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Chromogeometry and Relativistic Conics

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ABSTRACT

This paper shows how a recent reformulation of the basics of classical geometry and trigonometry reveals a three-fold symmetry between Euclidean and non-Euclidean (relativistic) planar geometries. We apply this *chromogeometry* to look at conics in a new light.

Key words: chromogeometry, conics, relativistic geometry

MSC 2010: 51N25, 83A05

Kromogeometrija i relativističke konike

SAŽETAK

U radu se pokazuje kako novija formulacija osnova klasične geometrije i trigonometrije otkriva trostruku simetriju između euklidske i neeuklidskih (relativističkih) ravninskih geometrija. Primjenjujemo kromogeometriju kako bismo konike vidjeli u novom svjetlu.

Ključne riječi: kromogeometrija, konike, relativistička geometrija

Pythagoras, area and quadrance

To measure a line segment in the plane, the ancient Greeks measured the *area of a square constructed on it*. Algebraically, the parallelogram formed by a vector $v = \overrightarrow{A_1 A_2} = (a, b)$ and its perpendicular $B(v) = (-b, a)$ has area

$$Q = \det \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = a^2 + b^2. \tag{1}$$

The Greeks referred to building squares as 'quadrature', and so we say that Q is the **quadrance** of the vector v, or the quadrance $Q(A_1, A_2)$ between A_1 and A_2. This notion makes sense over any field.
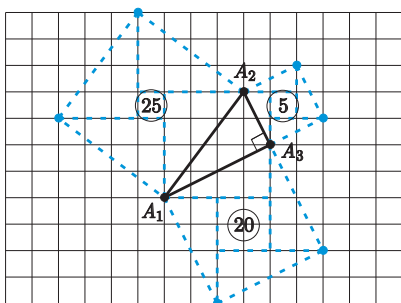

Figure 1: *Pythagoras' theorem:* $5 + 20 = 25$

If $Q_1 = Q(A_2, A_3)$, $Q_2 = Q(A_1, A_3)$ and $Q_3 = Q(A_1, A_2)$ are the quadrances of a triangle $\overline{A_1 A_2 A_3}$, then Pythagoras'

theorem and its converse can together be stated as: $A_1 A_3$ *is perpendicular to* $A_2 A_3$ *precisely when*

$$Q_1 + Q_2 = Q_3.$$

Figure 1 shows an example where $Q_1 = 5$, $Q_2 = 20$ and $Q_3 = 25$. As indicated for the large square, these areas may also be calculated by subdivision and (translational) rearrangement, followed by counting cells.

There is a sister theorem—the *Triple quad formula*—that Euclid did not know, but which is fundamental for *rational trigonometry*, introduced in [2]: $A_1 A_3$ *is parallel to* $A_2 A_3$ *precisely when*

$$(Q_1 + Q_2 + Q_3)^2 = 2 \left(Q_1^2 + Q_2^2 + Q_3^2 \right).$$

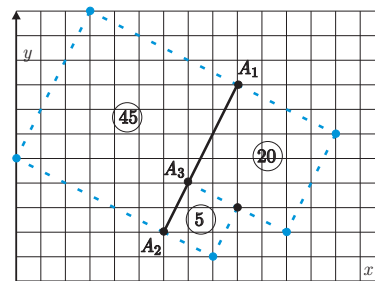Figure 2 shows an example where $Q_1 = 5$, $Q_2 = 20$ and $Q_3 = 45$.

Figure 2: *Triple quad formula:*
$$(5 + 20 + 45)^2 = 2 \left(5^2 + 20^2 + 45^2 \right)$$

In terms of side lengths $d_1 = \sqrt{Q_1}, d_2 = \sqrt{Q_2}$ and $d_3 = \sqrt{Q_3}$, and the semi-perimeter $s = (d_1 + d_2 + d_3)/2$, observe that

$$(Q_1 + Q_2 + Q_3)^2 - 2\left(Q_1^2 + Q_2^2 + Q_3^2\right)$$

$$= 4Q_1 Q_2 - (Q_1 + Q_2 - Q_3)^2 = 4d_1^2 d_2^2 - \left(d_1^2 + d_2^2 - d_3^2\right)^2$$

