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Orthologic Tetrahedra with Intersecting Edges

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ABSTRACT

Two tetrahedra are called orthologic if the lines through vertices of one and perpendicular to corresponding faces of the other are intersecting. This is equivalent to the orthogonality of non-corresponding edges. We prove that the additional assumption of intersecting non-corresponding edges ("orthosecting tetrahedra") implies that the six intersection points lie on a sphere. To a given tetrahedron there exists generally a one-parametric family of orthosecting tetrahedra. The orthographic projection of the locus of one vertex onto the corresponding face plane of the given tetrahedron is a curve which remains fixed under isogonal conjugation. This allows the construction of pairs of conjugate orthosecting tetrahedra to a given tetrahedron.

Key words: orthologic tetrahedra, orthosecting tetrahedra, isogonal conjugate

MSC 2010: 51M04

Ortologni tetraedri s bridovima koji se sijeku

SAŽETAK

Dva tetraedra nazivamo ortolognim ako se pravci koji prolaze vrhovima jednog i okomiti su na odgovarajuće stranice drugog međusobno sijeku. Ovo je ekvivalentno ortogonalnosti ne-odgovarajućih bridova. Mi dokazujemo kako dodatna pretpostavka da se ne-odgovarajući bridovi sijeku ("ortopresječni tetraedar") povlači da šest sjecišta leži na jednoj kugli. Za dani tetraedar postoji općenito jednoparametarska familija ortopresječnih tetraedara. Ortogonalna projekcija geometrijskog mjesta jednog vrha na pripadajuću ravninu danog tetraedra je krivulja koja ostaje fiksnom pod djelovanjem izogonalne konjugacije. Ovo dopušta konstrukciju parova konjugiranih ortopresječnih tetraedara za dani tetraedar.

Ključne riječi: ortologni tetraedar, ortopresječni tetraedar, izogonalno konjugiranje

1 Introduction

Ever since the introduction of orthologic triangles and tetrahedra by J. Steiner in 1827 [10] these curious pairs have attracted researchers in elementary geometry. The characterizing property of orthologic tetrahedra is concurrency of the straight lines through vertices of one tetrahedron and perpendicular to corresponding faces of the second. Alternatively, one can say that non-corresponding edges are orthogonal. Proofs of fundamental properties can be found in [7] and [8]. Quite a few results are known on special families of orthologic triangles and tetrahedra. See for example [5, 6, 9, 11] for more information on orthologic tetrahedra (or triangles) which are also perspective or [3] for a generalization of a statement on families of orthologic triangles related to orthopoles.

In this article we are concerned with *orthosecting tetrahedra* — orthologic tetrahedra such that non-corresponding edges intersect orthogonally. The concept as well as a few basic results will be introduced in Section 2. In Section 3 we show that the six intersection points of non-corresponding edges necessarily lie on a sphere (or a plane). While the computation of orthosecting pairs requires, in general, the solution of a system of algebraic equations, conjugate orthosecting tetrahedra can be constructed from a given orthosecting pair. This is the topic of Section 4. Our treatment of the subject is of elementary nature. The main ingredients in the proofs come from descriptive geometry and triangle geometry.

A few words on notation: By $A_1A_2A_3$ we denote the triangle with vertices A_1, A_2, and A_3, by $A_1A_2A_3A_4$ the tetrahedron with vertices A_1, A_2, A_3, and A_4. The line spanned by two points A_1 and A_2 is $A_1 \vee A_2$, the plane spanned by three points A_1, A_2, and A_3 is $A_1 \vee A_2 \vee A_3$. Furthermore, $\mathscr{I}_n$ denotes the set of all n-tuples with pairwise different entries taken from the set $\{1,\dots,n\}$.

