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The Dilation Coefficient of Complete Graphs

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Abstract. The *dilation coefficient* of a graph representation is defined as the quotient of the longest and the shortest edge representation. The minimum of the dilation coefficients over all planar representations of a graph G is called the *dilation coefficient* of the graph G. The dilation coefficient of different planar representations of complete graphs is considered and upper and lower bounds for the dilation coefficients of complete graphs are given. Two iterative graph-drawing algorithms that try to minimize the dilation coefficient of a given graph are given. The calculated upper bounds for the dilation coefficients of complete graphs are compared to the values obtained by the graph-drawing algorithms.

Keywords: dilation coefficient, graph representation, energy of a representation, algorithm, circular representation, complete graph, circular packing

INTRODUCTION

Molecular mechanics forms an important area of computational and mathematical chemistry.[1–5] Its main goal is to provide an energy estimate of the given molecular conformation by the use of Newtonian mechanics. The energy of a molecule is calculated using force fields and is determined from the distances of adjacent vertices (atoms), from the angles between the incident edges (bonds), from the dihedral angles and also by the van der Waals and electrostatic interactions.[6] The optimal geometry is then searched by minimizing the energy. As a consequence the coordinates of the atoms of a molecule, when only a topological information about the bonds is given, are produced.

In discrete mathematics and theoretical computer science similar ideas for automatic drawing of graphs in the plane or in the 3D-space are used. Usually the algorithms for automatic drawing of graphs are based on the local search method, where a total "energy" of the drawing is being minimized. Here the term "energy" is used, although it bears only a formal relation to the physical energy of the underlying structure. In graph drawing algorithms usually only the distances of adjacent and sometimes also non-adjacent vertices are taken into consideration when calculating the total "energy" of the graph.[1,3–8]

In this paper we introduce the theoretical background for such a computation that is based on graph representations.[2,6] The "energy" that is being investigated in this paper is simply the quotient of the longest and the shortest edge representation. Such quotient is called *the dilation coefficient* of the representation. The minimum of all dilation coefficients over all planar representations of graph G is called *the dilation coefficient* $\Delta(G)$ of a graph. It is a graph invariant.

Searching for $\Delta(G)$ seems quite a difficult task for general graphs. Graphs for which $\Delta(G) = 1$ are quite special as they can be drawn in the plane with all edges of the same length. Such graphs are called *unit-distance graphs*. However, one has to be careful with possible degeneracies.[9] For instance, if a graph G is bipartite, *i.e.* two-colorable, all vertices of the first color can be represented as the same point, the origin (0,0), and all vertices of the opposite color can be represented as the point (1,0). This makes all edges of the graph G of length exactly 1 and hence, the dilation coefficient of a bipartite graph is 1.

In this paper we investigate the dilation coefficient of complete graphs. As opposed to unit-distance graphs, these graphs have the maximal possible dilation coefficient for a given number of vertices. By studying "concentric representations" of complete graphs we give upper bounds for their dilation coefficients and compare them to the values, obtained by different graph-drawing algorithms. At the end, a lower bound for dilation coefficients of complete graphs is also given.

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REPRESENTATIONS OF GRAPHS

Let $\mathbb{R}^k$ be the k-dimensional Euclidean space with d denoting the usual distance between points:

$$d(a,b)=\|a-b\|_2=\sqrt{\sum_{i=1}^{k}(a_i-b_i)^2}.$$

Let $G=(V,E)$ be a graph and let $P(X)$ denote the power set of a given set X. The pair of mappings (ρ_V,ρ_E), $\rho_V: V(G)\to\mathbb{R}^k$, $\rho_E: E(G)\to P(\mathbb{R}^k)$ is called a *k-dimensional graph representation* or a $\mathbb{R}^k$-*representation* of a graph G if for any edge $e=u\sim v$ (u adjacent to v) its representation $\rho_E(e)$ is the line segment with endpoints $\rho_V(u)$ and $\rho_V(v)$. In other words:

$$\rho_E(e)=(1-\lambda)\rho_V(u)+\lambda\rho_V(v),\ \text{where } 0\le\lambda\le 1.$$

Since the edge representation is completely determined by the vertex representation, we usually drop the subscripts and denote both mappings (ρ_V,ρ_E) simply by ρ. We define the length $\|e\|$ of each edge $e=u\sim v$ in representation ρ as the length of the corresponding line segment, or equivalently as $\|e\|:=d\big(\rho(u),\rho(v)\big)$. In general, representation can be degenerate and map two different vertices into a single point.[9] In this paper only non-degenerate representations will be considered.

A $\mathbb{R}^2$-representation is called *planar* and a $\mathbb{R}^3$-representation *spatial*. A planar representation is therefore equivalent to the usual drawing of a graph in the plane with straight lines representing the edges of the graph. A representation is called a *unit-distance representation*, if all the line segments belonging to edges have the same length.