$$= \left(2d_1 d_2 - \left(d_1^2 + d_2^2 - d_3^2\right)\right)\left(2d_1 d_2 + \left(d_1^2 + d_2^2 - d_3^2\right)\right)$$

$$= \left(d_3^2 - (d_1 - d_2)^2\right)\left((d_1 + d_2)^2 - d_3^2\right)$$

$$= (d_3 - d_1 + d_2)(d_3 + d_1 - d_2)(d_1 + d_2 - d_3)(d_1 + d_2 + d_3)$$

$$= 16(s - d_1)(s - d_2)(s - d_3)s.$$

Thus Heron's formula in the usual form

$$\text{area} = \sqrt{s(s - d_1)(s - d_2)(s - d_3)}$$

may be restated in terms of quadrances as

$$16\,\text{area}^2 = (Q_1 + Q_2 + Q_3)^2 - 2\left(Q_1^2 + Q_2^2 + Q_3^2\right)$$
$$\equiv A(Q_1, Q_2, Q_3).$$

This more fundamental formulation deserves to be called **Archimedes' theorem**, since Arab sources indicate that Archimedes knew Heron's formula. The Triple quad formula is the special case of Archimedes' theorem when the area is zero. The function $A(Q_1, Q_2, Q_3)$ will be called **Archimedes' function**.

In Figure 3 the quadrances are $Q_1 = 13$, $Q_2 = 25$ and $Q_3 = 26$, so $16\,\text{area}^2 = 1156$, giving $\text{area}^2 = 289/4$ and $\text{area} = 17/2$.
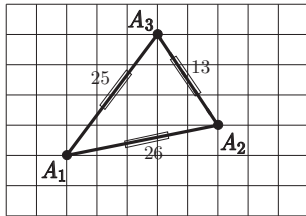

Figure 3: $16\,\text{area}^2 = (13 + 25 + 26)^2 - 2\left(13^2 + 25^2 + 26^2\right)$
$= 1156$

Irrational side lengths are not needed to determine the area of a rational triangle, and in any case when we move to more general geometries, we have no choice but to give up on distance and angle.

Blue, red and green geometries

Euclidean geometry will here be called **blue geometry**. We now introduce two relativistic geometries, called *red* and *green*, which arise from Einstein's theory of relativity.

These rest on alternate notions of perpendicularity, but they share the same underlying affine concept of area as blue geometry, and indeed the same laws of rational trigonometry, as will be explained shortly.
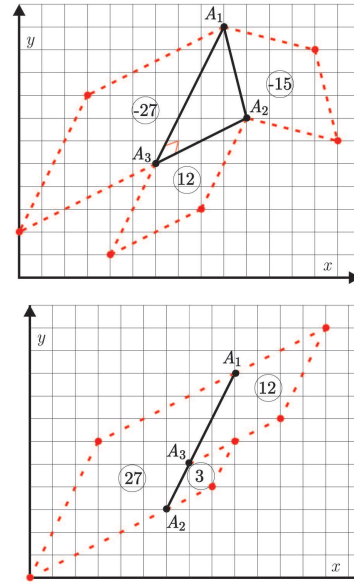

Figure 4: *Red Pythagoras' theorem and Triple quad formula*

Define the vector $v = \overrightarrow{A_1 A_2} = (a, b)$ to be **red perpendicular** to $R(v) = (b, a)$. This mapping is easily visualized: it corresponds to Euclidean reflection in a line of slope 1 or -1. A **red square** is then a parallelogram with sides v and $R(v)$, and hence (signed) area

$$Q^{(r)} = \det\begin{pmatrix} a & b \\ b & a \end{pmatrix} = a^2 - b^2 \tag{2}$$

which we call the **red quadrance** between A_1 and A_2. Figure 4 illustrates that both Pythagoras' theorem and the Triple quad formula hold also using red quadrances and red perpendicularity, where the areas of the red squares can be computed as before by subdivisions, (translational) rearrangement and counting cells—or by applying the algebraic formula for the red quadrance.

