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V-PERSPECTIVES, DIFFERENCES, PSEUDO-NATURAL NUMBER SYSTEMS AND PARTIAL ORDERS

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Abstract. In this paper, we generalise the notion of partial well-orderability and consider its relation to partial difference operations possibly definable. Results on these and generalised PWO–posets with systems of invariants for V–PWO posets are also formulated. These are relevant in partial algebras with differences and pseudonatural number systems for very generalised abstract model theory in particular.

1. Notations and terminology

For convenience the basic notations and terminology are presented below. Set will mean a set in ZFC unless stated otherwise. A subset T of a poset S is a *μ–subset* iff $\{x;\ \forall y \in S\ (x \leq y\ \vee\ x \parallel y)\} \subset T$ and $\forall x \in S$, $\exists y \in T$ $y \leq x$. A *minimal subset* T is a μ–subset which does not properly include any μ–subsets i.e. $\{x;\ \forall y \in S\ (x \leq y\ \vee\ x \parallel y)\} = T$ and $\forall x \in S\ \exists y \in T\ y \leq x$.

A poset $S = \langle S, \leq, (2)\rangle$ is *well founded* iff each nonempty subset has at least one minimal element. A *linear order* is a PO which satisfies $\forall x \forall y$ $x = y \vee x < y \vee y < x$. A *well ordered set* X is a linearly ordered set for which every nonempty set $Y \subseteq X$ has a least element, w.r.t. $<$.

A poset $S = \langle S, \leq\rangle$ is *partially well ordered* (PWO) iff every subset of S has a finite μ–subset (but not necessarily a minimal subset) iff for every infinite sequence (x_n) in S there exists i, j with $i < j$, $x_i \leq x_j$.

All PWO–posets are well founded but not conversely and the structure is so total that every infinite PWO poset S contains a chain C satisfying $\mathrm{card}(C) = \mathrm{card}(S)$. All posets contain at least one μ–subset but this is not so

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for minimal subsets. The above notions extend to quasi ordered sets (qosets) also.

Let $S = \langle S, \leq, -, (2,2)\rangle$ be a partial algebraic system with $\leq$ being a PO relation and $-$ being a binary partial operation satisfying $(x \leq y \rightarrow y - x \leq y)$; $(x \leq y \rightarrow y - (y - x) = x)$; $(x \leq y \leq z \rightarrow (z - y) \leq (z - x)$, $((z - x) - (z - y)) = (y - x))$, then S is called a *poset with difference*, (2,2) being the arities of the predicate and operation respectively. A *difference poset* is a poset with difference which includes two distinguished elements 0, 1 s.t. $\forall x\ 0 \leq x \leq 1$.

The *dimension* of a poset is the least cardinal k for which the partial order is an intersection of k linear orders on S. An *order ideal* S_1 of a poset S is a subset which satisfies $\forall y \in S_1\ (x \leq y \ \Rightarrow\ x \in S_1)$. The set of order ideals will be denoted by $\zeta(S)$. If both $\zeta(S)$ and S are PWO then S is called *normal*.

We use weak equalities in particular portions. Terms $t^{\underline{S}}$, $t'^{\underline{S}} \in \mathrm{Tm}_{\Sigma}(S)$, $t^{\underline{S}} \mapsto t'^{\underline{S}}$ will mean $\forall \overline{x} \in \mathrm{dom}(t^{\underline{S}} \subseteq \mathrm{dom}(t'^{\underline{S}})$, $t^{\underline{S}}(\overline{x}) = t'^{\underline{S}}$ and $'t^{\underline{S}} \stackrel{w^*}{=} t'^{\underline{S}}$ will mean $\forall \overline{x} \in \mathrm{dom}(t^{\underline{S}}) = \mathrm{dom}(t'^{\underline{S}})$, $t^{\underline{S}}(\overline{x}) = t'^{\underline{S}}(x)$. The usual weak equality $t^{\underline{S}} \stackrel{w}{=} t'^{\underline{S}}$ (iff $\forall \overline{x} \in \mathrm{dom}(t^{\underline{S}}) \cap \mathrm{dom}(t'^{\underline{S}})$, $t^{\underline{S}}(\overline{x}) = t'^{\underline{S}}(x))$ is also used.

2. V–PWO Posets with Difference Operations

One strong reason for introducing the notion of a V–perspective is that in many contexts the perspective can be properly related to the basis (predicative or otherwise) of existence of partiality in many contexts.