Figure 1 depicts two planar representations of the generalized Petersen graph $G(6,2)$, also known as the Dürer graph. The one on the right-hand side is a (planar) unit-distance representation. It is easy to see that any graph G has a unit-distance representation in $\mathbb{R}^k$ for some k. We may use the following reasoning. Since G is a subgraph of the complete graph K_n, it suffices to find a unit-distance representation of K_n. For any $k\ge n-1$, the complete graph K_n admits a unit-distance representation in $\mathbb{R}^k$; namely, every vertex of K_n is mapped to a distinct vertex of a $(n-1)$-dimensional face of the k-simplex.[10,11]

Some graphs admit unit-distance representations in smaller dimensions, *i.e.* any tree (of any size) has a planar unit-distance representation. However, there are graphs without planar unit-distance representations; take K_4 for example, the ratio between the longest and the shortest edge is always at least $\sqrt{2}$, which is attained if the four vertices of K_4 are placed in the four corners of the unit square.[10,11] Motivated by this example we give

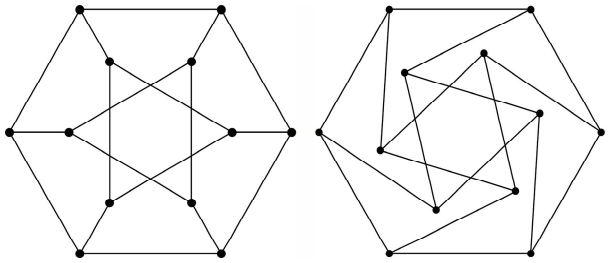

Figure 1. Two planar non-degenerate representations of the generalized Petersen graph $G(6,2)$.

the quotient a special name. Let $D(\rho)=\max_{e\in E(G)}\|e\|$ denote the maximum edge length of the representation ρ and similarly, let $d(\rho)=\min_{e\in E(G)}\|e\|$ denote the minimal edge length of the representation ρ. In the case when $\|e\|>0$ for every $e\in E(G)$, we define the *dilation coefficient* $\Delta(\rho)$ of a *k*-dimensional representation ρ, as follows:

$$\Delta(\rho)=\frac{D(\rho)}{d(\rho)}.$$

In any case $\Delta(\rho)\ge 1$. Observe that in the special case when $\Delta(\rho)=1$ the representation ρ is a (k-dimensional) unit-distance representation and G is a (k-dimensional) unit-distance graph. Note that some authors use the term unit-distance graph for graphs with faithful representations, which have a property that two vertices are adjacent if and only if their distance is 1.[11–13] A related parameter $\dim(G)$, *the dimension* of a graph G, namely the smallest integer k such that G can be represented as a k-dimensional unit-distance graph, was also considered in the past.[13,14]

A natural question arises: given a connected simple graph G, how to find a planar non-degenerate representation minimizing the dilation coefficient. Define the (planar) dilation coefficient $\Delta(G)$ of a graph G as

$$\Delta(G)=\min\left\{\Delta(\rho)\;\middle|\;\begin{array}{l}\rho\text{ is non-degenerate planar}\\ \text{representation of } G\end{array}\right\}.$$

For example, $\Delta(K_4)=\sqrt{2}$. Clearly, we may restrict our search to representations whose shortest edge length is 1. This can be described in a more formal way. Two representations ρ and ρ' of a graph G are equivalent if there exists a positive constant $c\ge 0$ such that for all pairs of vertices $u,v\in V(G)$ it follows that

$$d\big(\rho(u),\rho(v)\big)=c\,d\big(\rho'(u),\rho'(v)\big).$$

Obviously, any two equivalent representations have the same dilation coefficient. This means that we may indeed assume that the minimal edge-length is 1 when searching for $\Delta(G)$.

Several graph-drawing algorithms determine a representation of a graph G by an optimization process as described in the next section. Each representation obtained in this way gives an upper bound on $\Delta(G)$; however, lower bounds are much harder to obtain.

ENERGY OF REPRESENTATIONS

Let ρ be a representation of a graph G in $\mathbb{R}^k$ and $k, p > 0$. Usually we associate with each representation ρ a real number $\varepsilon(\rho) > 0$ called the *energy of the representation*.[6] The energy function $\varepsilon : \rho \to \mathbb{R}$ may be selected in different ways. In this paper we mainly focus on the case $\varepsilon_\infty(\rho) = \max_{e \in E(G)} \|e\| = D(\rho)$. Considering $\min_{e \in E(G)} \|e\| = 1 = d(\rho)$, algorithms that minimize the energy function ε_∞ also minimize the dilation coefficient $\Delta(\rho)$. Note that ε_∞ is the limit example of

$$\varepsilon_{\mathrm{p}}(\rho) = \left(\sum_{e \in E(G)} \|e\|^p \right)^{1/p},$$

when $p \to \infty$.[6] More general energy functions are used in various graph drawing algorithms and as a tool for comparing representations.[6,15,16,19] Finding an optimal representation of a given type can then be viewed as an energy minimization problem for a suitable energy model.

An important family of graph-drawing algorithms is the so-called *spring embedders* also known as force-directed placement algorithms. The main idea is to consider edges as springs, define two forces, one pushing vertices apart and another one attracting adjacent vertices. The force between two vertices depends on the Euclidean distance between them. An energy function is defined that takes into account the contributions of all pairs of vertices. A minimum of the energy function corresponds to an equilibrium point of the forces. The problem of energy minimization is then solved by local optimization or simulated annealing techniques.[8]

An algorithm that works well on general graphs was devised by Fruchterman and Reingold.[15] However, they speak of attractive and repulsive forces but never explicitly state their energy function. Let l be the distance between the representations of two vertices of a graph G. Then the attractive force $f_{\mathrm{a}}(l) = \frac{l^2}{k}$ and the repulsive force $f_{\mathrm{r}}(l) = \frac{k^2}{l}$. The force $f_{\mathrm{r}}(l)$ acts on every pair of vertices while force $f_{\mathrm{a}}(l)$ acts only between two adjacent vertices. By Fruchterman and Reingold,[15] the constant k represents the "ideal" length of all the edges and is normally chosen to be the radius of the empty area around the representation of a vertex. The following formula represents an energy function for their algorithm:

$$\varepsilon_{\mathrm{FR}}(\rho) = \sum_{v \in V} \left(\sum_{u \in N(v)} \frac{\|\rho(u) - \rho(v)\|^3}{3k} - k^2 \sum_{u \in V} \log \|\rho(u) - \rho(v)\| \right),$$

where $N(v)$ denotes the set of vertices adjacent to v.