In a similar fashion the vector $v = \overrightarrow{A_1 A_2} = (a, b)$ is **green perpendicular** to $G(v) = (-a, b)$. This corresponds to Euclidean reflection in a vertical or horizontal line. A **green square** is a parallelogram with sides v and $R(v)$, and hence (signed) area

$$Q^{(g)} = \det\begin{pmatrix} a & b \\ -a & b \end{pmatrix} = 2ab \tag{3}$$

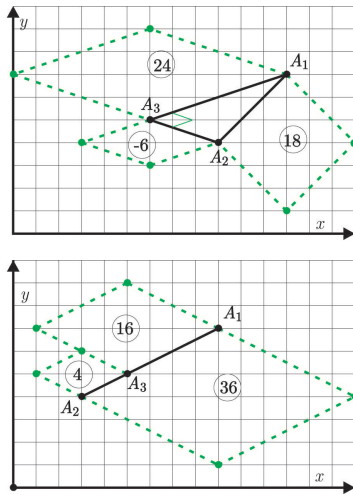which we call the **green quadrance** between A_1 and A_2.

Figure 5: *Green Pythagoras' theorem and Triple quad formula*

Figure 5 shows Pythagoras' theorem and the Triple quad formula in the green context. This version of relativistic geometry corresponds to a basis of null vectors in red geometry.

All three geometries can be defined over a general field, not of characteristic two.

Spreads and rational trigonometry

The three quadratic forms

$$Q^{(b)}(a,b) = a^2 + b^2$$
$$Q^{(r)}(a,b) = a^2 - b^2$$
$$Q^{(g)}(a,b) = 2ab$$

have corresponding dot products

$$(a_1, b_1) \cdot_b (a_2, b_2) \equiv a_1 a_2 + b_1 b_2$$
$$(a_1, b_1) \cdot_r (a_2, b_2) \equiv a_1 a_2 - b_1 b_2$$
$$(a_1, b_1) \cdot_g (a_2, b_2) \equiv a_1 b_2 + a_2 b_1.$$

Together with $a_1 b_2 - a_2 b_1 = 0$ describing parallel vectors, these are the four simplest bilinear expressions in the four variables.

In rational trigonometry, one wants to work over general fields, so the notion of angle is not available, but it is important to realize that the dot product is not necessarily the best replacement. Instead we introduce the related notion of *spread* between two lines (not between rays), which in the blue framework is the square of the sine of the angle between the lines (there are actually many such angles, but the square of the sine is the same for all).

The **blue, red** and **green spreads** between lines l_1 and l_2 with equations

$$a_1 x + b_1 y + c_1 = 0 \qquad \text{and} \qquad a_2 x + b_2 y + c_2 = 0$$

are respectively the numbers

$$s^{(b)}(l_1, l_2) = \frac{(a_1 b_2 - a_2 b_1)^2}{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}$$

$$s^{(r)}(l_1, l_2) = -\frac{(a_1 b_2 - a_2 b_1)^2}{(a_1^2 - b_1^2)(a_2^2 - b_2^2)}$$

$$s^{(g)}(l_1, l_2) = -\frac{(a_1 b_2 - a_2 b_1)^2}{4 a_1 b_1 a_2 b_2}.$$

These quantities are undefined when the denominators are zero. The negative signs in front of $s^{(r)}$ and $s^{(g)}$ insure that, for each of the colours, the *spread at any of the three vertices of a right triangle (one with two sides perpendicular) is the quotient of the opposite quadrance by the hypotenuse quadrance*. See [3] for a proof of this, and other facts about rational trigonometry, in a wider context.

In Figure 1 the spreads at A_1 and A_2 are $1/5$ and $4/5$ respectively, in the upper diagram of Figure 4 the spreads at A_1 and A_2 are $-4/5$ and $9/5$ respectively, and in the upper diagram of Figure 5 the spreads at A_1 and A_2 are $-1/3$ and $4/3$ respectively. In each case the spread at the right vertex is 1, and the other two spreads sum to 1.
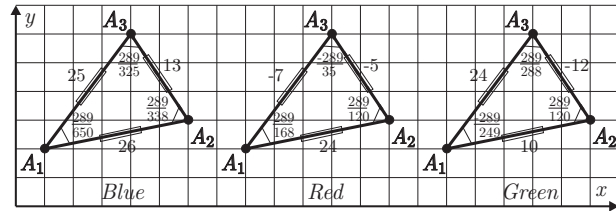

Figure 6: *Blue, red and green quadrances and spreads*

Figure 6 allows you to compare the various quadrances and spreads of a fixed triangle in each of the three geometries. Note the common numerators of the spreads, arising because $A(Q_1, Q_2, Q_3) = \pm 4 \times 289$ is up to sign the same in each geometry, with a plus sign in the blue situation and a negative sign in the red and green ones.