2 Preliminaries

Two triangles $A_1A_2A_3$ and $B_1B_2B_3$ are called *orthologic*, if the three lines

$$a_i\colon A_i \in a_i,\ a_i \perp B_j \vee B_k; \quad (i,j,k) \in \mathscr{I}_3 \tag{1}$$

intersect in a point O_A, the *orthology center* of $A_1A_2A_3$ with respect to $B_1B_2B_3$. In this case, also the lines

$$b_i\colon B_i \in b_i,\ b_i \perp A_j \vee A_k; \quad (i,j,k) \in \mathscr{I}_3 \tag{2}$$

intersect in a point O_B, the orthology center of $B_1B_2B_3$ with respect to $A_1A_2A_3$. The concept of orthologic tetrahedra is similar. Two tetrahedra $\mathbf{A} = A_1A_2A_3A_4$ and $\mathbf{B} = B_1B_2B_3B_4$ are called *orthologic*, if the four lines

$$a_i\colon A_i \in a_i,\ a_i \perp B_j \vee B_k \vee B_l; \quad (i,j,k,l) \in \mathscr{I}_4 \tag{3}$$

intersect in a point O_A, the orthology center of $\mathbf{A}$ with respect to $\mathbf{B}$. In this case, also the lines

$$b_i\colon B_i \in b_i,\ b_i \perp A_j \vee A_k \vee A_l; \quad (i,j,k,l) \in \mathscr{I}_4 \tag{4}$$

intersect in a O_B, the orthology center of $\mathbf{B}$ with respect to $\mathbf{A}$. Orthologic triangles and tetrahedra have been introduced by J. Steiner in [10]; proofs of fundamental properties can be found in [7, 8] or [1, pp. 173–174].

The symmetry of the two tetrahedra in the definition of orthology is a consequence of the following alternative characterization of orthologic tetrahedra. It is well-known but we give a proof which introduces concepts and techniques that will frequently be employed throughout this paper.

Proposition 1 *The two tetrahedra* $\mathbf{A}$ *and* $\mathbf{B}$ *are orthologic if and only if non-corresponding edges are orthogonal:*

$$A_i \vee A_j \perp B_k \vee B_l, \quad (i,j,k,l) \in \mathscr{I}_4. \tag{5}$$

Proof. We only require that the lines a_i through A_i and orthogonal to the plane $B_j \vee B_k \vee B_l$ intersect in a point O_A. The plane $A_i \vee A_j \vee O_A$ contains two lines orthogonal to the line $B_k \vee B_l$, $(i,j,k,l) \in \mathscr{I}_4$. Therefore, all lines in this plane, in particular $A_i \vee A_j$, are orthogonal to $B_k \vee B_l$.

Assume conversely that the orthogonality conditions (5) hold. Clearly, any two perpendiculars a_i intersect. We have to show that all intersection points $A_{ij} = a_i \cap a_j$ coincide. Using our freedom to translate the tetrahedron $\mathbf{A}$ without destroying orthogonality relations we can ensure, without loss of generality, the existence of the intersection points

$$\begin{aligned} V_{12} &:= (A_1 \vee A_2) \cap (B_3 \vee B_4),\\ V_{13} &:= (A_1 \vee A_3) \cap (B_2 \vee B_4),\\ V_{23} &:= (A_2 \vee A_3) \cap (B_1 \vee B_4). \end{aligned} \tag{6}$$

Consider now the orthographic projection onto the plane $A_1 \vee A_2 \vee A_3$ (Figure 1). We denote the projection of a point X by X'. By the Right-Angle Theorem of descriptive geometry,[1] the points B'_1, B'_2 and B'_3 lie on the perpendiculars through B'_4 onto the sides of the triangle $\mathbf{A} = A_1A_2A_3$. Moreover, since the plane $V_{12} \vee V_{13} \vee V_{23}$ appears in true size, the lines a_i through A_i and orthogonal to the respective face planes of $\mathbf{A}$ have projections a'_1, a'_2, a'_3 orthogonal to the edges of the triangle $\mathbf{V} = V_{23}V_{13}V_{12}$. The triangles $\mathbf{V}$ and $\mathbf{A}$ are orthologic. Therefore the lines a'_i intersect in a point O'_A which is necessarily the projection of a common point O_A of the lines a_1, a_2, and a_3. □

