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Izvorni znanstveni članak

Mathematical Analysis of Measured Displacements with Emphasis on Polynomial Interpolation

Boštjan KOVAČIČ, Rok KAMNIK – Maribor[1], Zdravko KAPOVIĆ – Zagreb[2]

ABSTRACT. At the Faculty of Civil Engineering, the University of Maribor, in the Centre for Geodesy, we have been primarily dealing with the study of displacements and application of results on deformations and mathematical analyses for the last 10 years. According to these results, we wanted to find a connection between displacements and resulting deformations in mathematical terms. For this purpose, experimental load-bearing test of a concrete girder was carried out step by step up to crack. Additionally, concrete girder was analytically modelled and also photogrammetric, geodetic, and physical methods were compared. During the process of analyzing the results, further polynomial interpolation was performed through which the displacements in the intermediate stages were calculated. Polynomial interpolation can also help in accomplishment of load-bearing experiments of larger structures.

Keywords: geodesy, vertical movement, deformation, mathematical analysis.

1. Introduction

A control measurement can be performed in a number of ways depending on objects. In practice, geodetic methods, which can detect all the geometric changes of a measured object, are usually used. Thus, displacements and deformations are measured according to object form considering environment and time. Nowadays, our attention is focusing on the predictable maintenance of existing objects, so more and more researches are shifted again back to the laboratory.

There are many authors who are engaged in structure testing and are publishing their achievements in various scientific and professional literatures. Each contri-

[1] Doc. dr. Boštjan Kovačič, Rok Kamnik, univ. dipl. inž. geod., Fakulteta za gradbeništvo, Univerza v Mariboru, Smetanova ulica 17, 2000 Maribor, Slovenija, e-mail: mail: bostjan.kovacic@uni-mb.si, rok.kamnik@uni-mb.si

[2] Prof. dr. sc. Zdravko Kapović, Geodetski fakultet, Sveučilište u Zagrebu, Kačićeva 26, 10000 Zagreb, e-mail: zkapovic@geof.hr.

bution is of course a part of a mosaic in science of measurement and interpretation of displacements and, consequently, deformations that occur on constructions. In the field of measuring displacements, we also follow technological development of equipment and construction building, since such bridging objects have been built in the last decade which seemed to be impossible years ago. Of course, it is necessary to test such complex structures where reliable high-capacity instruments are required. Therefore, every two years the latest equipment is tested in a laboratory and on a basis of result comparison we give opinion on the appropriateness of a certain method and equipment. Even though there are ideal conditions in the laboratory which can not be provided on the field (constant temperature, pressure, humidity, lighting, measuring distances and sighting angles) laboratory results serve for mathematical analyses and solution of certain problems. This time we used a classical geodetic method of measuring displacements based on trigonometric heights, a method of measuring displacements with inductive meter, mounted on a hydraulic press and the photogrammetric method of obtaining data, which we used for the first time as a means of determining the displacements. All the results were compared to the calculated (expected) displacements which are believed to be relevant and are a basis for a comparative analysis.

2. Displacement Measurements on the Beam and Analysis of Test Results

2.1. Properties of the Test

In order to compare the above described measurement methods, a prefabricated pre-stressed concrete plate beam was used (Figure 3). A predicted vertical displacement was analytically calculated with the help of static model with the vertical force at the middle of the span as shown in Figure 2.

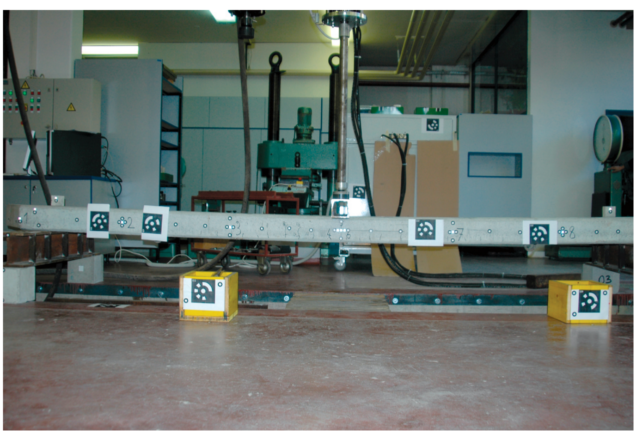


Fig. 1. *Prefabricated prestressed concrete plate with measured marks.*

Deflections of a plate depend on geometrical and material characteristics of the test sample.