Aside from Pythagoras' theorem and the Triple quad formula, the main laws of rational trigonometry are: for a triangle with quadrances Q_1, Q_2 and Q_3, and spreads s_1, s_2 and s_3 :

$$\frac{s_1}{Q_1} = \frac{s_2}{Q_2} = \frac{s_3}{Q_3} \quad \text{(Spread law)}$$

$$(Q_1 + Q_2 - Q_3)^2 = 4 Q_1 Q_2 (1 - s_3) \quad \text{(Cross law)}$$

$$(s_1 + s_2 + s_3)^2 = 2(s_1^2 + s_2^2 + s_3^2) + 4 s_1 s_2 s_3 \quad \text{(Triple spread formula)}.$$

As shown in [2], these laws are derived using only Pythagoras' theorem and the Triple quad formula. Since these latter two results hold in all three geometries, the Spread law, Cross law and Triple spread formula also hold in all three geometries.

For any points A_1 and A_2 the square of $Q^{(b)}(A_1, A_2)$ is the sum of the squares of $Q^{(r)}(A_1, A_2)$ and $Q^{(g)}(A_1, A_2)$, and for any lines l_1 and l_2

$$\frac{1}{s^{(b)}(l_1, l_2)} + \frac{1}{s^{(r)}(l_1, l_2)} + \frac{1}{s^{(g)}(l_1, l_2)} = 2.$$

The first statement follows from the Pythagorean triple identity

$$\left(x^2 + y^2\right)^2 = \left(x^2 - y^2\right)^2 + (2xy)^2$$

while the latter follows from the identity

$$\left(a_1^2 + b_1^2\right)\left(a_2^2 + b_2^2\right) - \left(a_1^2 - b_1^2\right)\left(a_2^2 - b_2^2\right) - 4a_1 b_1 a_2 b_2$$
$$= (a_1 b_2 - a_2 b_1)^2.$$

So in Figure 6 there are three linked (signed) Pythagorean triples, namely $(13, -5, -12)$, $(25, -7, 24)$ and $(26, 24, 10)$, and three triples of harmonically related spreads.
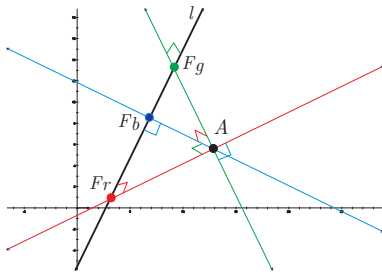

Figure 7: *Three altitudes from a point to a line*

Figure 7 shows the three coloured altitudes from a point A to a line l, and the feet of those altitudes. Note that the blue and red altitudes are green perpendicular and similarly for the other colours. The three triangles formed by the four points are each **triple right triangles**, containing each a blue, red and green right vertex.

Most theorems of planar Euclidean geometry have universal versions, valid in each of the three geometries. *This is a large claim that deserves further investigation*. In the red and green geometries, circles and rotations become rectangular hyperbolas and Lorentz boosts. There are no equilateral triangles in the red and green geometries, so results like Napoleon's theorem or Morley's theorem will not have (obvious) analogs.

To see some chromogeometry in action, let's have a look at conics in this more general framework.