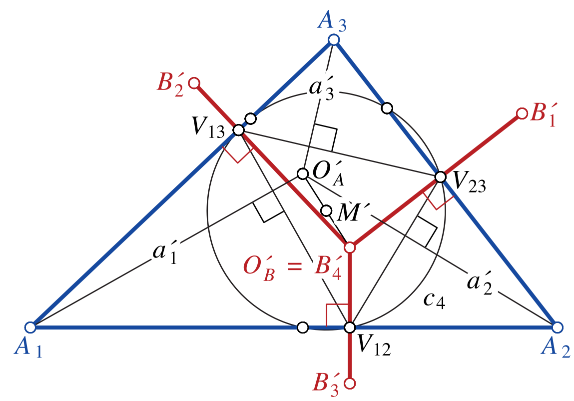

Figure 1: *Orthographic projection onto a face plane*

3 The six intersection points

The new results in this paper concern pairs of orthologic tetrahedra $\mathbf{A} = A_1A_2A_3A_4$ and $\mathbf{B} = B_1B_2B_3B_4$ such that non-corresponding edges are not only orthogonal but also intersecting. That is, in addition to (5) we also require existence of the points

$$V_{ij} := (A_i \vee A_j) \cap (B_k \vee B_l) \neq \varnothing, \quad (i,j,k,l) \in \mathscr{I}_4. \tag{7}$$

Definition 1 *We call two tetrahedra* $\mathbf{A}$ *and* $\mathbf{B}$ orthosecting *if their vertices can be labelled as* $A_1A_2A_3A_4$ *and* $B_1B_2B_3B_4$*, respectively, such that* (5) *and* (7) *hold.*

Theorem 1 *If two tetrahedra are orthosecting, the six intersection points of non-corresponding edges lie on a sphere (or a plane, if flat tetrahedra are permitted). The sphere center is the midpoint between the orthology centers.*

[1] In an orthographic projection the right angle between two lines appears as right angle if and only if one line is in true size (parallel to the image plane) and the other is not in a point-view (orthogonal to the image plane).

Proof. Denote the two tetrahedra by $\mathbf{A} = A_1A_2A_3A_4$ and $\mathbf{B} = B_1B_2B_3B_4$ such that the lines $A_i \vee A_j$ and $B_k \vee B_l$ intersect orthogonally in V_{ij} for $(i, j, k, l) \in \mathscr{I}_4$. As in the proof of Proposition 1 we consider the orthographic projection onto the plane $A_1 \vee A_2 \vee A_3$ (Figure 1). Clearly, B_4' equals the projection O_B' of the orthology center O_B of $\mathbf{B}$ with respect to $\mathbf{A}$. If it lies on the circumcircle of $A_1A_2A_3$, all perpendiculars from B_4' onto the sides of $A_1A_2A_3$ are parallel. In this case the tetrahedron $B_1B_2B_3B_4$ is flat and the theorem's statement holds. Otherwise, the points V_{12}, V_{13}, and V_{23} define a circle c_4 — the pedal circle of the point B_4' with respect to the triangle $A_1A_2A_3$. By the Right-Angle Theorem the projection O_A' of the orthology center O_A of $\mathbf{A}$ with respect to $\mathbf{B}$ is the orthology center of the triangle $A_1A_2A_3$ with respect to the triangle $V_{23}V_{13}V_{12}$. Moreover, from elementary triangle geometry it is known that the center M' of c_4 halves the segment between B_4' and O_A' [4, pp. 54–56]. Hence all circles c_i drawn in like manner on the faces of $\mathbf{A}$ have axes which intersect in the midpoint M of the two orthology centers O_A and O_B. Moreover, any two of these circles share one of the points V_{ij}. Hence, these circles are co-spherical and the proof is finished. □

The proof of Theorem 1 can also be applied to a slightly more general configuration where only five of the six edges intersect orthogonally. We formulate this statement as a corollary:

Corollary 1 *If* $\mathbf{A} = A_1A_2A_3A_4$ *and* $\mathbf{B} = B_1B_2B_3B_4$ *are two orthologic tetrahedra such that five non-corresponding edges intersect, the five intersection points lie on a sphere (or a plane).*

4 The one-parametric family of solution tetrahedra

So far we have dealt with properties of a pair of orthosecting tetrahedra but we have left aside questions of existence or computation. In this section $\mathbf{A} = A_1A_2A_3A_4$ is a given tetrahedron to which an orthosecting tetrahedron $\mathbf{B} = B_1B_2B_3B_4$ is sought.

4.1 Construction of orthologic tetrahedra

At first, we consider the simpler case of orthologic pairs. Clearly, translation of the face planes of $\mathbf{B}$ will transform an orthologic tetrahedron into a like tetrahedron (unless all planes pass through a single point). Therefore, we consider tetrahedra with parallel faces as equivalent.