Geometrical characteristics:

Calculated static length:	$L = 3750\ mm$
Width:	$b = 500\ mm$
Height:	$h = 150\ mm$
Effective depth of a cross-section:	$d = 120\ mm$
Cross sectional area of tensile reinforcement:	$A_{s1}^{+} = 7.70\ \mathrm{cm}^2$ (5Φ14)
	$A_{s2}^{+} = 0.724\ \mathrm{cm}^2$ (3x3Φ3.2)
Cross sectional area of compressive reinforcement:	$As^{-} = 3.93\ \mathrm{cm}^2$ (5Φ10)

Material characteristics:

Concrete:

Strength class:	*C 30/37*
Mean tensile strength:	$f_{ctm} = 2.9\ MPa$
Secant modulus of elasticity:	$E_{cm} = 32 \cdot 10^6\ kN/m^2$

Reinforcement:

Strength class:	*S 400* (A_{s1}^{+}, A_s^{-}) , *S 1680* (A_{s2}^{+})
Characteristic yield strength:	$f_{yk} = 400\ MPa$
Design value of modulus of elasticity:	$E_s = 200 \cdot 10^6\ kN/m^2$

2.2. Analytical Results

While well known analytical methods can be used for calculation of non-cracked cross-section displacements, further problems are encountered at cracked cross-sections where cracks occur in the tensile area due to the low tensile strength of a concrete. This results in reduction of the second moment of area of concrete section and consequently in deflection increasing. As it is difficult to determine the location as well as the size of the cracks exactly, they are usually approximately stated using different national codes. Recently Eurocode 2 [9] has been most frequently applied in Europe and therefore considered in our analysis.

According to the presented characteristics the second moment of area of the un-cracked cross-section is $I_y^{(I)} = 15345.478\ cm^4$.

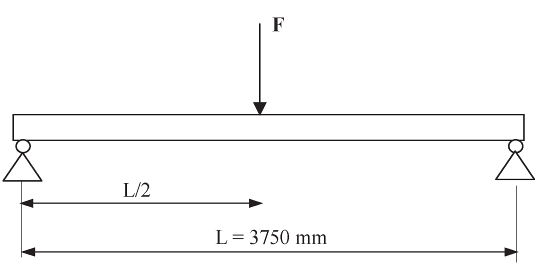


Fig. 2. *Static system of the test sample.*

The bending moment forming the first crack in the tensile concrete section ($M_y^{(I)}$) is calculated in the form of:

$$M_y^{(I)} = f_{ctm} \cdot \frac{2 \cdot I_y^{(I)}}{h} = 0.29 \cdot \frac{2 \cdot 15345.478}{15} = 593.36\, kNcm = 5.9336\, kNm. \quad (1)$$

For the un-cracked cross-section ($M_{y0} \leq M_y^{(I)}$) the maximal vertical displacement (w_{inst}) is the sum of bending moment (M_{y0}), shear force (V_{z0}) and contribution (N_0):

$$w_{init} = w_{init,M} + w_{init,V} + w_{init,N} =$$
$$= \int_S \frac{M_{y0}(x) \cdot \overline{M}_{y1}(x)}{E_{cm} \cdot I_y^{(I)}} \mathrm{d}x + \int_S \frac{V_{z0}(x) \cdot \overline{V}_{z1}(x)}{G_{cm} \cdot A_{cs}} \mathrm{d}x + \int_S \frac{N_0(x) \cdot \overline{N}_1(x)}{E_{cm} \cdot A_c} \mathrm{d}x \quad (2)$$

The treated static design (*Figure 2*) Eq. (1) results in:

$$w_{init} = \frac{F \cdot L^3}{48 \cdot E_{cm} \cdot I_{yI}} + \frac{1.2 \cdot F \cdot L}{4 \cdot G_{cm} \cdot A_{cs}} \quad (3)$$

where $G_{cm} = \dfrac{E_{cm}}{2 \cdot (1 + v_c)}$ and $A_{cs} = \dfrac{A_c}{1.2}$.