Three graph-drawing algorithms were implemented and their results compared with theoretical bounds for the dilation coefficient of a complete graph (see Table 1), which will be discussed in the following section. We implemented the well-known Fruchterman and Reingold type spring embedder (FRAAlg) to serve as a benchmark for two other algorithms; first one (SAAlg) is minimizing the energy $\varepsilon_\infty(\rho)$ of the representation ρ while the second one (SPEAlg) attempts to generate the optimal representation of the complete graph by numerically minimizing a sum-of-squares error function (such as Kruskal's stress or Sammon's stress)[16] in a linear time.

The graph drawing algorithm SAAlg is using a paradigm of simulated annealing[8] to minimize the energy function $\varepsilon_\infty(\rho)$ and hence the dilation coefficient. When simulated annealing is used, graph representations are considered to be states in the discrete search space. We define a graph representation ρ' to be a *neighbor* of a graph representation ρ, if ρ' differs from ρ in exactly two points – the representations of two neighbor vertices u,v, with the property that $\|\rho'(u) - \rho'(v)\| = 1$. Each step of the SAAlg algorithm replaces the current state ρ by a random "neighbor" state ρ', chosen with a probability that depends on the difference between the corresponding dilation coefficients and on a global parameter T (also called the *temperature*), that is gradually decreased during the process. When the energy function $\varepsilon_\infty(\rho) > \varepsilon_\infty(\rho')$ the system always moves "downhill" to state ρ'. In the opposite case, when $\varepsilon_\infty(\rho) \le \varepsilon_\infty(\rho')$, the system decides to move "uphill" to state ρ' with the probability $e^{\frac{\varepsilon_\infty(\rho) - \varepsilon_\infty(\rho')}{T}}$. When T is large, the current representation gets replaced almost always, but as T goes to zero, the probability for "uphill" moves decreases. The allowance for "uphill" moves saves the method from becoming stuck at local minima.

The SPEAlg algorithm is using the idea of stochastic proximity embedding.[16] SPEAlg for a predescribed number of steps extends or contracts edges; at every step, the algorithm modifies a random edge to be of length one. The algorithm starts with an initial representation and iteratively refines it by repeatedly selecting an edge at random and adjusting its coordinates, so

Table 1. Upper and lower bounds for the dilation coefficient of the complete graph K_n on n vertices compared to dilation coefficients of the representations obtained by graph-drawing algorithms SAAlg, FRAlg and SPEAlg and to the dilation coefficient of the underlying graph of the circular packing