The maybe simplest construction of an equivalence class of solutions consists of the choice of the orthology center O_A. This immediately yields the face normals $\mathbf{n}_i$ of $\mathbf{B}$ as connecting vectors of O_A and A_i. The variety of solution classes is of dimension three, one solution to every choice of O_A. Since five edges determine two face planes of a tetrahedron and, in case of suitable orthogonality relations, also the orthology center O_A, we obtain

Theorem 2 *If the vertices of two tetrahedra can be labelled such that five non-corresponding pairs of edges are orthogonal then so is the sixth.*

The variety of all solution classes contains a two-parametric set of trivial solutions $\mathbf{n}_1 = \mathbf{n}_2 = \mathbf{n}_3 = \mathbf{n}_4$. They correspond to orthology centers at infinity, the solution tetrahedra are flat. Note that the possibility to label the edges such that non-corresponding pairs are orthogonal is essential for the existence of non-flat solutions. If, for example, corresponding edges are required to be orthogonal only flat solutions exist.

4.2 Conjugate pairs of orthosecting tetrahedra

Establishing algebraic equations for solution tetrahedra is straightforward. Six orthogonality conditions and six intersection condition result in a system of six linear and six quadratic equations in the twelve unknown coordinates of the vertices of $\mathbf{B}$. Because of Theorem 2, only five of the six linear orthogonality conditions are independent. Therefore, we can expect a one-dimensional variety of solution tetrahedra. This expectation is generically true, as can be confirmed by computing the dimension of the ideal spanned by the orthosecting conditions by means of a computer algebra system.

The numeric solution of the system induced by the orthosecting conditions poses no problems. We used the software Bertini,[2] for that purpose. Symbolic approaches are feasible as well. One of them will be described in Subsection 4.3. It is based on a curious conjugacy which can be defined in the set of all tetrahedra that orthosect the given tetrahedron $\mathbf{A}$.

Assume that $\mathbf{B} = B_1B_2B_3B_4$ is a solution tetrahedron and denote the orthographic projection of B_i onto the face plane $A_j \vee A_k \vee A_l$ by $B_i^\star$, $(i, j, k, l) \in \mathscr{I}_4$. By the Right-Angle theorem the pedal points of all points $B_i^\star$ on the edges of $A_jA_kA_l$ are precisely the intersection points defined in (7). Three intersection points on the same face of $\mathbf{A}$ form a pedal triangle. This observation gives rise to:

[2] D. J. Bates, J. D. Hauenstein, A. J. Sommese, Ch. W. Wampler: Bertini: Software for Numerical Algebraic Geometry, `http://www.nd.edu/ sommese/bertini/`.

Definition 2 *A pedal chain on a tetrahedron is a set of four pedal triangles, each with respect to one face triangle of the tetrahedron, such that any two pedal triangles share the vertex on the common edge of their faces (Figure 2). If all vertices of pedal triangles lie on a sphere (or a plane), we speak of a spherical pedal chain.*

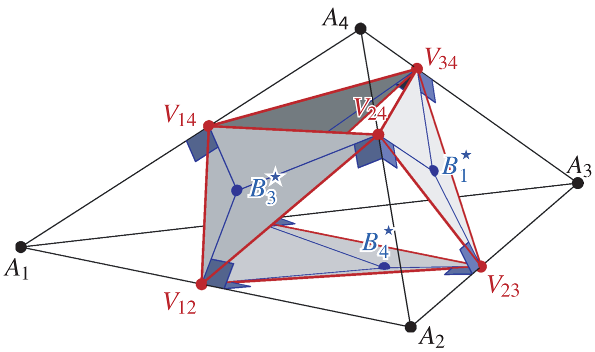

Figure 2: *A pedal chain*

If $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ are orthosecting, the proof of Theorem 1 shows that six intersection points are the vertices of a spherical pedal chain. The converse is also true:

Theorem 3 *Given the vertices V_{ij} of a spherical pedal chain on a tetrahedron $\mathbf{A} = A_1A_2A_3A_4$ there exists a unique orthosecting tetrahedron $\mathbf{B} = B_1B_2B_3B_4$ such that $A_i \vee A_j \cap B_k \vee B_l = V_{ij}$ for all $(i, j, k, l) \in \mathscr{I}_4$.*

Proof. If a solution tetrahedron **B** exists at all it must be unique since its faces lie in the planes $\beta_i := V_{ij} \vee V_{ik} \vee V_{il}$ (i, j, $k \in \{1,2,3,4\}$ pairwise different).[3]