For the cracked cross-section ($M_{y0} > M_y^{(I)}$) the maximal vertical displacement (*winst*) is calculated in the form of:

$$w_{init} = w_{init,M} + w_{init,V} + w_{init,N} =$$
$$= \int_S \frac{M_{y0}(x) \cdot M_{y1}(x)}{E_{cm} \cdot I_{y,eff}^{(II)}} \mathrm{d}x + \int_S \frac{V_{z0}(x) \cdot V_{z1}(x)}{G_{cm} \cdot A_{cs,eff}^{(II)}} \mathrm{d}x + \int_S \frac{N_0(x) \cdot N_1(x)}{E_{cm} \cdot A_{c,eff}^{(II)}} \mathrm{d}x. \quad (4)$$

The effective second moment of area of the cracked cross-section ($I_{y,eff}^{(II)}$) is determined according to *Eurocode 2* in the form of:

$$I_{y,eff}^{(II)} = \xi \cdot I_y^{(II)} + (1 - \xi) \cdot I_y^{(I)}$$
$$\xi = 1 - \beta_1 \cdot \beta_2 \cdot \left(\frac{M_{yI}}{M_{y0,\max}} \right)^2 \quad (5)$$
$$M_{y0,\max} = \frac{F \cdot L}{4}; \quad \beta_1 = 1.0, \quad \beta_2 = 1.0$$

The value of $I_y^{(II)}$, which represents constant moment of cross-section in the place of the biggest crack, is calculated according to the neutral axis position (x_{II}) using the scheme from Figure 3. Hereby a pressed part of the cross-section of concrete as well as bending and pressure reinforcement is taken into consideration in the form of:

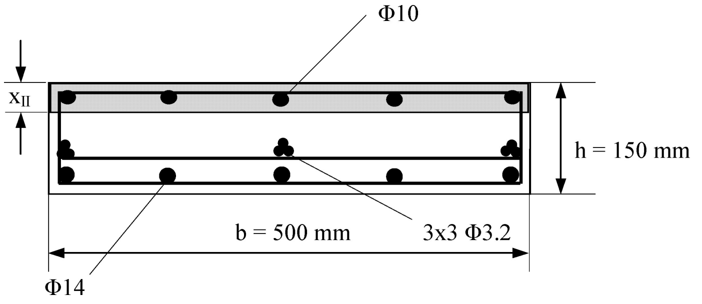


Fig. 3. *Considered elements for the cracked cross section.*

$$\frac{b\cdot x_{II}^2}{4}+(n-1)\cdot A_s^-\cdot(x_{II}-a_3)-n\cdot A_{s1}^+\cdot(h-a_1-x_{II})-n\cdot A_{s2}^+\cdot(h-a_2-x_{II})=0$$

$$x_{II}=4.948\,cm \tag{6}$$

$$I_y^{(II)}=\frac{b\cdot x_{II}^3}{3}+(n-1)\cdot A_s^-\cdot(x_{II}-a_3)^2+ \\ +n\cdot A_{s1}^+\cdot(h-a_1-x_{II})^2+n\cdot A_{s2}^+\cdot(h-a_2-x_{II})^2=4757.344\,cm^4$$

2.3. Measured Results Using Different Methods

For characteristic point observations trigonometric height was used. Measurements were performed from one station point. For the sake of measurement reliability, control points were observed during each load increasing which were stabilized out of the reach of deformation transfer. Electronic calibrated tachimeter Nikon series 800 was used for measurement.