Definition 2.1. *Let* $\mathrm{Proj}\,(\cup_{\alpha \geq \omega} S^{\alpha}, S)$ *denote the set of all projection functions* $\cup_{\alpha \geq \omega} S^{\alpha} \rightarrow S$. *Then a V–perspective on a poset S is a subset V of the union* $\cup_{\alpha \geq \omega} S^{\alpha}$ *which satisfies* $\forall x \in V\ \exists e_{k'},\ e_{k'} \in \mathrm{Proj}\,(\cup_{\alpha \geq \omega} S^{\alpha}, S)$ $e_k x > e_{k'} x$, $k > k'$. *Posets with V–perspectives will be called V–PWO posets.*

For V–PWO posets, the notions of dimension, normality, order ideals and M-decompositions of a PWO set can be generalised/directly adapted. Apart from the examples obtainable from diverse fields, most types of V–PWO posets are explicity definable using set theoretic operations on PWO posets and posets.

Lemma 2.2. *Every PWO–poset is a* $(\cup_{\alpha \geq \omega} S^{\alpha})$*–PWO–poset.*

Definition 2.3. *A perspective V will be called separable (finitely) iff V is representable as a union (finite union) of sets of the form S_i^{α}, $\alpha \geq \omega$ and* $(i \neq j \rightarrow S_i \cap S_j = \emptyset)$.

Definition 2.4. *A perspective V will be called nearly separable iff V is representable as an extension of a separable perspective V_0 contained in V using the set theoretical operations (finite)* $\cap$, $\cup$, $\setminus$, Δ *alone on the elements of V_0.*

THEOREM 2.5. (i) *The different notions of separability are all distinct and finite separability implies separability.*
(ii) *If a separable perspective exists then it generates maximal nearly separable extensions.*

We formulate a principle which is particularly suitable for net based approaches and generalisations.

PERSPECTIVITY PRINCIPLE: On every poset S a unique maximal perspective V_m is definable relative to which S is a V_m–PWO poset. (Poset is replaceable with qoset in the principle.)

This principle is quite distinct from AC and WO principles since it is a sentence of a predicative nature given a PO and is closer to the maximality principle (HMP). It can be used to obtain interesting generalisations based on weakening over ZF and semiset theories. The equivalence of the perspectivity principle with HMP, AC and WO in ZFC is easy to prove.

THEOREM 2.6. *$PP \Leftrightarrow HMP \Leftrightarrow AC \Leftrightarrow WO$ in ZFC.*

REMARK 2.7. Theorem 2.5 (ii) can also be regarded as a foundational principle. This is not equivalent to transfinite induction.

DEFINITION 2.8. *If S and $\zeta(S)$ are V–PWO and T–PWO respectively with $T \prec 2^V$ then S will be called (V, T)–normal, $(V, 2^V)$–normality will be normality.*

The notions of PWO–posets with difference and V–PWO posets with difference will be direct extensions from posets with difference. Different structure theoretic results on these are proved in the next six results.

THEOREM 2.9. *A finite dimensional PWO–poset with difference is embeddable in a finite product of well–ordered sets with differences and conversely.*

PROOF. Let $S = \langle S, \leq, -, (2, 2)\rangle$ be a PWO–poset with difference and let its dimension be $n < \infty$. Consider the forgetful PWO poset $\widehat{S} = \langle \underline{S}, \leq, (2)\rangle$ of dimension n. There exist well order extensions $(T_i)_1^n$ of $\leq$ with $\leq = \cap T_i^n$ (for PWO every linear extension must be a well–order).

If $\langle H, \leq\rangle = \prod_1^n(\underline{S}, T_i)$ then defining $f_x : \{1, \ldots, n\} \to S$ for $x \in S$ via $f_x(i) = x$, $i = 1, \ldots, n$ it follows that $\langle \underline{S}, \leq\rangle$ is isomorphic to the subset $\{f_x,\ x \in S\} \subset H$. This allows the product representation of an extension preserving the difference. The converse is obvious.

The proof of the existence of a compatible order coherent extension depends on the existence of a linear extension for every PO on a set S and the WO principle. □

REMARK 2.10. Uniqueness is not ensurable.