		Standard uniform concentric representation	General concentric representation		Circular packing[20]				
n	Lower bound[18,19]	Upper bound $\Delta_{\{m_1,\ldots\}}(K_n)$	Upper bound $\Delta_{[m_1,\ldots]}(K_n)$	Ordered integer partition $[m_1,\ldots]$	Upper bound $\Delta_{CP}(K_n)$	$\frac{\Delta_{CP}(K_n)}{\Delta_{[m_1,\ldots]}(K_n)}$	SAAlg	FRAlg	SPEAlg
2	0.4850	2	1	[2]	1	100.00 %	1	1	1
3	0.8188	2	1	[3]	1	100.00 %	1	1	1
4	1.1002	2	1.4142	[4]	1.4142	100.00 %	1.4142	1.4164	1.4201
5	1.3480	2	1.6180	[5]	1.6180	100.00 %	1.6297	1.6383	1.6374
6	1.5722	2	1.9021	[5, 1]	2	105.15 %	1.9044	2.0781	1.9906
7	1.7782	2	2	[6, 1]	2	100.00 %	2.0055	2.0566	2.0795
8	1.9701	4	2.2470	[7, 1]	2.2470	100.00 %	2.2568	2.3307	2.3591
9	2.1502	4	2.6131	[8, 1]	2.6131	100.00 %	2.6979	2.8162	2.7269
10	2.3206	4	2.8794	[9, 1]	2.7938	97.03 %	3.3205	3.3825	3.0530
11	2.4827	4	2.9544	[9, 2]	2.8794	97.46 %	3.3523	3.6682	3.2387
12	2.6376	4	3.1068	[9, 3]	2.9960	96.43 %	3.1614	3.7134	3.5829
13	2.7861	4	3.2361	[10, 3]	3.2361	100.00 %	3.4372	4.0166	3.7671
14	2.9290	4	3.4142	[10, 4]	3.3251	97.39 %	3.6075	4.3921	4.0907
15	3.0669	4	3.5133	[11, 4]	3.5202	100.19 %	3.9151	4.2499	4.5065
16	3.2003	4	3.6636	[11, 5]	3.5933	98.08 %	3.7454	4.6231	4.8885
17	3.3296	4	3.8637	[12, 5]	3.7837	97.93 %	4.0499	4.6431	4.6870
18	3.4551	4	3.9593	[11, 6, 1]	3.8637	97.59 %	3.9928	5.1883	5.2157
19	3.5772	4	4	[12, 6, 1]	3.8637	96.59 %	4.2001	5.1174	5.6607
20	3.6961	6	4.1481	[13, 6, 1]	4.1015	98.88 %	4.4755	5.2817	5.9600
21	3.8121	6	4.2734	[13, 7, 1]	4.2348	99.10 %	5.2243	5.7495	6.5630
22	3.9253	6	4.4940	[14, 7, 1]	4.4389	98.77 %	5.4136	5.6523	6.8137
23	4.0360	6	4.6131	[14, 8, 1]	4.5445	98.51 %	5.3458	5.8755	6.8752
24	4.1443	6	4.7834	[15, 8, 1]	4.6449	97.10 %	5.6482	5.7917	7.4315
25	4.2504	6	4.8968	[15, 9, 1]	4.7526	97.05 %	5.8130	6.4269	7.6314
26	4.3544	6	4.9726	[15, 9, 2]	4.8281	97.09 %	5.8125	6.7805	7.9892
27	4.4564	6	5.1258	[16, 9, 2]	4.8856	95.31 %	5.8360	6.5777	7.8196
28	4.5565	6	5.1547	[16, 9, 3]	5.0149	97.29 %	6.1285	6.9569	8.5486
29	4.6548	6	5.2361	[16, 10, 3]	5.1384	98.13 %	6.5785	6.4743	7.9189
30	4.7515	6	5.4142	[16, 10, 4]	5.1977	96.00 %	6.3015	7.3133	9.5919
31	4.8466	6	5.4190	[17, 10, 4]	5.2915	97.65 %	6.5744	7.4074	8.5796
32	4.9401	6	5.5258	[17, 11, 4]	5.4274	98.22 %	7.0711	7.9482	10.3989
33	5.0322	6	5.6770	[17, 11, 5]	5.4697	96.35 %	6.8092	8.0789	10.5514
34	5.1229	6	5.7588	[18, 11, 5]	5.6108	97.43 %	7.2694	7.6486	10.6985
35	5.2123	6	5.8637	[18, 12, 5]	5.6962	97.14 %	7.5832	7.9259	12.4338
36	5.3005	6	5.9744	[17, 12, 6, 1]	5.7467	96.19 %	7.1789	7.6474	11.7903
37	5.3874	6	6	[18, 12, 6, 1]	5.7588	95.98 %	7.6798	7.6583	11.5260
38	5.4731	8	6.0548	[19, 12, 6, 1]	5.9532	98.32 %	7.8716	8.0515	11.6702
39	5.5577	8	6.1575	[19, 13, 6, 1]	6.0482	98.23 %	7.7182	9.0842	12.6976
40	5.6413	8	6.2832	[19, 13, 7, 1]	6.1091	97.23 %	8.1228	9.9491	13.8120

that the edge length becomes one. The magnitude of these adjustments is controlled by a "temperature", which decreases during the course of the simulation to avoid oscillations. The algorithm is programmatically simple, robust, convergent and scales linearly with respect to sample size.

COMPLETE GRAPHS

In this section we consider the dilation coefficient of the complete graph K_n on n vertices. Suppose that m vertices of K_n form a convex polygon with m sides. Since the denominator of $\Delta(K_m)$ is equal to $\min_{e\in E(K_m)} \|e\|$,

in order to obtain an upper bound on the dilation coefficient, it makes sense to make the numerator smaller. To attain minimal $\max_{e\in E(K_m)} \|e\|$, the vertices should be spread on the boundary of the polygon evenly and at the same time, they should lie on the common circle (or an oval). This gives a motivation to study special representations of complete graphs, where all vertices lie on concentric circles.

First, let us compute the dilation coefficient of a planar representation of the complete graph K_m on m vertices, when the vertices are placed equidistantly along a circle in order to form a regular m-gon. This representation of the complete graph is called the *(regular) circular representation*, and is denoted by $\rho_{\{m\}}(K_m)$. It can be easily observed that if vertices of the complete graph K_m describe the regular m-gon, then the dilation coefficient $\Delta_{\{m\}}(K_m)$ of the regular circular representation $\rho_{\{m\}}(K_m)$ can be calculated as

$$\Delta_{\{m\}}(K_m)=\begin{cases}\dfrac{\sin(\frac{(m-1)\pi}{2m})}{\sin(\frac{\pi}{m})}, & m \text{ is odd},\\[2ex] \dfrac{1}{\sin(\frac{\pi}{m})}, & m \text{ is even}.\end{cases}$$

This formula represents the quotient of the length of the longest diagonal of a regular m-gon and the length of its side. The proposed notation $\Delta_{\{m\}}$ has a meaning; there are m vertices on a single cycle (or orbit). Note that in a regular m-gon, where m is at least 4, a side is shorter than any of diagonals, and, where m is at least 7, a side is shorter than the radius of the circumscribed circle.