All together, 210 sight points were observed in a local coordinate system. In Figure 4 reflective tape targets 4, 5 and 6 in the middle of the concrete plate (where the breakage was expected) are shown.

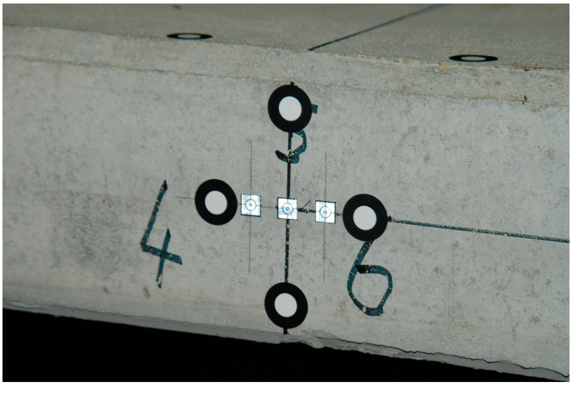


Fig. 4. *Reflective tape targets 4, 5 and 6 in the middle of the concrete plate.*

Referential points (targets) have been used as a means to compare measurement methods (trigonometric heightening, stereo-photogrametry) which were glued in vicinity of reflective tape targets so that the comparison of measured displacements from both methods is possible (Fig. 4). For determination of camera position and position of referential points, the coded marks have been used. Each of them had a 15-bit type code presented by circle sectors which identifies them in every snap-shot. The correct distance between measuring points is determined with scale bars whose distance is known and does not change during the measurement.

A stereo-photogrammetry system TRITOP was used (manufacturer GOM mbH, Braunschweig, Germany) with high resolution CCD camera Fuji Pro S3 (4256 x 2848 pixels) and software for calculation and analysis of object coordinates (Figure 5). In each loading stage minimum 8 snap-shots from different angles and positions were made so that all the points were visible. Snap-shots of each loading stage have been processed during the measurement so that the deflections from previous stages were already known (Fig. 5). Maximum standard deviation of measuring results was 0.08 mm in each single stage.

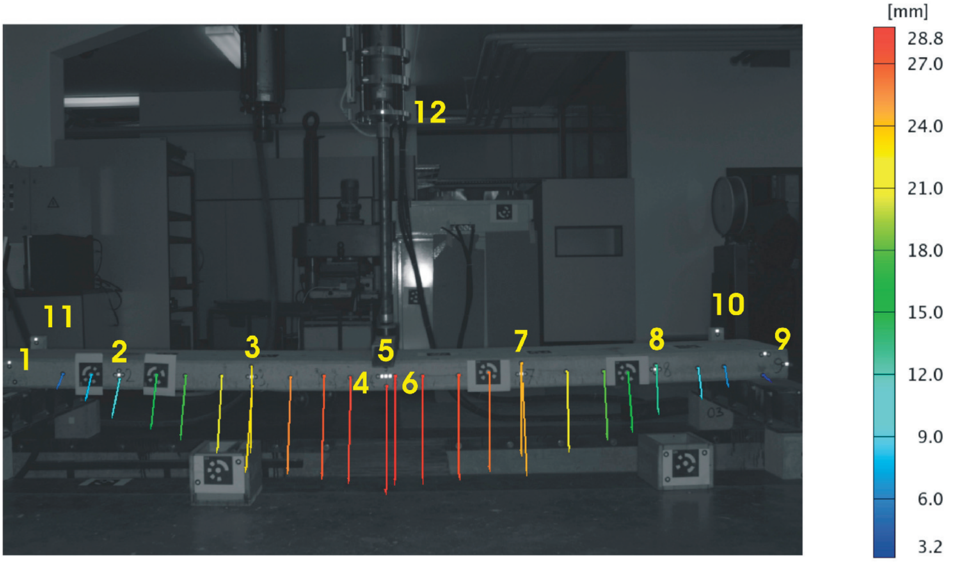


Fig. 5. *Reference points, coded marks, scale bars and displacement vectors of a concrete plate.*