THEOREM 2.11. *Theorem 2.9 is not true for V–PWO sets in general.*

THEOREM 2.12. *The order ideal of a PWO–poset with difference is also an order ideal with difference.*

PROOF. Let S_1 be an order ideal of the PWO–poset with difference $S = \langle \underline{S}, \leq, - \rangle$. Let $\forall\, y \in S_1\ (x \leq y \rightarrow x \in S_1) \equiv \Phi$. Then

$$x \leq y \rightarrow \exists\, z\ (y - x) = z \quad \text{and} \quad y - x \leq y,\ \Phi \rightarrow x \in S_1.$$

So the restriction of the difference from S to S_1 is also closed. The other conditions including $(a \leq b \rightarrow b - (b - a) = a)$; $(a \leq b \leq c \rightarrow c - b \leq c - a,\ (c-a)-(c-b) = b-a)$ are directly verifiable. S_1 is a closed subalgebraic system also. ∎

THEOREM 2.13. *The order ideal S_1 of a V–PWO–poset S with difference is also a $V_{|S_1}$–PWO set with difference ($V_{|S_1}$ being the set of infinite sequences over S_1 in V).*

PROOF. This is fairly in direct verification. S_1 is not necessarily a closed subalgebra but can be termed a $V_{|S_1}$–relative subalgebra. ∎

THEOREM 2.14. *Finite direct products of PWO–posets with difference $(S_k)_1^n$ are also PWO–posets with difference.*

THEOREM 2.15. *Transfinite products of normal PWO–posets with difference are also PWO–posets with difference.*

We consider the relation between V–PWO–posets and other difference–operation endowed partial algebraic systems in what follows.

THEOREM 2.16. (i) *A PWO–poset with difference is not necessarily a difference poset.*
(ii) *The order ideal of a normal PWO–difference poset is not necessarily a difference poset but is a generalised difference poset.*

PROOF. (ii) refers a counter example. This is provided by a forgetful countable/ finite MV–algebra S with difference operation defined by $(x \leq y \rightarrow y - x = (x + y^*)^*)$. As $\forall\, x \in S\ 0 \leq x \leq 1$, S is a difference poset. It is also a PWO–poset with $\zeta(S)$ being obviously a PWO–poset. Order ideals are however not difference posets but $\alpha \in \zeta(S) \ \Rightarrow\ 0 \in \alpha$ and $(a \leq b\ a \leq c,\ c - a = b - a \rightarrow b = c)$ are satisfied. ∎

Clones are partial algebras of the form $S = \langle \underline{S}, \oplus, 0, (2,0) \rangle$ which satisfy $a \oplus b \stackrel{w^*}{=} b \oplus a$, $(a \oplus b) \oplus c \mapsto a \oplus (b \oplus c)$; $(a \oplus b = a \oplus c \rightarrow b = c)$; $a \oplus 0 = a$; and $(a \oplus b = 0 \rightarrow a = b = 0)$ (cf [2] for example). *Generalised orthoalgebras* are clones satisfying $(a \oplus a = b \rightarrow b = 0)$.

THEOREM 2.17. *Finite dimensional PWO–posets with difference are all clones.*

Proof. It is proved in [2] in essence that the class of clones are categorically equivalent to the class of posets with cancellative difference $[(b - a = c - a \rightarrow b = c)$ and $(a - b = c \leftrightarrow a \oplus c = b)]$. It, therefore, suffices to prove the cancellativeness aspect.

Let L be an arbitrary finite dimensional PWO–poset with difference. Then

(i) L is normal and $\zeta(L)$ is a normal PWO set.

(ii) $L \cong F \subseteq \prod_1^n W_i$ where W_i are well ordered sets and $n < \infty$.

Since $-$ is a partial difference operation on L, its composition with projection functions e_i on restriction must also be partial difference operations. But each of these compositions restricted suitably determines a cancellative difference obviously. Let (K_i) be the sequence of subsets of W_i over which e_i- is inconsistent for the difference definition, then by (ii) (or equivalently as the PO is the intersection of n number of well orders on L), the only possible form of $x \in K_i$ is (a, a) but by the PWO all subsets have minimal elements, so K_i must be empty.

Cancellativeness and the other condition of $\oplus$–definition are consequences. $\square$

Remark 2.18. For difference posets in the context there is nothing to prove.