The ellipse as a grammola

In the real number plane, one usually defines an *ellipse* as the locus of a point X whose ratio of distance from a fixed point (focus) to distance from a fixed line (directrix) is constant and less than one, and hyperbolas and parabolas similarly with eccentricities greater than one and equal to one. By squaring this condition, we can discuss the locus of a point whose ratio of *quadrance* from a fixed point to a fixed line is constant. By quadrance from a point X to a line l we mean the obvious: construct the altitude line n from X to l, find its foot F and measure $Q(X, F)$. Let's call such a locus a **conic section**. Over a general field we cannot distinguish 'ellipses' from 'hyperbolas', although parabolas are always well defined.
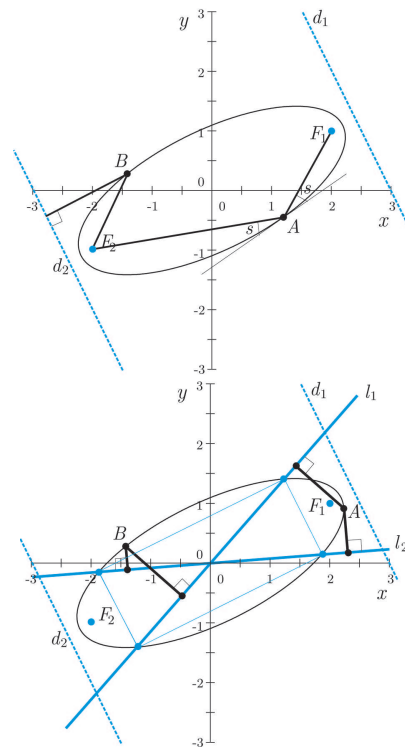

Figure 8: *Two views of the ellipse $2x^2 - 4xy + 5y^2 = 6$*

The upper diagram in Figure 8 shows the central ellipse

$$2x^2 - 4xy + 5y^2 = 6$$

with foci at $F_1 = [2, 1]$ and $F_2 = [-2, -1]$, corresponding directrices d_1 and d_2 with respective equations $2x - y + 6 = 0$ and $2x - y - 6 = 0$, and eccentricity $e = \sqrt{5/6}$. The familiar reflection property may be recast as: *spreads* between a tangent and lines to the foci from a point on the ellipse are equal.

The lower diagram in Figure 8 illustrates a (perhaps?) novel definition of an ellipse. It is motivated by the fact that a circle is the locus of a point X whose quadrances to two fixed perpendicular lines add to a constant. Define a **grammola** to be the locus of a point X such that the sum of the quadrances from X to two fixed non-perpendicular intersecting lines l_1 and l_2 is constant. This definition works for each of the three colours. It turns out that the lines l_1 and l_2 are unique; we call them the **diagonals** of the grammola (see [2, Chapter 15]). The **corners** of the grammola are the points where the diagonal lines intersect it, and determine the **corner rectangle**. In the blue setting over the real numbers a grammola is always an *ellipse*, while in the red and green settings a grammola might be an ellipse, or it might be a hyperbola.

The ellipse of Figure 8 is a blue grammola with blue diagonals

$$\left(14 + 5\sqrt{6}\right)x - 23y = 0 \quad \text{and} \quad \left(14 - 5\sqrt{6}\right)x - 23y = 0.$$

The blue quadrances of the sides of the corner rectangle are 12 and 2, whose product 24 is the squared area. The lower diagram in Figure 8 shows the usual foci and directrices of the grammola and its diagonals and corners. The quadrances from any point on the conic to the two diagonals sum to 6. The blue spread between the two diagonals is an invariant of the ellipse—in this case $s_b = 24/49$.
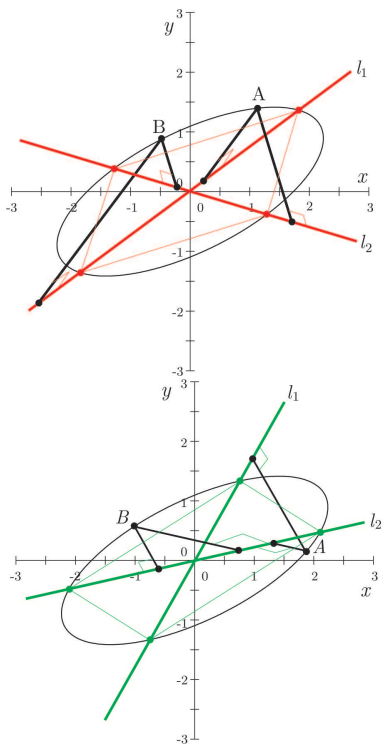

The ellipse can also be described as a red grammola, as in the upper diagram of Figure 9. The red diagonals are

$$\left(\sqrt{22} + 2\right)x - 9y = 0 \quad \text{and} \quad \left(\sqrt{22} - 2\right)x + 9y = 0$$

and the **red corner rectangle** has sides parallel to the red axes of the ellipse, and red quadrances $3 + \sqrt{33}$ and $3 - \sqrt{33}$, whose product is -24. The four red corners have rather complicated expressions in this case. The red spread between the red diagonals is $s_r = -8/3$.