2.4. Result Analysis

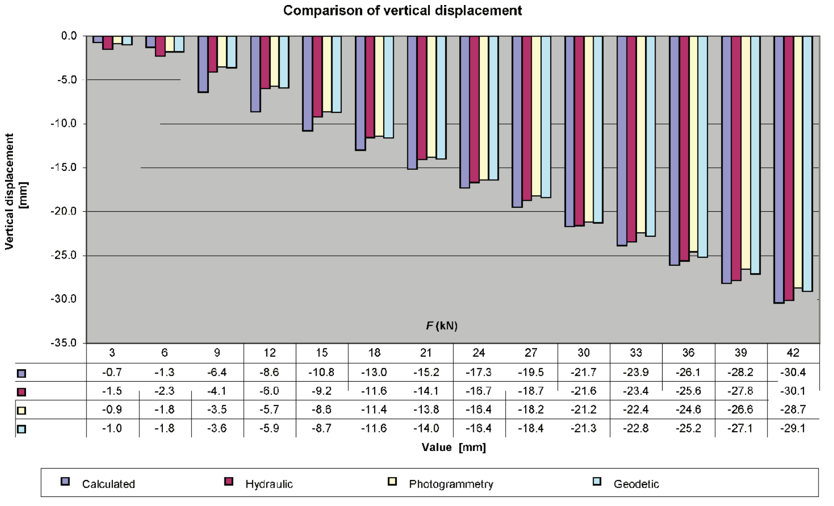
A comparison of the calculated analytical and (measured) geodetic, (measured) photogrammetric vertical displacements and (measured) hydraulic cylinder vertical displacements was made. Table 1 shows results after each loading phase for a point in the middle of the concrete plate. In practice there is an unwritten rule that the ratio between calculated and measured value should not be less then 75 % (Breznikar 1999). The first two epochs were neglected so that the mean ratio between calculated and measured values was 88.5 %.

Table 1. *Vertical displacement by epochs for point 5 in the middle of the concrete plate.*

[kN]	Point	Vertical displacement			
		Analytical	**Hydraulic**	**Photo**	**Geodetic**
3	5	–0.7	–1.5	–0.9	–1.0
6	5	–1.3	–2.3	–1.8	–1.8
9	5	–6.4	–4.1	–3.5	–3.6
12	5	–8.6	–6.0	–6.7	–6.9
15	5	–10.8	–9.2	–8.6	–8.7
18	5	–13.0	–11.6	–11.4	–11.6
21	5	–15.2	–14.1	–13.8	–14.0
24	5	–17.3	–16.7	–16.4	–16.4
27	5	–19.5	–18.7	–18.2	–18.4
30	5	–21.7	–21.6	–21.2	–21.3
33	5	–23.9	–23.4	–22.4	–22.8
36	5	–26.1	–25.6	–24.6	–25.2
39	5	–28.2	–27.8	–26.6	–27.1
42	5	–30.4	–30.1	–28.7	–29.1
average ratio analytical/ measured			90.5 %	86.8 %	88.2 %

The comparison of measurements of vertical displacements for the whole concrete plate was made. Therefore, the geodetic and photogrammetric results were comparable (some tenth of mm) (Graph 1). It can be seen that the surveyed vertical displacement was always smaller then the calculated values. There is a small deviation in the first two epochs because the local contact surface under hydraulic piston was undefined.

The purpose of our experiment was not only a comparative analysis but also to use measured results in scientific-research field and to establish a mathematical model simulating displacements. In order to achieve our research aim, polynomial approximation was used. The obtained results have given answers to our presumptions about displacements: what is a displacement between the intermediate unplanned stages, can the result of polynomial approximation give us feedback on displacement and can this method be applied in practice at load tests on complex constructions?