Since $\Delta_{\{i\}}(K_i) > \Delta_{\{j\}}(K_j)$ for $i > j$, placing i vertices on a single orbit would not always yield an optimal dilation coefficient. Hence, it makes sense to place j points in the outer orbit and $(i-j)$ points inside the circle in such a way that:

- the distance between any two inner points is at least 1,
- the distance between any point on the outer orbit and any inner point is at least 1.

Let $n = m_1 + m_2 + \ldots + m_t$ where $m_1 > 0$ and let $m_2 \geq \ldots \geq m_t > 0$ be the partition of an integer n. We will refer to it as a *uniform integer partition* and denote it with $\{m_1, m_2, ..., m_t\}$. Define a *standard uniform concentric representation* $\rho_{\{m_1,m_2,\ldots,m_t\}}(K_n)$ of the complete graph K_n with respect to the uniform integer partition $\{m_1, m_2, ..., m_t\}$, by considering two cases: $m_t > 1$ and $m_t = 1$:

- Case 1, $m_t > 1$: Place vertices uniformly spaced on t concentric circles centered at the origin O using increasing radii: $1, 2, \ldots, t$. Explicitly, the coordinates of the j-th vertex of the i-th circle with radius $t+1-i$, where $1 \leq j \leq m_i$, are:

$$(t+1-i)\cdot\left\{\cos\left(j\frac{2\pi}{m_i}\right), \sin\left(j\frac{2\pi}{m_i}\right)\right\}.$$

  We may assume that $m_t < 7$ (otherwise at least two of the vertices are at distance less than 1).

- Case 2, $m_t = 1$: In this case one vertex is placed in the origin O and other vertices on $(t-1)$ concentric circles centered at O using increasing radii: $1, 2, \ldots, t-1$. We may think that radius r_t is equal to 0 and hence again observe case 1 for the partition $n-1 = m_1 + m_2 + \ldots + m_{t-1}$.

The standard uniform concentric representation $\rho_{\{m_1,m_2,\ldots,m_t\}}(K_n)$ will give an upper bound $\Delta_{\{m_1,m_2,\ldots,m_t\}}(K_n)$ for $\Delta(K_n)$, which can be easily computed. Clearly, the minimal distance between any two points is at least 1: $\min_{e\in E(K_n)} \|e\| \geq 1$. On the other hand, the distance between any two points is bounded by the diameter of the largest circle. Hence $\Delta_{\{m_1,m_2,\ldots,m_t\}}(K_n) \leq 2t$. In case 2 this crude upper bound can be improved to $2t-2$.

In a degenerate case, exactly one vertex lies on every cycle, thus $\Delta_{\{m_1,m_2,\ldots,m_n\}}(K_n) \leq n-1$. This bound can be greatly improved as follows. From now on, let $m_t > 1$ (*i.e.*, we consider case 1). To minimize the numerator of the dilation coefficient $\Delta_{\{m_1,m_2,\ldots,m_t\}}(K_n)$, we are interested to find the smallest number of orbits t.

The (circum-) radius of the regular m-gon with side of length s is given by $r_m(s) = \dfrac{s}{2\sin(\frac{\pi}{m})}$, and it can be easily observed that $r_i > r_j$ if $i > j$. For a fixed radius R, the maximal number of vertices of the inscribed regular m_R-gon with side of length s, can be thus calculated as $m_R(s) = \left\lfloor \dfrac{\pi}{\arcsin(\frac{s}{2R})} \right\rfloor$. It makes sense to put as many vertices as possible on a single orbit (starting from the smallest one with label t) and, at the same time, provide that distances on all orbits are at least one. Hence, we may assume that the side length s of a regular m_R-gon is one. We denote

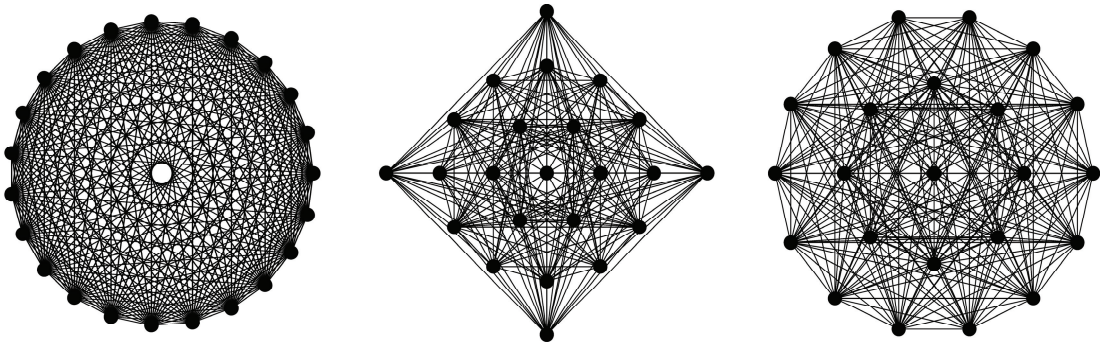

Figure 2. Planar representations of the complete graph K_{23}, with $\underset{\{23\}}{\Delta}(K_{23}) = 7.32682$, $\underset{\{4,12,6,1\}}{\Delta}(K_{23}) = 6$ and $\underset{[14,8,1]}{\Delta}(K_{23}) = 4.61313$, respectively.

$$m_R = m_R(1) = \left\lfloor \frac{\pi}{\arcsin(\frac{1}{2R})} \right\rfloor \text{ and } r_m = r_m(1).$$