The same ellipse may also be viewed as a green grammola, as in the lower diagram of Figure 9. The green diagonals are

$$\left(-5 + \sqrt{15}\right)x + 5y = 0 \quad \text{and} \quad \left(-5 - \sqrt{15}\right)x + 5y = 0$$

and the **green corner rectangle** has sides parallel to the green axes of the ellipse, and green quadrances $4 + 2\sqrt{10}$ and $4 - 2\sqrt{10}$, whose product is again -24. Except for a sign, the three squared areas of the blue, red and green corner rectangles are the same. The green spread between the green diagonals is $s_g = -3/2$.

The relationship between the blue, red and green spreads of an ellipse is

$$\frac{1}{s_b} + \frac{1}{s_r} + \frac{1}{s_g} = 1.$$

The ellipse as a quadrola

Another well known definition of an ellipse is as the locus of a point X whose sum of distances from two fixed points F_1 and F_2 is a constant k. To determine a universal analog of this, we consider the locus of a point X such that the quadrances $Q_1 = Q(F_1, X)$ and $Q_2 = Q(F_2, X)$, together with a number K, satisfy Archimedes formula $A(Q_1, Q_2, K) = 0$. This is the quadratic analog to the equation $d_1 + d_2 = k$, just as the Triple quad formula is the analog to a linear relation between three distances.

Such a locus we call a **quadrola**. This algebraic formulation applies to the relativistic geometries, and also extends to general fields. The notion captures both that of ellipse and hyperbola in the Euclidean setting, and while it is in general a different concept than a grammola, it is possible for a conic to be both, as is the case of an ellipse in Euclidean (blue) geometry.

The upper diagram in Figure 10 shows that in the red geometry, a new phenomenon occurs: our same ellipse as a quadrola has *two pairs* of foci $\{F_1, F_2\}$ and $\{G_1, G_2\}$. Each of these points is also a focus in the context of a conic section, and there are two pairs of corresponding directrices $\{d_1, d_2\}$ and $\{h_1, h_2\}$. Directrices are parallel or red perpendicular, and intersect at points on the ellipse, and

Figure 9: *Ellipse as red and green grammola*

tangents to these **directrix points** pass through two foci, forming a parallelogram which is both a *blue* and a *green rectangle*. It turns out that the red spreads between a tangent and lines to a pair of red foci are equal, as shown at points *A* and *B*.
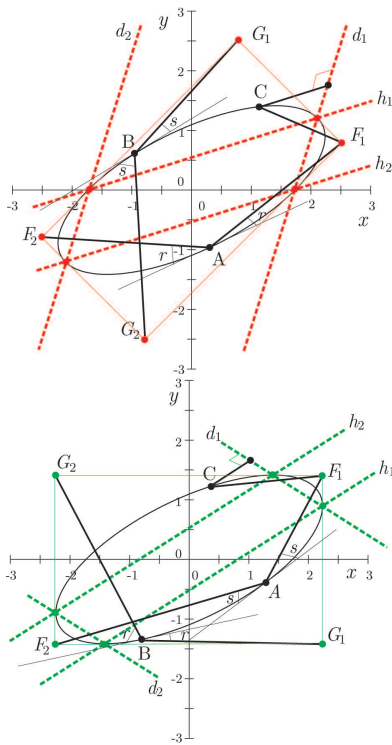

Figure 10: *Red and green foci and directrices*

The lower diagram in Figure 10 shows the same ellipse as a green quadrola, with again two pairs of green foci, two pairs of corresponding green directrices (which are parallel or green perpendicular), and the tangents at directrix points forming a blue and red rectangle.

The red and green directrix points are easy to find: they are the limits of the ellipse in the null and the coordinate directions. So the red and green directrices and foci are also then simple to locate geometrically. This is not the case for the usual (blue) foci and directrices, and suggests that considering ellipses from the relativistic perspectives can be practically useful. In algebraic geometry the 'other' pair of blue foci are not unknown; they require complexification and a projective view (see for example [1, Chapter 12]).