Theorem 2.19. *There exist normal PWO difference posets of finite dimension with complementation which are not generalised orthoalgebras or orthoalgebras.*

Proof. Generalised orthoalgebras are clones satisfying $(a \oplus a = b \rightarrow b = 0)$ and this need not hold in finite dimensional PWO–difference posets. $\square$

Theorem 2.20. *Every chain in a PWO–set with difference has a generalised poset structure and is necessarily complemented.*

Proof. Every chain in a PWO poset with difference has a minimal element as every subset must have a minimal subset [3, 4]. The complementation is easy. $\square$

A PO will be called *faintly linear* iff $\exists!\, o\, \forall x\ (o < x \vee x < o \vee x = o)$ while PO will be called *skew linear* iff $\exists\, o\, \forall x, y\ (o \leq x,\ o \leq y \rightarrow x \leq y$ or $y \leq x)$. Examples of such orders are abundant. The notions are related to positivity of partial orders w.r.t. binary operations.

Theorem 2.21. *There exist faintly linear PWO sets with difference which are not generalised difference posets.*

Proof. A counter example for the proposition can be based at J_2 as defined in [7]. Let $Y = X \cup J_2$; X being a set and $J_2 = \omega \times \omega$. If $a = (a_1, a_2)$,

$b = (b_1, b_2) \in J_2$ then $(a_1 = b_1 \implies a \leq b \overset{\Delta}{\longleftrightarrow} a_2 \leq b_2)$ and $(a_1 < b_1 \to a < b \overset{\Delta}{\longleftrightarrow} a_1 + a_2 \leq b_1)$. It sufficies to consider a two element X for ensuring that Y is not a difference poset and the proposition. ∎

Theorem 2.22. *Skew–linear finite dimensional upper bounded PWO sets are all endowable with orthoalgebra structure. The converse is not necessarily true.*

Proof. It suffices to show that $(a = b - a \to a = 0)$ is also defined/is true nontrivially in skew–linear finite dimensional PWO upper bounded posets whenever a generalised co-difference poset structure is defined. The existence of the minimal subset and skew linearity along with a contradiction argument is one strategy. ∎

Remark 2.23. In Theorems 2.21, 2.22, o is not necessarily a difference 0.

Remark 2.24. PWO is necessary for the definability.

3. Intervals, Convex Sets and V–PWO Posets

The structure of collections of intervals and convex intervals has been well–studied for lattices. Important extensions to posets have been obtained in [3, 4]. These include a classification of interval posets based on particular types of binary relations and the relation between posets with isomorphic convex interval collections. The implications of those results on PWO and V–PWO sets are naturally very relevant.

An *interval* in a poset S is a subset of the form $[a, b] = \{x;\ a \leq x \leq b\}$. A *convex interval* or a strict interval is an interval $[a, b]$ with $\forall\, x, y \in [a, b]$ $x \leq y$ or $y \leq x$. A *convex* set A is a subset for which $\forall\, x_1, x_2 \in A\ \forall x \in S (x_1 \leq x \leq x_2 \to x \in A)$. $\operatorname{Int} S$, $\mathrm{CINT}(S)$ and $\mathrm{CNV}(S)$ will respectively be the associated collections of sets of the type. Posets S_1, S_2 are convexly isomorphic iff $\mathrm{CNV}(S_1) \equiv \mathrm{CNV}(S_2)$.

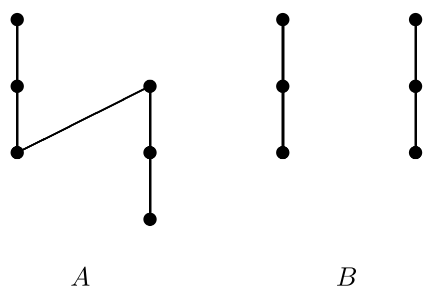
Theorem 3.1. *If $S = \langle \underline{S}, \leq \rangle$ is a poset, then the posets convexly isomorphic to S are just those, (up to isomorphism) obtainable by the successive application of the following three constructs*

1) $S_1 = \langle \underline{S}, \leq_1 \rangle$ *where* $x \leq_1 y$ *iff* $x \leq y$ *and* $(x, y) \notin P$ *for a subset* P *of* $\{(x, y);\ (x, y) \in S^2;\ x \prec y,\ x \in \mathrm{Min}(S),\ y \in \mathrm{Max}(S)\}$.
2) *Given* S_1, S_2 *is definable via* $S_2 = \langle \underline{S}, \leq_2 \rangle$, *where* $x \leq_2 y$ *iff* $[[x, y \in C,\ x \leq_1 y]$ *or* $[x, y \in D,\ y \leq_1 x]]$, *for a decomposition* $\underline{S} = C \cup D$ *of* $\underline{S}$ *with* $\forall\, c \in C,\ d \in D\ c \parallel_1 d$.
3) *Given* S_2, S_3 *is definable via* $S_3 = \langle \underline{S}, \leq_3 \rangle$, *where* $x \leq_3 y$ *iff* $x \leq_2 y$ *or* $(x, y) \in Q$, *for a subset* Q *of* $\{(x, y) \in S^2;\ x \parallel_2 y,\ x \in \mathrm{Min}(S_2),\ y \in \mathrm{Max}(S_2)\}$ *under* (α).