For an integer $i \geq 1$ one can verify by using Taylor series expansion that $\sin\left(\frac{\pi}{6i}\right) \geq \frac{1}{2i}$ and hence $m_i \geq 6i$. We place the i-th vertex on the j-th orbit into the point

$$\left\{ j \cdot \cos\left(i \, \frac{2\pi}{6j} \right), j \cdot \sin\left(i \, \frac{2\pi}{6j} \right) \right\},$$

where $0 \leq i \leq 6j-1$. The number of vertices m_i on the i-th orbit is $m_i = 6(t-i+1)$, where $1 < i \leq t$ and $m_1 \leq 6t$. Hence, $n = m_1 + 6(t-1) + 6(t-2) + \ldots + 6$, and $n = m_1 + 3t(t-1)$. Since $0 \leq m_1 \leq 6t$, $n \geq 3t^2 + 3t$. Solving $3t^2 + 3t - n = 0$ yields $t_{1,2} = \frac{1}{2}\left(-1 \pm \sqrt{1 + \frac{4n}{3}} \right)$. Since $1 + \frac{4n}{3} > 1$, the equation has two solutions, one negative and one positive. Considering $m_1 \geq 0$, the smallest integer greater than or equals to the positive solution, that is $t = \left\lceil \frac{1}{2}\left(\sqrt{1 + \frac{4n}{3}} - 1 \right) \right\rceil$, represents the number of orbits. Since we can always put one vertex in the center O (on a circle with radius 0),

$$\underset{\{m_1, m_2, \ldots, m_t\}}{\Delta}(K_n) \leq 2\left\lceil \sqrt{1 + \frac{4(n-1)}{3}} - 1 \right\rceil = 2\left\lceil \sqrt{\frac{4n-1}{3}} - 1 \right\rceil.$$

Calculated bounds are presented in Table 1. It is sometimes possible to place $6i+1$ vertices instead of $6i$ on the orbit with radius i. That could minimize the dilation coefficient even more. However, by allowing the radii in a concentric representation to differ by a different amount than one, a smaller number of orbits could be achieved and further improve the dilation coefficient. In such a representation we concentrate on the lowest number of orbits, hence on maximizing the number of vertices on the first orbit, then on the second one, *etc*.; see, for example, the representations of K_{23} on Figure 2.

Define an *ordered integer partition* $n = m_1 + m_2 + \ldots + m_t$ where $m_1 > m_2 > \ldots > m_t > 0$. The *general* (or non-uniform) *concentric representation* of the complete graph K_n with respect to the ordered integer partition $n = m_1 + m_2 + \ldots + m_t$, denoted by $\underset{[m_1, m_2, \ldots, m_t]}{\rho}(K_n)$, is defined as follows: we place n vertices on t concentric cycles, m_i vertices into m_i evenly spaced points on the i-th cycle with radius R_i, such that every pair of such points is at distance at least one and such that two neighboring cycles are at least one apart; *i.e*., $R_i = \max\left\{ r_{m_i}, R_{i+1} + 1 \right\}$. We denote the dilation coefficient of the general concentric representation of the complete graph K_n with $\underset{[m_1, m_2, \ldots, m_t]}{\Delta}(K_n)$.

Let $\{n_1, n_2, \ldots, n_s\}$ and $[m_1, m_2, \ldots, m_t]$ be a standard and an ordered integer partition of an integer n, respectively. Apparently,

$$\underset{\{n\}}{\Delta}(K_n) > \underset{\{n_1, n_2, \ldots, n_s\}}{\Delta}(K_n) \geq \underset{[m_1, m_2, \ldots, m_t]}{\Delta}(K_n) \geq \Delta(K_n).$$

An ordered integer partition $[m_1, m_2, \ldots, m_t]$ of an integer n that gives the optimal $\underset{[m_1, m_2, \ldots, m_t]}{\Delta}(K_n)$ could be calculated using a dynamic programming method with the *top-down approach* and the *memorization* (also known as memoization).[17] In computing, when the top-down approach with the memorization is used, the problem is broken into subproblems, these subproblems are solved and the solutions remembered, in case they need to be solved again. The Bellman equation for recursively calculating the smallest radius R_n of the outer cycle of an optimal general concentric representation of the complete graph K_n, is given by

$$R_n = \min_{0 \leq m < n} \left\{ \left\{ r_{n-m} \,\middle|\, r_{n-m} - R_m > 1 \right\} \bigcup \left\{ R_m + 1 \,\middle|\, r_{n-m} - R_m \leq 1 \right\} \right\},$$

where again

$$r_m = \frac{1}{2\sin\left(\frac{\pi}{m}\right)}$$

is the radius of the circumscribed circle of the regular unit side m-gon. The idea behind the presented Bellman equation is to place some vertices onto the outer orbit with the smallest radius as possible and to optimally place all remaining vertices (in a recursive way) inside of the outer orbit. To achieve this, we observe two situations:

1) We try to place $n-m$ vertices into points of a regular $(n-m)$-gon with side one and remaining m vertices inside a disc with the radius smaller than $r_{n-m}-1$.

2) We try to place $n-m$ vertices into points of a regular $(n-m)$-gon with points circularly embedded onto a circle with radius R_m+1, where R_m is the radius of the circumscribed circle of the smallest disc containing the remaining m vertices.