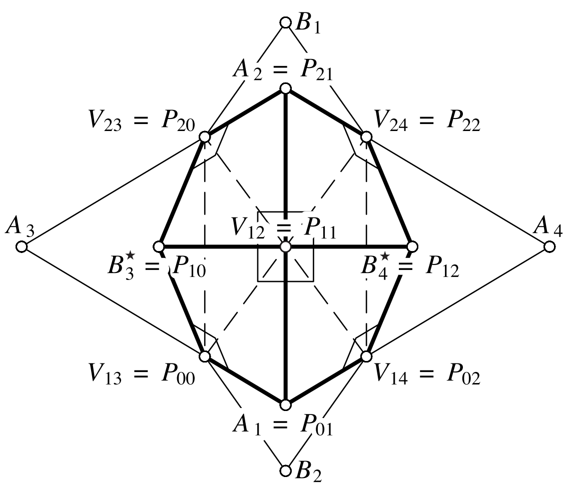

Figure 3: *Proof of Theorem 3*

In order to prove existence, we have to show that the lines $A_i \vee A_j$ and $\beta_i \cap \beta_j$ are, indeed, orthogonal for all pairwise different i, $j \in \{1,2,3,4\}$. We denote the point from which the pedal triangle on the face $A_iA_jA_k$ originates (the "anti-pedal point") by $B_l^\star$ and show orthogonality between $A_1 \vee A_2$ and $\beta_1 \cap \beta_2$ for $(i, j, k, l) \in \mathscr{I}_4$. Relabelling according to

$$\begin{array}{lll} P_{00} := V_{13}, & P_{01} := A_1, & P_{02} := V_{14}, \\ P_{10} := B_4^\star, & P_{11} := V_{12}, & P_{12} := B_3^\star, \\ P_{20} := V_{23}, & P_{21} := A_2, & P_{22} := V_{24} \end{array} \tag{8}$$

(Figure 3) we obtain a net of points P_{ij}. In every elementary quadrilateral the angle measure at two opposite vertices equals $\pi/2$. Thus, the net is *circular.* Such structures are extensively studied in the context of discrete differential geometry [2]. Our case is rather special since two pairs of quadrilaterals span the same plane. This does, however, not hinder application of [2, Theorem 4.21] which states that our assumptions on the co-spherical (or co-planar) position of the points P_{00}, P_{02}, P_{11}, P_{20}, and P_{22} is equivalent to the fact that the net P_{ij} is a *discrete isothermic net*. These nets have many remarkable characterizing properties. One of them, stated in [2, Theorem 2.27], says that the planes $P_{00} \vee P_{11} \vee P_{02}$, $P_{10} \vee P_{11} \vee P_{12}$, and $P_{20} \vee P_{11} \vee P_{22}$ have a line in common. In our original notation this means that the line $\beta_1 \cap \beta_2$ intersects the face normal of $A_1 \vee A_2 \vee A_3$ through $B_4^\star$ and the face normal of $A_1 \vee A_2 \vee A_4$ through $B_3^\star$. Therefore, it is orthogonal to $A_1 \vee A_2$. □

As a consequence of Theorem 3 it can be shown that tetrahedra which orthosect **A** come in conjugate pairs: Given **A** and an orthosecting tetrahedron **B** it is possible to construct a second orthosecting tetrahedron **C**. The same construction with **C** as input yields the tetrahedron **B**. This conjugacy is related to the pedal chain originating from **B**. The key ingredient is the following result from elementary triangle geometry [4, pp. 54–56]:

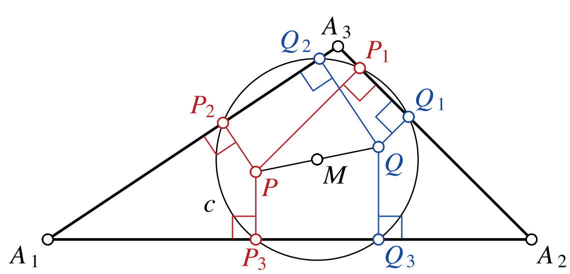

Figure 4: *Pedal circles in a triangle*

Proposition 2 *If P is a point in the plane of the triangle $A_1A_2A_3$ and c its pedal circle, the reflection Q of P in the center M of c has the same pedal circle c (Figure 4).*

[3] The case of collinear or coinciding points V_{ij} leads to degenerate solution tetrahedra whose faces contain one vertex of **A**.