(α) $\forall\, u,\, v,\, w \in Q \sim [(u,v) \in Q;\ (v,w) \in Q]$. *So if posets* $A = \langle \underline{A}, \leq \rangle$ *and* $B = \langle \underline{B}, \leq^* \rangle$ *are convexly isomorphic then there exists a poset* $A' = \langle \underline{A}, \leq \rangle$ *isomorphic to* B *s.t.* $\mathrm{CNV}(A) = \mathrm{CNV}(A')$.

Proposition 3.2. *If a finite dimensional PWO–poset* A *is convexly isomorphic to a finite dimensional PWO–poset* B*, it does not necessarily follow that* B *is isomorphic to* A *or its dual.*

Proof. Consider, the figure below, A, B are convexly isomorphic finite dimensional PWO–posets but B is not isomorphic to A or its dual.



∎

Remark 3.3. In Proposition 3.2 the posets can also be endowed with difference operations. The constructs of Theorem 3.1 are also interesting from the view point of modification of difference operations (especially in the sense of internalised valuation).

Proposition 3.4. (a) *In the context of Theorem 3.1, if* $-$ *is a difference operation on* S*, then the restricted difference operation* $-_1$ *on* S_1 *is obtainable from* $-$ *via* $a -_1 b = x$*, iff* $a - b = x$ *and* $(b,a) \notin P$.

(b) *In the context of Theorem 3.1 if* $-$ *is a difference operation on* S *and if the second construction is directly applied on* S*, then a new difference operation* $-_2$ *is definable on* S_2 *via* $x -_2 y = b$ *iff* $\{x, y \in C,\ x - y = b\}$ *or* $\{x, y \in D,\ y - x = b\}$.

(c) *In the context of Theorem 3.1 3), if* $-$ *is a difference operation on* S *and if the* $\leq_3$ *definition is interpreted relative* $\leq$ *itself then a set of* $\leq_3$ *"extensions* $-'$*" of* $-$ *are definable under,* $x \leq y$ $y - x = z \to x \leq_3 y$*,* $y -' x = z$. $(x,y) \in Q \to x \leq_3 y$*,* $y -' x$ *is definable. There is at least one nontrivial extension within* S.

Proof. The proofs consist in verification and are not difficult. ∎

In general some strong connections between the nature of a V–PWO, PWO–poset and their set of convex subsets are expectable. A study of such connections under different conditions including cardinality is of interest.

The distribution of intervals and convex intervals in a PWO or V–PWO–posets are relatively more easily determined under normality or finite dimensionality. The classificatory theorem proved in [3, 4] becomes simpler for finite dimensional PWO and V–PWO posets (when V is an union of intervals or maximal intervals).

Let U, V be tolerances (reflexive and symmetrical relations) on a poset S, under

(P1) $U, V \subseteq \{(x,y) \in S \times S; \sim (x \parallel y)\}$;
(P2) $\forall\, x, y\ (x \leq y \rightarrow \exists!\, p, q \in [x,y],\ pVxUqVyUp)$;
(P3) $\forall\, x, y, u\ (u \leq x, y,\ xVuUy \rightarrow u = \inf\{x,y\},\ \exists\, v = \mathrm{Sup}\{x,y\} yVvUx))$;
(P3') $\forall\, x, y, v\ (x, y \leq v,\ yVvUx \rightarrow v = \mathrm{Sup}\{x,y\}\ \exists\, u,\ u = \inf\{x,y\},\ xVuUy)$;
(P4) $a = a_1 U a_2 U \ldots U a_n = a',\ a = aVa'_2V \ldots Va'_m = a' \rightarrow a = a';\ n, m \in \mathbf{N}$;
(P5) $\forall\, a, a' \in S, \exists\, n, m \in \mathbf{N}, \exists\, a_1 \ldots a_n, a'_1 \ldots a'_m \in S\ a = a_1 U a_2 U \ldots U a_n = a'_1 V a'_2 V \ldots V a'_m = a'$.

