcs, such that $HS(X)$ is not homeomorphic to the Hilbert cube ([7, 5.3]).

Now, we compare n-fold hyperspace suspensions with cones.

Theorem 4.5. *If X is a finite-dimensional continuum, then $HS_n(X)$ is not homeomorphic to* Cone(X) *for any integer n greater than one.*

Proof. Let n be an integer greater than one. Suppose $HS_n(X)$ is homeomorphic to Cone(X). With an argument similar to the one given in the proof of Theorem 3.8, we obtain that $n = 2$ and no nondegenerate proper subcontinuum of X is decomposable.

Suppose each proper subcontinuum of X is indecomposable. Then, by [19, 3.1], X is hereditarily indecomposable. Hence, $\text{Cone}(X)$ is uniquely arcwise connected. On the other hand, since $HS_2(X)$ contains 2-cells ([16, 3.7]), $HS_2(X)$ is not uniquely arcwise connected.

Therefore, $HS_n(X)$ is not homeomorphic to $\text{Cone}(X)$ for any integer n greater than one. □

The following lemma is easy to prove.

Lemma 4.6. *Let Z be an arcwise connected continuum. If $x_1, x_2 \in \text{Cone}(Z)$, then $\text{Cone}(Z) \setminus \{x_1, x_2\}$ is arcwise connected.*

Observe that $HS_2([0,1])$ is homeomorphic to $[0,1]^4$ ([20, 4.6]). Hence, $HS_2([0,1])$ is homeomorphic to $\text{Cone}([0,1]^3)$. In connection with this we have:

Theorem 4.7. *Let X be a continuum. If Z is a finite-dimensional continuum such that $\text{Cone}(Z)$ is homeomorphic to $HS_n(X)$, for some integer n greater than one, then X is hereditarily decomposable, and X does not contain nondegenerate proper terminal subcontinua. Also, Z is arcwise connected.*

Proof. Let n be an integer greater than one and let $h\colon HS_n(X) \twoheadrightarrow \text{Cone}(Z)$ be a homeomorphism. Since $HS_n(X)$ is not arcwise disconnected by any of its points, $\text{Cone}(Z)$ is not arcwise disconnected by any of its points. Hence, Z is arcwise connected.

Suppose X contains an indecomposable continuum A. Since Z is finite-dimensional, we have that $HS_n(X)$ is finite-dimensional. This implies that $\dim(HS_n(X)) = \dim(\mathcal{C}_n(X))$ ([16, 3.6]). Thus, $\mathcal{C}_n(X) \setminus \{A\}$ is not arcwise connected ([19, 3.4]). Hence, $HS_n(X) \setminus \{q_X^n(A), F_X^n\}$ is not arcwise connected (compare with [18, 4.3]). This implies that $\text{Cone}(Z) \setminus \{h(q_X^n(A)), h(F_X^n)\}$ is not arcwise connected, a contradiction to Lemma 4.6. Therefore, X is hereditarily decomposable.

Now, assume X contains a nondegenerate proper terminal subcontinuum B. Then $\mathcal{C}_n(X) \setminus \{B\}$ is not arcwise connected ([14, 6.4]). Repeating the argument of the previous paragraph, we obtain, again, a contradiction to Lemma 4.6. Therefore, X does not contain nondegenerate proper terminal subcontinua. □

Next, we compare n-fold hyperspace suspensions with n-fold symmetric products.

Theorem 4.8. *If X is a finite-dimensional continuum, then $HS_n(X)$ is not homeomorphic to $\mathcal{F}_n(X)$ for any integer n greater than one.*

Proof. The proof is similar to the proof of Theorem 3.14. We just need to mention that $\dim(HS_n(X)) = \dim(\mathcal{C}_n(X))$ ([16, 3.6]) and that $HS_n(X)$ is arcwise connected ([16, 3.3]). □

The proofs of the following two theorems are similar to the ones given in [13, Theorems 9 and 11], respectively. We include the proofs, a little bit simplified, of these theorems for the convenience of the reader.