When we put all three coloured pictures together, as shown in Figure 11, another curious phenomenon appears—there are three pairs of coloured foci that appear to be close to the intersections of directrices of the opposite colour. The reason for this will become clearer later when we consider parabolas.
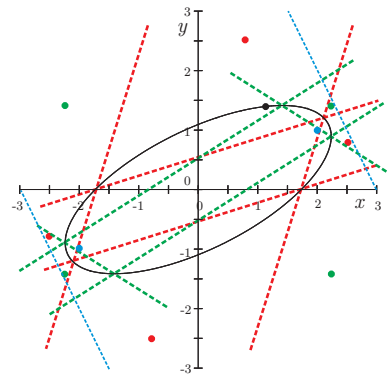

Figure 11: *Three sets of foci and directrices*

Hyperbolas

Over the real numbers some of what we saw with ellipses extends also to hyperbolas, although there are differences. The central hyperbola shown in Figure 12 with equation

$$7x^2 + 6xy - 17y^2 = 128$$

is a red quadrola with red foci $F_1 = [3,1]$ and $F_2 = [-3,-1]$, meaning that it is the locus of a point $X = [x,y]$ such that

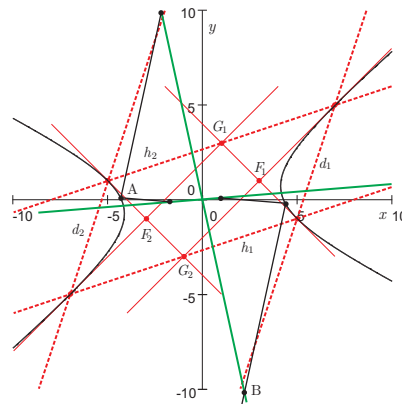$$A\left((x-3)^2 - (y-1)^2, (x+3)^2 - (y+1)^2, 64\right) = 0.$$

Figure 12: *A hyperbola as red quadrola and green grammola*

As a conic section the corresponding directrices have equations $3x - y - 16 = 0$ and $3x - y + 16 = 0$. This hyperbola also has another pair of red foci $G_1 = [1,3]$ and $G_2 = [-1,-3]$, with associated directrices $x - 3y + 8 = 0$ and $x - 3y - 8 = 0$.

As in the case of the ellipse we considered earlier, in each case the focus is the pole of the corresponding directrix,

meaning that it is the intersection of the tangents to the hyperbola at the directrix points. These tangents pass through two foci at a time, and are parallel to the red null directions. The parallelogram formed by the four foci is a blue and green rectangle.

So we could have found the red foci and directrices purely geometrically, by finding those points on the hyperbola where the tangents are parallel to the red null directions, and then forming intersections between these points. This is again quite different from finding the usual blue foci and directrices. Note that if we try to find green foci, vertical tangents are easy to find, but there are no horizontal tangents, thus the situation will necessarily be somewhat different.

Is the hyperbola also a grammola? It cannot be a blue grammola, since these are all ellipses, and it turns out not to be a red grammola either. But it *is* a green grammola with equation

$$\frac{\left(\left(119+8\sqrt{238}\right)x+51y\right)^2}{2\left(119+8\sqrt{238}\right)51} + \frac{\left(\left(119-8\sqrt{238}\right)x+51y\right)^2}{2\left(119-8\sqrt{238}\right)51}$$
$$= \frac{128}{3}.$$

The green diagonals are shown in Figure 12.

The parabola

From the viewpoint of universal (affine) geometry, the most interesting conic is the *parabola*. Given a point F and a generic line l not passing through F, the locus of a point X such that $Q(X,F) = Q(X,l)$ is what we usually call a parabola, independent of which geometry we are considering. The generic parabola has a distinguished blue, red and green focus, and also a blue, red and green directrix.
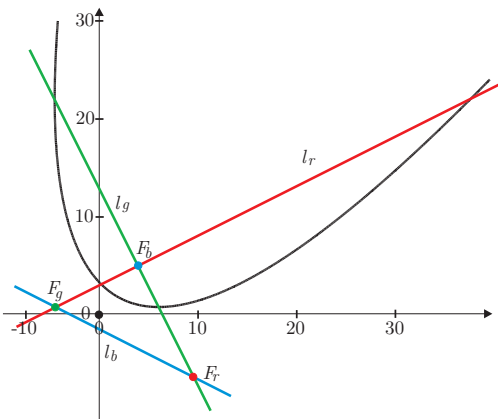