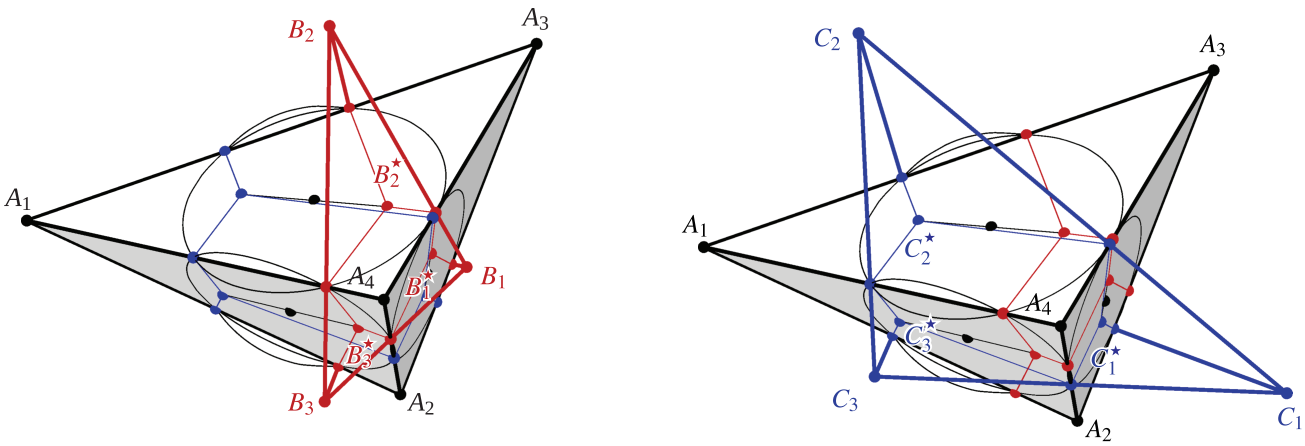

Figure 5: *A conjugate pair* **B**, **C** *of orthosecting tetrahedra.*

Suppose that **A** and **B** are orthosecting tetrahedra. The orthographic projections $B_i^\star$ of the vertices of **B** onto corresponding face planes of **A** are points whose pedal triangles form a spherical pedal chain. By reflecting $B_i^\star$ in the centers of the pedal circles on the faces of **A** we obtain points $C_i^\star$ which, according to Proposition 2, give rise to a second spherical pedal chain (with the same sphere of vertices) and, by Theorem 3, can be used to construct a second orthosecting tetrahedron **C** (Figure 5).

The points P and Q of Proposition 2 are called *isogonal conjugates* with respect to the triangle $A_1A_2A_3$. The above considerations lead immediately to

Theorem 4 *Given a tetrahedron* $\mathbf{A} = A_1A_2A_3A_4$*, the orthographic projection of all vertices* $B_i^\star$ *of orthosecting tetrahedra onto the face plane* $A_j \vee A_k \vee A_l$ *of* **A** *(with* $(i,j,k,l) \in \mathscr{I}_4$*) is a curve which is isogonally self-conjugate with respect to the triangle* $A_jA_kA_l$*.*

4.3 Computational issues

We continue with a few remarks on the actual computation of the isogonal self-conjugate curves of Theorem 4 with the help of a computer algebra system. Our first result concerns the construction of pedal chains.

Theorem 5 *Consider a tetrahedron* $\mathbf{A} = A_1A_2A_3A_4$ *and six points* $V_{ij} \in A_i \vee A_j$, $(i,j,k,l) \in \mathscr{I}_4$*. If three of the four triangles* $V_{ij}V_{jk}V_{ki}$*, with* $(i,j,k) \in \mathscr{I}_3$*, are pedal triangles with respect to the triangle* $A_iA_jA_k$ *then this is also true for the fourth.*