Then

THEOREM 3.5. (i) *Let S be a connected poset. Then there exists a mapping ϕ of the system of all couples of relations U, V on S under (P1)–(P3) onto the system of all isomorphism classes of posets B with* $\mathrm{Int}\, B \cong \mathrm{Int}\, S$. *If (U,V) satisfies (P1)–(P5), then $\phi(U,V)$ consists of all posets isomorphic to $S_1^\delta \times S_2$ for a direct decomposition $S_1 \times S_2$ of S. Conversely the class of all posets isomorphic to $S_1^\delta \times S_2$ for a direct decomposition $S_1 \times S_2$ of S is $\phi(U,V)$ for some (U,V) under (P1)–(P5).*

(ii) *If S is a directed poset, and B a poset with* $\mathrm{Int}\, S \cong \mathrm{Int}\, B$ *then there exist posets C, D with $S \cong C \times D$ and $B \cong C^S \times D$. Given S, B as above the converse is also true.*

THEOREM 3.6. *Let S_1, S_2 be two V–PWO posets (when V– is a union of powers of covering maximal intervals) with* $\mathrm{Int}\, S_1 \cong \mathrm{Int}\, S_2$, *then there exist posets $C, D, S_1 \cong C \times D$ and $S_2 \cong C^D \times D$. The union of the maximal intervals is S_1.*

THEOREM 3.7. *In Theorem 3.6, V–PWO is replaceable by finite dimensional posets.*

PROOF OF THEOREMS 3.6, 3.7. In both cases it suffices to take the base sets to be the same $\underline{S}$ and the orders as $\leq_1$, $\leq_2$, S_1, S_2 are decomposable into maximal connected sets $(S_{1\delta})$ and $(S_{2\delta})_{\delta \in D}$ with, $\mathrm{Int}\, S_{1\delta} = \mathrm{Int}\, S_{2\delta}$ necessarily. Applying Theorem 3.5 to these $S_{1\delta}$, $S_{2\delta}$ pairs, it remains to prove the reconstructibility of C, D which is possible in both the contexts. $\square$

Definition 3.8. *A PWO–interval will be an interval, partially well–ordered as a poset. The set of all PWO–intervals of a poset S will be* $\mathrm{PWI}(S)$. *A Co–μ–subset X of a poset Y is a subset satisfying* $\{x;\ \forall\, y \in S;\ y \leq x \text{ or } x \parallel y\} \subset Y$ *and* $\forall\, y \in S\ \exists \in X\ y \leq x$.

Clearly,

Proposition 3.9. (a) $\mathrm{PWI}(S) \subset \mathrm{Int}(S)$. $\mathrm{CNV}(S) \cap \mathrm{PWI}(S) \subset \mathrm{CINT}(S)$.

(b) *If a subcollection $\xi \prec \mathrm{PWI}(S)$ is s.t. $\cup\xi = S$ then S is a PWO–poset.*

(c) *If every subinterval of an element of* $\mathrm{PWI}(S)$ *is also in* $\mathrm{PWI}(S)$ *then* $\mathrm{PWI}(S)$ *is endowable with a partial lattice structure, otherwise it is a poset in general. In particular when the Co–μ–subsets of* $\mathrm{PWI}(S)$ *are normal,* $\mathrm{PWI}(S)$ *has a partial lattice structure.*

The notion of isomorphism determined by proposition 3.9 allows the possible equivalence $\mathrm{PWI}(S_1) \cong \mathrm{PWI}(S_2)$ between two posets. A problem is the characterisation of S_1 and S_2 when such an equivalence is true.

4. Generalised Closure Operators, Invariant Systems

In general posets can be characterised up to different desired levels by different sets of invariants. These include the dimension, height, cardinalities of maximal antichains, invariants associated with order ideals and collections of intervals and invariants related to different types of denseness among others. For PWO–posets and V-PWO–posets (with/without differences) most of these are relevant, V-PWO–posets are naturally more difficult to characterise via invariants. A modified set of partial invariants are developed below. These are partial in the characterisation of V–perspectives and also so from the dual semantic (preservation by special morphism) point of view.

In all that follows $S = \langle \underline{S}, \leq, -, (2,2)\rangle$ will be a V-PWO–poset with partial difference operation. Four different generalised closure operators are initially defined. These also lead to corresponding notions of simpler types of PWO–posets. The proper invariant system for a V-PWO–poset must correctly be considered contextually, but the fragment developed below is almost always useful.

The first operator CW is motivated by the connections with order ideals.