Figure 13: *Three foci and directrices of a parabola*

Figure 13 shows a parabola in the Cartesian plane and all three foci and directrices. A remarkable phenomenon appears: F_b is the intersection of l_r and l_g, F_r is the intersection of l_b and l_g, and F_g is the intersection of l_b and l_r. Furthermore l_r and l_g are blue perpendicular, l_b and l_g are red perpendicular, and l_b and l_r are green perpendicular— in other words we get a triple right triangle of foci. This means that once we know one of the focus/directrix pairs, the other two can be found simply by constructing the appropriate altitudes from the focus to the directrix together with their feet.
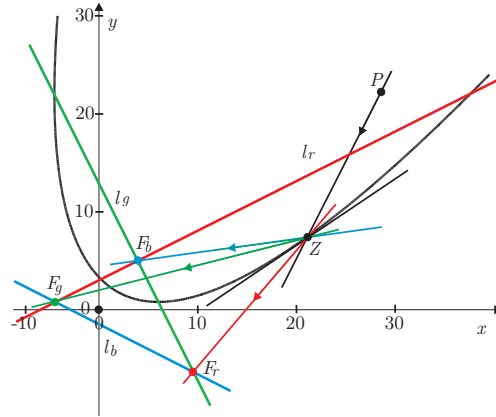

Figure 14: *Reflection properties of a parabola*

Although the various directrices are in different directions, the *axis* direction, defined as being perpendicular to the directrix, is common to all. Figure 14 shows the familiar reflection property of the parabola, where a particle P approaching the parabola along the axis direction and reflecting off the tangent (in either a blue, red or green fashion) always then passes through the corresponding focus.

The Figure 15 shows some interesting collinearities associated to a parabola, involving coloured vertices V (intersections of axes with the parabola), bases X (intersections of axes with directrices) and points Y formed by tangents to vertices.

Finally we show the three parabolas which have a given focus F and a given directrix l, both in black, each interpreted in one of the three geometries. Each of the three parabolas that share this focus and directrix have a focal triangle consisting of F and two of the feet of the altitudes from F to l, labelled F_b, F_r and F_g. The dotted line passes through the intersections of the red and green parabolas. Various vertices and axes are shown, and we leave the reader to notice interesting collinearities, and to try to prove them.
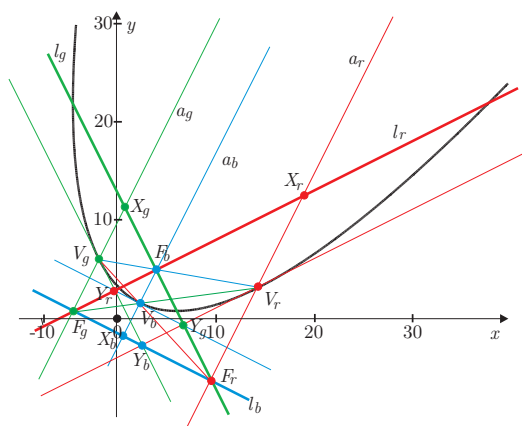
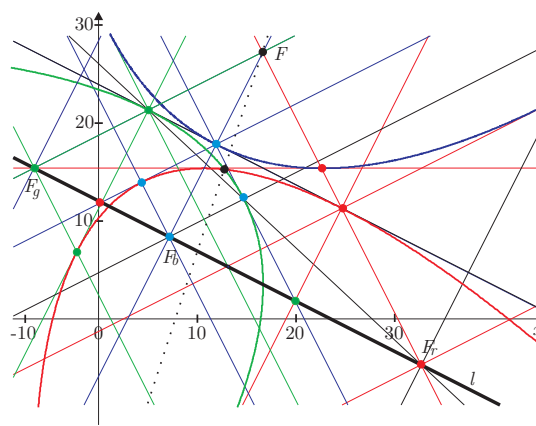Figure 15: *Collinearities for a chromatic parabola*

Figure 16: *Three parabolas with a common focus and directrix*

In conclusion, there may very well be other useful metrical definitions of conics; there are certainly still many rich discoveries to be made about these fascinating and most important geometric objects. Chromogeometry extends to many other aspects of planar geometry, for example to triangle geometry in [4].

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