Graph 1. *Comparison of vertical displacement in point 5.*

3. Mathematical Polynomials Approximation

For every load case (3 kN, 9 kN, 15 kN, ...) polynomials of degree 3 through the monitoring points were fitted with the Mathematica 5.0 software. The polynomials of degree 3 are in the form of:

$$p_i(x) = a_\mathrm{i} + b_\mathrm{i}\, x + c_\mathrm{i}\, x^2 + d_\mathrm{i}\, x^3, \tag{7}$$

$i = 1, 2, \dots n$, where n is the number of observed control points on the concrete plate (7 points) and x is the length of the concrete plate. Fig. 6 shows the magnitude of vertical displacements at different loads. The calculated polynomials are also given in Table 2. (For the sake of better presentation, the units for x and y axis are metres [m] and millimetres [mm], respectively.) Note that the $p(x)$ represents the vertical displacement of the plate at point x caused by the force $\vec{F}$.

Table 2. *Polynomials through points 1-9 for different loads.*

Load	$p_i(x)$ in mm, x in m
	polynomial
3 kN	$0.164286 - 1.00571\, x + 0.237714\, x^2 - 1.54405 \cdot 10^{-16}\, x^3$
9 kN	$0.233333 - 3.79175\, x + 1.00876\, x^2 - 0.0113778\, x^3$
15 kN	$0.430952 - 8.58159\, x + 2.0541\, x^2 + 0.0455111\, x^3$
21 kN	$0.883333 - 14.1352\, x + 3.3981\, x^2 + 0.0682667\, x^3$
27 kN	$1.70952 - 19.066\, x + 4.59886\, x^2 + 0.0910222\, x^3$
33 kN	$2.42857 - 24.5156\, x + 6.15924\, x^2 + 0.0568889\, x^3$
39 kN	$3.33333 - 29.4546\, x + 7.33562\, x^2 + 0.0910222\, x^3$

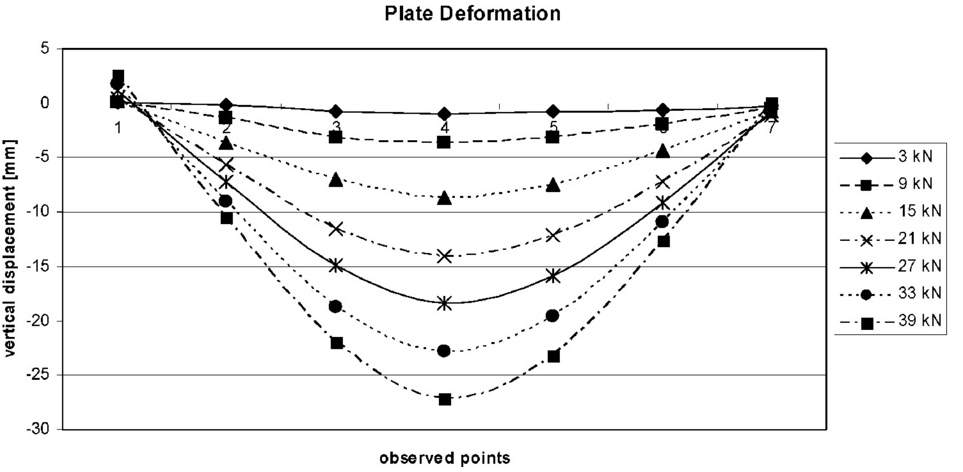


Fig. 6. *Graphs of fitted polynomials for seven load cases (3 kN,..., 39 kN).*

Additionally, the polynomials were calculated for the deformations at seven monitoring points versus load. This time, the x and y represents the load (in kN) and the vertical displacement (in mm) caused by that force at the given monitoring points.

The results of polynomial approximations show the correlation with the measured values so that we can read from the diagram in Figure 6 displacements for intermediate unloaded phases and can also calculate a displacement for higher load than it was performed. This gave us a further idea about displacement simulation for a bridging object since a beam, which is simply supported and loaded in the middle point of the span, represents one field of a bridging object.

3.1. Using a Polynomial in Practice

As a means to confirm the presumptions of the previous section, a polynomial approximation on a concrete example, the Puch's bridge in Ptuj (Fig. 8) was applied. To implement a loading test, the trucks of the total weight of 256 tons were used and displacements were measured by electronic tachimeters. As an example, the results of loading of intermediate (3) field are stated.