Concentric radii give the general concentric ordered integer partition, which defines the general concentric representation and $\underset{[m_1,m_2,\ldots,m_t]}{\Delta}(K_n) = \underset{\{m_1\}}{\Delta}(K_{m_1})$.

The general concentric representation $\underset{[5,1]}{\rho}(K_6)$ of K_6, see Table 1, best describes the second situation. The distance between the two orbits of $\underset{[5,1]}{\rho}(K_6)$ is exactly one and at the same time, since the radius of the circumscribed circle of the regular heptagon is app. 0.8506, the smallest distance between two vertices on the outer rim is also greater than one. It can be easily verified that the dilation coefficient of $\underset{[5,1]}{\rho}(K_6)$ equals $\underset{[5,1]}{\Delta}(K_6) = 1.9044 \le 2 = \underset{\{6\}}{\Delta}(K_6)$, see Figure 3.

Sometimes the proposed representation maps a vertex, say v, into a point that is placed inside a representation of an edge, v does not belong to. Such highly-degenerated cases can be seen in Figure 4. Since for finite (of countable infinite) number of vertices, there are finitely (or countable) many edges and hence finitely

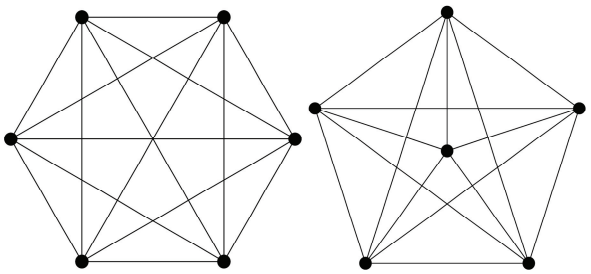

Figure 3. Planar representations of the complete graph K_6, with $\underset{\{6\}}{\Delta}(K_6) = 2$ and $\underset{[5,1]}{\Delta}(K_6) = 1.9044$, respectively.

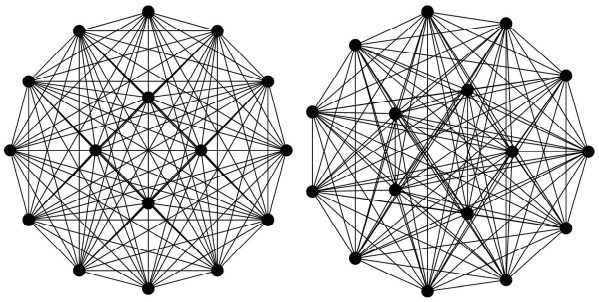

Figure 4. The highly-degenerate planar representation of the complete graph K_{16} with $\underset{[12,4]}{\Delta}(K_{16}) = 3.8637$, and the planar representation of the complete graph K_{16} with $\underset{[11,5]}{\Delta}(K_{16}) = 3.66363$. Note that $r_{11} - r_5 < 1$.

(or countable) many degeneracies, rotating the inner orbit could help avoiding them.

Rotating the vertices on some orbit around the circumcenter may allow moving the orbit closer to its neighbor orbit; in some cases, we may select different (non-concentric) integer partition and possibly decrease the radius of the greater orbit. Thus, rotating the vertices on the orbits could help us minimize the greatest edge-length and find even tighter upper bound on the dilation coefficient $\Delta(K_n)$. Therefore, a tight bound on the dilation coefficient of the complete graph still remains an open question.

Now we present an estimate for the lower bound of $\Delta(K_n)$, that is obtained with only elementary reasoning. Suppose we have a planar representation of the graph K_n with the minimal possible dilation coefficient. Let the minimal distance between two vertices be one. Then we can place in the plane a grid composed of squares with the diagonal a little less than 1 such that no vertex of K_n lies on the grid, see Figure 5. Since the

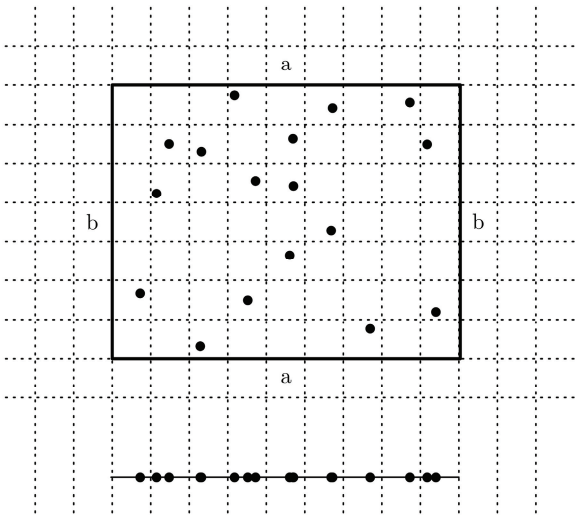

Figure 5. A square grid in the plane that gives a lower bound on dilation coefficient of K_n, which is obtained with only elementary reasoning.

minimal distance between the two vertices is 1, in any square there lies at most one vertex of K_n. Therefore, the number of squares in the minimal rectangle with sides a and b, that contains all the vertices of K_n, is at least n and so $ab \geq n$. Without loss of generality we may assume that $a \geq b$. Then $a \geq \lceil \sqrt{n} \rceil$. To find the maximal distance between two vertices we project all the vertices of K_n on the side a of the rectangle. The distance between two vertices that are the furthest apart on the projection is at least $\frac{\sqrt{2}}{2}(a-1)$ otherwise we could move the grid to the right a little and obtain a shorter side a. Finally, we have

$$\frac{\sqrt{2}\left(\lceil \sqrt{n} \rceil - 1\right)}{2} \leq D(K_n),$$

which is also a lower bound for the dilation coefficient of K_n.