THEOREM 4.9. *Let X be a finite-dimensional continuum. If $HS(X)$ is homeomorphic to $\mathcal{F}_2(X)$, then X is homeomorphic to* $[0,1]$.

PROOF. Suppose $HS(X)$ is homeomorphic to $\mathcal{F}_2(X)$. Since X is finite-dimensional, by the proof of [2, 3.1], we have that $\dim(\mathcal{F}_2(X)) \leq 2\dim(X)$. Since $HS(X)$ is homeomorphic to $\mathcal{F}_2(X)$, $\dim(HS(X)) \leq 2\dim(X)$. Thus, $\dim(X) = 1$ ([20, 3.1]). Hence, $2 \leq \dim(\mathcal{C}(X)) = \dim(HS(X)) = \dim(\mathcal{F}_2(X)) \leq 2$, the first inequality follows from [6, Theorem 1]. Therefore, $\dim(\mathcal{C}(X)) = \dim(HS(X)) = 2$. Now, by [28, Theorem 1], X is atriodic. Since $HS(X)$ is arcwise connected ([16, 3.3]), $\mathcal{F}_2(X)$ is arcwise connected. Thus, X is arcwise connected ([2, 2.2]).

Now, suppose X is not unicoherent. Then by the proof of [10, 1.1], there exists a map $f\colon \mathcal{F}_2(X) \to \mathcal{S}^1$ such that f is not homotopic to a constant map. Since $HS(X)$ has property (b) ([24, (2.2)]), $\mathcal{F}_2(X)$ has property (b), a contradiction. Therefore, X is unicoherent. Thus, by Sorgenfrey's Theorem ([26, 11.34]), X is an irreducible continuum. Since X is arcwise connected, X is an arc. □

THEOREM 4.10. *Let X be a finite-dimensional continuum. If n is an integer greater than two, then $HS(X)$ is not homeomorphic to $\mathcal{F}_n(X)$.*

PROOF. Let n be an integer greater than two. Suppose $HS(X)$ is homemorphic to $\mathcal{F}_n(X)$. Since X is finite-dimensional, by the proof of [2, 3.1], we have that $\dim(\mathcal{F}_n(X)) \leq n\dim(X)$. Since $HS(X)$ is homeomorphic to $\mathcal{F}_n(X)$, $\dim(HS(X)) < \infty$. Hence, $\dim(X) = 1$ ([20, 3.1]). Therefore, $\dim(\mathcal{F}_n(X)) \leq n$. Since X is finite-dimensional, $\dim(\mathcal{F}_n(X)) = \dim(X^n)$ ([27, 22.12]). Also, since $\dim(X) = 1$, $\dim(X^n) \geq n$ ([8, (a), p. 197]). Thus, $n = \dim(\mathcal{F}_n(X)) = \dim(HS(X)) = \dim(\mathcal{C}(X))$. By [28, Theorem 1], X does not contain $(n+1)$-ods. Hence, X contains a free arc ([13, Theorem 11]). This implies that $\mathcal{C}(X)$ and $HS(X)$ contain a 2-dimensional subset with nonempty interior. But, since $n \geq 3$, $\mathcal{F}_n(X)$ does not contain 2-dimensional subsets with nonempty interior. Therefore, $HS(X)$ is not homemorphic to $\mathcal{F}_n(X)$. □

We turn our attention to the comparison of n-fold hyperspace suspensions and products of continua.

The following lemma is easy to prove.

LEMMA 4.11. *If Y and Z are nondegenerate arcwise connected continua, then $Y \times Z \setminus \{(y_1, z_1), (y_2, z_2)\}$ is arcwise connected for any two points (y_1, z_1) and (y_2, z_2) of $Y \times Z$.*

The proof of the following theorem is similar to the proof of [19, 4.1] (see Remark 3.10), we need to use Lemma 4.11. To conclude that Y and Z are acyclic, we apply Theorem 3.11, [3, 8.1] and [16, 4.1].