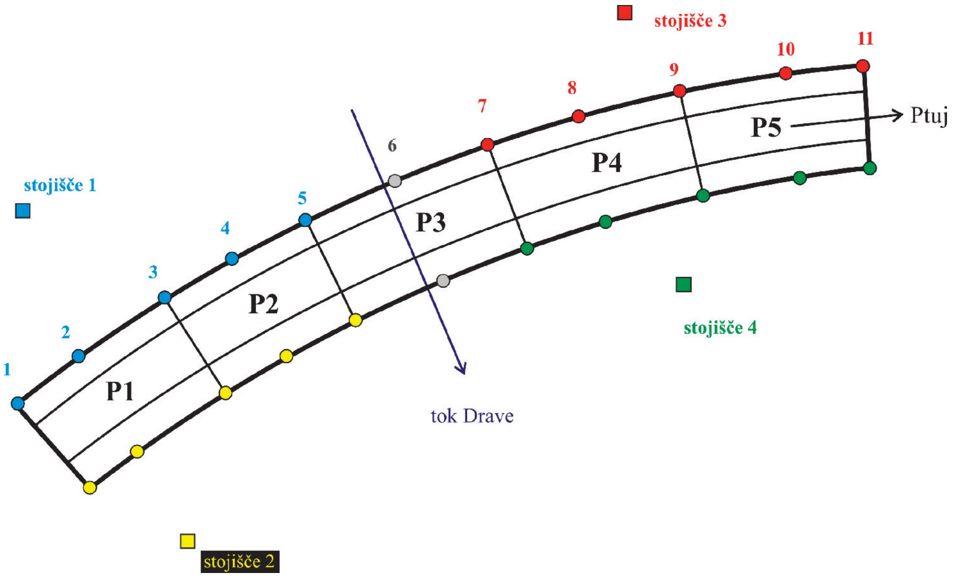


Fig. 7. *Situation and arrangement of measuring marks and stands.*

Table 3. *Measured vertical displacements at the loading of the field 3.*

Point	**Displacement [mm]**
1	0.3
2	–1.9
3	–0.8
4	11.9
5	0.8
6	**–49.3**
7	0.5
8	13.0
9	0.2
10	–1.9
11	–0.8

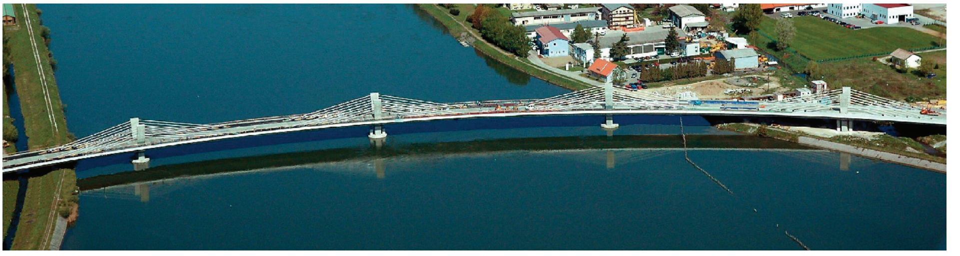
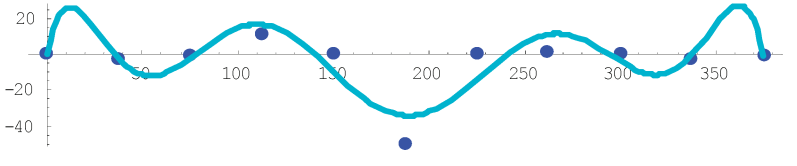


Fig. 8. *Puch's Bridge.*

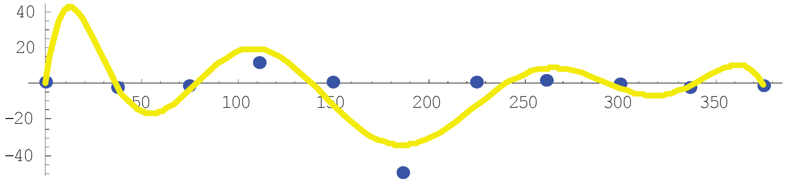
According to the number of observation points (11) and the theory of polynomial interpolation, it is sensible to consider polynomials of degree 10 or less. Additionally, best accordance of polynomial functions with the observed points is obtained at the polynomial of degree 9 and 10; increasing the polynomial degree above 10 does not bring results in polynomial adaptation to the field measurements.