A better lower bound $\sqrt{\frac{2\sqrt{3}}{\pi} n} - 1 \leq \Delta(K_n)$ was obtained by A. Bezdek and F. Fodor[18] in different context, see also Kaminski, Medvedev and Milanič.[19] This lower bound is presented in Table 1.

RESULTS AND DISCUSSION

Our problem is also related to the problem of packing unit circles into a circle with smallest radius.[20–22] Any circle with n unit circles embedded, provides an upper bound for the dilation coefficient of K_n. Namely, if we represent the vertices of K_n as the centers of the unit circles, every vertex is at least at distance $1/2$ from the boundary of the larger circle and so the maximum distance between any two vertices of K_n is at most $r-1$, where r is the radius of the larger circle. An even better bound is obtained if one calculates the dilation coefficient of the complete graph obtained from the given configuration of the unit circles in an optimal (or near optimal) packing.

The circular packing problem uses only discs with diameter exactly one and hence, places centers of each two discs at distance at least one. This is not necessary when obtaining the dilation coefficient of the complete graph; see Table 1 and compare dilation coefficients of the underlying graphs of circular packings with dilation coefficients of general concentric representations for $n = 6$ and $n = 15$. It was proved by Pirl[22] that the bound for $n = 6$ is exact. This shows that the problems are not the same.

Table 1 contains values that result from our computations compared with the results obtained with graph-drawing algorithms. Table 1 lists also the upper bounds, obtained from the best known packings of of n unit circles into a larger circle; the data is taken from the web site.[20] The comparison between our upper bound with general concentric representation and the upper bound from the circle packing problem, denoted as $\underset{\text{CP}}{\Delta}(K_n)$, shows that the bounds from packings are usually a little better; their quotient is around 0.96. However, most of the packings from[20] were obtained by hand for every n separately, while our upper bound is calculated automatically.

Let us now discuss the last three columns of Table 1. The presented graph drawing algorithms usually place vertices of a complete graph in worse positions; *i.e.*, dilation coefficients of representations of a complete graph obtained by graph drawing algorithms are in general considerably larger (worse) than the dilation coefficient of the complete graph. On the other hand, upper bounds for the dilation coefficient of a complete graph are also upper bounds for the dilation coefficient of all its subgraphs. Although the presented graph drawing algorithms perform quite poorly (concerning the dilation coefficient) compared to the general concentric representation, they can be used to represent graphs that are not complete. Hence, using graph-drawing algorithms, representations of simple graphs that are not complete with even smaller dilation coefficients can be obtained.

The reader may notice that the dilation coefficients obtained from algorithms do not monotonically increase; for example, compare results for $n = 35$ and $n = 36$. This can be explained by the heuristic nature of the algorithms. All three algorithms start with random vertex positioning (the so-called *initial representation*) of a given graph and refine it for a pre-described number of iterations. Since the results were obtained starting from a random vertex positioning, all algorithms were run 30 times and for each *n* and each algorithm, a representation with the minimal dilation coefficient was selected.

Let n be the number of vertices of a given graph G, let $N > 0$ be a chosen number of iterations and let $S > 0$ be a chosen number of steps to be performed within each iteration. Time complexities of FRAlg, SAAlg and SPEAlg are $T_{FR}(n) = O(N n^2)$, $T_{SA}(n) = O(N S)$ and $T_{SPE}(n) = O(N S n)$, respectively. Each final representation results heavily on the initial (random) positioning, and at the same time, since their time complexities differ, a detailed comparison of the algorithms would not be fair. The reader must also take into the consideration, that (intentionally) algorithms of different types were selected. SPEAlg for example, works well when a given graph can be drawn with edges of pre-described lengths and, on the other hand, does not perform well when representing a graph with high

(graph) dimension into an Euclidean space of too low (space) dimension.

Recently, the authors of the manuscript[19] introduced a new graph invariant, called plane-width $pw(G)$, that is similar to the dilation coefficient. The plane-width $pw(G)$ of a graph G is defined to be the minimum diameter of the image of the vertex set of G over all mappings of the vertices of G to (not necessarily distinct) points of the plane, so that two adjacent vertices are mapped at least a unit-distance apart. Note that for the complete graphs both invariants coincide. This means that our results hold also for $pw(K_n)$.

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SAŽETAK

Dilatacijski koeficijent potpunog grafa

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Dilatacijski koeficijent reprezentacije grafa definiran je kao kvocijent duljina najdulje i najkraće grane u reprezentaciji. Najmanji dilatacijski koeficijent među svim planarnim reprezentacijama grafa G naziva se *dilatacijskim koeficijentom grafa G*. U radu su razmatrani dilatacijski koeficijenti različitih planarnih reprezentacija potpunih grafova i određene su donje i gornje granice dilatacijskih koeficijenata potpunih grafova. Prikazana su dva iterativna algoritma za minimizaciju dilatacijskog koeficijenta zadanog potpunog grafa. Izračunate gornje granice za dilatacijske koeficijente potpunih grafova uspoređene su s vrijednostima iz algoritama za crtanje grafova.