THEOREM 4.12. *Let X be a continuum, and let n be a positive integer. If Y and Z are nondegenerate finite-dimensional continua such that $Y \times Z$ is homeomorphic to $HS_n(X)$, then X is hereditarily decomposable and does not contain nondegenerate proper terminal subcontinua. Also, Y and Z are arcwise connected an acyclic.*

The proof of the following theorem is similar to the one given for Theorem 3.13, we need to use Theorem 4.12, [16, 3.6], [3, 8.1] and [16, 4.1].

THEOREM 4.13. *Let X and Z be finite-dimensional continua and let n be an integer greater than one. If $HS_n(X)$ is homeomorphic to $X \times Z$, then X is a tree.*

The following theorem shows that the n-fold hyperspace suspension of an absolute retract is an absolute retract.

THEOREM 4.14. *If X be an absolute retract and n is a positive integer, then $HS_n(X)$ is an absolute retract.*

PROOF. Without loss of generality, we assume that X is embedded in $\mathcal{Q}$ (see [17, 1.1.16]). Since X is an absolute retract, there exists a retraction $r\colon \mathcal{Q} \twoheadrightarrow X$. Then it is easy to see that the induced map $HS_n(r)\colon HS_n(\mathcal{Q}) \twoheadrightarrow HS_n(X)$, given by:

$$HS_n(r)(\chi) = \begin{cases} q_X^n(\mathcal{C}_n(r)((q_{\mathcal{Q}}^n)^{-1}(\chi))), & \text{if } \chi \neq F_{\mathcal{Q}}^n; \\ F_X^n, & \text{if } \chi = F_{\mathcal{Q}}^n; \end{cases}$$

is a retraction (note that, by [4, 4.3, p. 126], $HS(r)$ is continuous). Since $HS_n(\mathcal{Q})$ is homeomorphic to $\mathcal{Q}$ ([16, 5.7]), $HS_n(X)$ is an absolute retract ([11, Theorem 7, p. 341]). □

Since absolute retracts have the fixed point property ([11, Theorem 11, p. 343]), we have the following:

COROLLARY 4.15. *If X be an absolute retract and n is a positive integer, then $HS_n(X)$ has the fixed point property.*

As a consequence of Theorem 3.16 and [16, 3.6], we have:

THEOREM 4.16. *If X is either an arc-like or a circle-like continuum and n is a positive integer, then* $\dim(HS_n(X)) \leq 2n$.

The proof of the following theorem is similar to the proof of Theorem 3.1. We include the proof for the convenience of the reader, to present the appropriate references needed.

THEOREM 4.17. *Let X and Y be continua, where X is indecomposable with the property of Kelley, and let n be a positive integer. If $HS_n(X)$ is homeomorphic to $HS_n(Y)$, then Y is indecomposable.*

PROOF. Let $h\colon HS_n(X) \twoheadrightarrow HS_n(Y)$ be a homeomorphism. Since X is indecomposable and has the property of Kelley, X is the only point at which $\mathcal{C}_n(X)$ is locally connected ([15, 3.7]). Hence, T_X^n and F_X^n are the only two points at which $HS_n(X)$ is locally connected (by Remark 2.1, and [16, 3.2]). Thus, $h(T_X^n)$ and $h(F_X^n)$ are the only two points at which $HS_n(Y)$ is locally connected. Since $HS_n(Y)$ is always locally connected at T_Y^n and F_Y^n (by [21, 2.3], Remark 2.1, and [16, 3.2]), we have that $\{h(T_X^n), h(F_X^n)\} = \{T_Y^n, F_Y^n\}$. Since X is indecomposable, $HS_n(X) \setminus \{T_X^n, F_X^n\}$ is not arcwise connected ([16, 6.2]). Hence, $HS_n(Y) \setminus \{h(T_X^n), h(F_X^n)\} = HS_n(Y) \setminus \{T_Y^n, F_Y^n\}$ is not arcwise connected. Therefore, Y is indecomposable ([16, 6.2]). □

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