A polynomial of degree 10 is written in the form of:

$$0.23832 + 8.18992\,x - 0.51461\,x^2 + 0.011952\,x^3 - 0.00014\,x^4 + \\ + 9.08206\,x^5 - 3.49141\,x^6 + 7.82930\,x^7 - 9.46829\,x^8 + 4.76588\,x^9$$



Graph 2. *Polynomial of degree 9.*



Graph 3. *Polynomial of degree 10.*

A displacement of –40.8 mm is obtained from polynomial approximation which is 9mm less than measured one. The analysis of the results leads to the conclusion that the method loses its applicative value due to increased degree of polynomial, which also results in the worse accordance since the polynomials approximate worse in points on the edge. Therefore, it is more suitable to apply different interpolation methods as for instance interpolation with piecewise polynomial function as well as cubic splines.

4. Conclusion

Mathematical analysis of the measured values is very important for the end values which are found in reports on measurements. At the beginning of our research, we primarily dealt with micro-displacements and the displacement measurements within instrument accuracy as the extremely low response to the concrete structure load is met in several smaller structures. In these cases, it is difficult to distinguish a displacement to declared equipment precision, as can be seen in the calculations of standard deviations. So, displacements of range up to 1 mm are considered. Many authors have written that a measurement is accurate within around 10% of the expected displacement. This means that if an expected displacement is 2 mm, the accuracy of measurement 0.2 mm should be ensured. But this is not the case when classical geodetic equipment is used and conditions on the field are considered, thus the accuracy is possible only with repeated measurements and under conditions the method can tolerate. The problem occurs at bridging objects crossing high ravines or wide rivers where an instrument cannot be placed. Therefore, other methods should be used and this is a reason why they are tested in a laboratory. Recently, however, more time is devoted to the mathematical analysis since we want to find solutions for the measurement where measuring is unfeasible or unreliable due to almost impossible constructions in the last time. Mathematical analysis of polynomial interpolation is just one of the methods by which simulated values can be obtained. Previously, the integral comparison was used that is suitable for smaller and less complex constructions as well as for comparison of the gained values with different methods. So it would be useful to use it prior to the measurements on the test structure in order to find the most accurate method with which we carry out the measurements in practice.

To conclude, we believe that with this method we rounded up analysis of measurements, added a stone in the mosaic of this scientific field and helped others to solve scientific problems both in the analysis as well as at measuring in practice.

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Matematička analiza mjerenih pomaka s naglaskom na polinomnoj interpolaciji

SAŽETAK. Na Građevinskom fakultetu Sveučilišta u Mariboru, na Odsjeku za geodeziju, bavili smo se, u posljednjih 10 godina, prvenstveno proučavanjem pomaka i primjenom rezultata deformacija te matematičkim analizama. Prema ovim rezultatima, željeli smo otkriti vezu između pomaka i nastalih deformacija u matematičkom smislu. U tu svrhu, postupno je provedeno pokusno ispitivanje nosivosti betonskog nosača sve do njegova pucanja. Osim toga, betonski nosač analitički je modeliran te su uspoređene fotogrametrijske, geodetske i fizikalne metode. Tijekom postupka analize rezultata provedena je nadalje polinomna interpolacija, kojom su izračunati pomaci u srednjim fazama. Polinomna interpolacija također može pomoći u provođenju pokusa nosivosti većih građevinskih objekata.

Ključne riječi: geodezija, vertikalni pomak, deformacija, matematička analiza.

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