Definition 4.1. *A subset $\mathcal{H} \subseteq \zeta(S)$ will be called relevant for S_0 ($\underline{S}_0 \subset \underline{S}$) iff*

(i) $\exists\, x_0 \in \mathcal{H}\ \forall\, x \in \mathcal{H}\ x \subseteq x_0$,

(ii) $\forall\, x \in \mathcal{H}\ S_0 \subseteq x$,

(iii) $\forall\, \mathcal{H},\ \mathcal{H}' \subseteq \zeta(S)\ (\mathcal{H} \subset \mathcal{H}' \rightarrow \exists\, x_0 \in \mathcal{H}'\ \forall\, x' \in \mathcal{H}'\ x' \subseteq x_0' \neq S$, $\cap\mathcal{H}' = \cap\mathcal{H})$.

Definition 4.2. CW : $P(S) \to \zeta(S)$ *will be an operator s.t.*

$$\mathrm{CX}(S_0) = \begin{cases} \cap\mathcal{H}, & \text{if all relevant subcollections for } S_0 \\ & \text{have identical intersection.} \\ \cap\alpha, & \text{otherwise } (S_u \subseteq \alpha \text{ and } \alpha \in \zeta(S)). \end{cases}$$

Proposition 4.3. *In the contexts of Definitions 4.1, 4.2, the statements* (i)–(iv) *hold*

(i) $\mathrm{CW}(\phi) = \phi$; $\mathrm{CW}(S) = S$,
(ii) $\mathrm{CW}(\mathrm{CW}(S_0)) = S_0$,
(iii) $S_0 \subseteq \mathrm{CW}(S_0)$,
(iv) $S_0 \subseteq S_0' \subseteq \mathrm{CW}(S_0) \to \mathrm{CW}(S_0) \subseteq \mathrm{CW}(S_0')$.

Proof. The verification of (i)–(iii) is obvious, (iv) is a case by case verification. ∎

Remark 4.4. CW does not satisfy monotony in general. In a weaker setting this operator has been used by the present author to obtain a concrete representation theorem for nonmonotonic consequence operators satisfying inclusion, idempotence and cautious monotony.

Definition 4.5. *Let S be a difference poset, then the operators* $\mathrm{CT}_c, \mathrm{CF}, \mathrm{CT} : 2^S \to 2^S$ *will be called C–top closure, F–closure and top closure respectively, whenever they satisfy*

(i) $\forall S_1 \in 2^S\ \mathrm{CF}(S_1) = \{1 - x;\ x \in S_1\} \cup \underline{S}_1$ *with the induced difference operation;*
(ii) $\forall S_1 \in 2^S\ \mathrm{CT}_c(S_1)$ *is the least closed forgetful subalgebra (w.r.t. $-$) containing* $\underline{S}_1 \cup \{1\}$;
(iii) $\forall S_1 \in 2^S\ \mathrm{CT}(S_1) = \underline{S}_1 \cup \{1\}$ *with the induced difference operation.*

Remark 4.6. Definition 4.5 is extendable to posets with difference by adjoining a top element 1 if not present under $\forall x\ x \leq 1$ or $x \parallel 1$ and suitably extending the difference operation on $\underline{S}$ to $\underline{S} \cup \{1\}$.

Proposition 4.7. *If S is a difference poset and* CT_c, CF, CT *c–top closure, F–closure and top closure operators on it respectively, then the statements* (i)–(ix) *are satisfied in S.*

(i) $\mathrm{CT}_c(\phi) \neq \phi$; $\mathrm{CT}_c(S) = S$; $S_1 \subset \mathrm{CT}_c(S_1)$,
(ii) $\mathrm{CT}_c\,\mathrm{CT}_c(S_1) = \mathrm{CT}_c(S_1)$,
(iii) $S_1 \subseteq S_2 \to \mathrm{CT}_c(S_1) \subseteq \mathrm{CT}_c(S_2)$,
(iv) $\mathrm{CF}(\phi) \neq \phi$, $\mathrm{CF}(S) = S$, $S_1 \subseteq \mathrm{CT}_c(S_1)$,
(v) $\mathrm{CF}\,\mathrm{CF}(S_1) \supseteq \mathrm{CF}(S_1)$,
(vi) $S_1 \subseteq S_2 \to \mathrm{CF}(S_1) \subseteq \mathrm{CF}(S_2)$,
(vii) $\mathrm{CT}(\phi) = 1$, $\mathrm{CT}(S) = S$, $S_1 \subseteq \mathrm{CT}(S_1)$
(viii) $\mathrm{CT}\,\mathrm{CT}(S_1) = \mathrm{CT}(S_1)$,
(ix) $S_1 \subseteq S_2 \to \mathrm{CT}(S_1) \subseteq \mathrm{CT}(S_2)$.