Proof. Assume that the triangles $V_{12}V_{24}V_{14}$, $V_{23}V_{24}V_{34}$, and $V_{13}V_{34}V_{14}$ are pedal triangles of their respective face triangles. We have to show that $V_{12}V_{23}V_{13}$ is a pedal triangle of $A_1A_2A_3$. As usual, the anti-pedal points are denoted by $B_1^\star$, $B_2^\star$, and $B_3^\star$. Clearly, we have $B_i^\star \vee B_j^\star \perp A_k \vee A_4$ for $(i,j,k,4) \in \mathscr{I}_4$. Denote by B_4° a point in the intersection of the three planes incident with V_{ij} and perpendicular to $A_i \vee A_j$, $(i,j,k) \in \mathscr{I}_3$. By Proposition 1 the tetrahedra **A** and $B_1^\star B_2^\star B_3^\star B_4^\circ$ are orthologic. Therefore, the face normals n_l of $A_i \vee A_j \vee A_k$ through $B_l^\star$ have a point B_4 in common ($l \neq 4$, $(i,j,k,l) \in \mathscr{I}_4$). By the Right-Angle Theorem, the intersection point $B_4^\star$ of the orthographic projections of n_1, n_2, and n_3 onto $A_1 \vee A_2 \vee A_3$ has $V_{12}V_{23}V_{13}$ as its pedal triangle. □

In order to construct a pedal chain on a tetrahedron $\mathbf{A} = A_1A_2A_3A_4$ on can proceed as follows:

1. Prescribe an arbitrary pedal triangle, say $V_{12}V_{23}V_{13}$.
2. Choose one anti-pedal point, say $B_3^\star$, on a neighbouring face. It is restricted to the perpendicular to $A_1 \vee A_2$ trough V_{12}.
3. The remaining pedal points are determined. Theorem 5 guarantees that the final completion of V_{34} is possible without contradiction.

In order to construct a spherical pedal chain, the choice of $B_4^\star$ and $B_3^\star$ needs to be appropriate. A simple computation shows that there exist two possible choices (in algebraic sense) for $B_3^\star$ such that the points V_{12}, V_{13}, V_{23}, V_{14}, and V_{24} are co-spherical (or co-planar). Demanding that the remaining vertex V_{34} lies on the same sphere yields an algebraic condition on the coordinates of $B_4^\star$ — the algebraic equation of the isogonally self-conjugate curve i_4 from Theorem 4. We are currently not able to carry out the last elimination step in full generality. Examples suggest, however, that i_4 is of degree nine. Once a point on i_4 is determined, the computation of the corresponding orthosecting tetrahedron is trivial.

5 Conclusion and future research

We introduced the concept of orthosecting tetrahedra and presented a few results related to them. In particular we characterized the six intersection points as vertices of a spherical pedal chain on either tetrahedron. This characterization allows the construction of conjugate orthosecting tetrahedra to a given tetrahedron **A**.

In general, there exists a one-parametric family of tetrahedra which orthosect **A**. The orthographic projection of their vertices on the plane of a face triangle of **A** is an isogonally self-conjugate algebraic curve. Maybe it is worth to study other loci related to the one-parametric family of orthosecting tetrahedra. Since every sphere that carries vertices of one pedal chain also carries the vertices of a second pedal chain, the locus of their centers might have a reasonable low algebraic degree.

Moreover, other curious properties of orthosecting tetrahedra seem likely to be discovered. For example, the repeated construction of conjugate orthosecting tetrahedra yields an infinite sequence $\langle \mathbf{B}_n \rangle_{n\in\mathbb{Z}}$ of tetrahedra such that $\mathbf{B}_{n-1}$ and $\mathbf{B}_{n+1}$ form a conjugate orthosecting pair with respect to $\mathbf{B}_n$ for every $n \in \mathbb{Z}$. All intersection points of non-corresponding edges lie on the same sphere and only two points serve as orthology centers for any orthosecting pair $\mathbf{B}_n$, $\mathbf{B}_{n+1}$. General properties and special cases of this sequence might be a worthy field of further study.

Finally, we would like to mention two possible extensions of this article's topic. It seems that, with exception of Steiner's result on orthologic triangles on the sphere, little is known on orthologic triangles and tetrahedra in non-Euclidean spaces. Moreover, one might consider a relaxed "orthology property" as suggested by the anonymous reviewer: It requires that the four lines a_1, a_2, a_3, a_4 defined in (3) lie in a regulus (and not necessarily in a linear pencil). This concept is only useful if the regulus position of the lines a_i also implies regulus position of the lines b_j of (4). We have some numerical evidence that this is, indeed, the case.

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