PROPOSITION 4.8. *If S is a difference V–PWO–poset and g is any one of the four operators defined above, then the relation ρ_g defined via $(x, y) \in \rho_g$ iff $g(x) = g(y)$ is an equivalence on the power set 2^S.*

CT_c, CF and CT operators allow interesting notions of generalised simplicity in PWO–posets. These posets have better structural properties.

DEFINITION 4.9. *A difference PWO–poset S will be called CT_c– simple (respectively CF; CT) as $\forall\, \alpha \in \zeta(S) \setminus \{\phi\}$ $\mathrm{CT}_c(\alpha) = S$ (respectively $\mathrm{CF}(\alpha) = S$; $\mathrm{CT}(\alpha) = S$) is satisfied in S.*

THEOREM 4.10. *For difference PWO–posets CT–simplicity implies CT_c–simplicity while CF–simplicity implies CT_c–simplicity.*

PROOF. Let $\alpha \in \zeta(S) \setminus \{\phi\}$, then $\mathrm{CT}(\alpha) = \alpha \cup \{1\}$ with the induced difference operation, so that the underlying set in α is essentially $S \setminus \{1\}$. Clearly this α satisfies that $\mathrm{CT}_c(\alpha)$ is the closed algebraic closure of $\underline{\alpha} \cup \{1\}$, which is S. The converse obviously fails.

$\mathrm{CF}(\alpha) = S$ implies $\{1 - x;\ x \in \alpha\} \cup \underline{\alpha}$ when endowed with the difference operation on S. This yields $\mathrm{CT}_c(\alpha) = S$ as $\mathrm{CT}_c(\alpha)$ is the closed algebraic closure of $\{1-x;\ x \in \alpha\} \cup \underline{\alpha} \cup \{1\}$, which must coincide with S. Counterexamples for the failure of the converse are easy. □

THEOREM 4.11. *If g is one of CT_c, CT or CF, then closed (isotone) morphic images of g–simple difference PWO–posets are g–simple.*

Some representation theory based at g–simple difference PWO–posets are possible [5]. The following conjecture appears possible.

CONJECTURE 4.12. *All normal g–simple difference PWO–posets are finite-dimensional.*

DEFINITION 4.13. *The PWO–type $L = \langle \underline{L}, \wedge, \vee, \varphi, \theta, (2, 2, 1, 0)\rangle$ of a poset S will be a lower, complete partial lattice endowed with a partial unary operation φ s.t.*

(i) *$\underline{L}$ is a bijective image of 2^S (i.e. $\underline{L}$ is forgetfully isomorphic to 2^S in the category of sets).*
(ii) *$\varphi x = x$ iff x is a PWO–poset, else φ is undefined.*
(iii) *If x' denotes the natural bijective image of x in 2^S, then for $x, y, a \in 2^S$*
$x \cap y = a,\ \varphi^{\underline{L}}(a') = a' \ \longleftrightarrow\ x' \wedge y' = a'$.
$x \cup y = a,\ \varphi^{\underline{L}}(x') = x',\ \varphi^{\underline{L}}(y') = y',\ \varphi^{\underline{L}}(a') = a' \ \longleftrightarrow\ x' \vee y' = a'$.
(iv) $x \wedge y \overset{w^*}{=} y \wedge x$; $x \vee y \overset{w^*}{=} y \vee x$.
$x \wedge (y \wedge z) \overset{w^*}{=} (x \wedge y) \wedge z$; $x \vee (y \vee z) \overset{w^*}{=} (x \vee y) \vee z$.
$x \wedge 0 = 0$, $x \vee 0 = x$, $x \wedge (y \vee z) \overset{w^*}{=} (x \vee y) \wedge (x \vee z)$.
$(\varphi(x \vee y) = \varphi x \vee \varphi y = x \vee y \to \varphi(x \wedge y) = \varphi x \wedge \varphi y = x \wedge y)$.

REMARK 4.14. $\wedge$, $\vee$ are restrictions of $\cap$, $\cup$ in 2^S. Partial complementations can be induced on L by the complementation c on 2^S. But widely different abstractions including set–valued partial–complementation (poly complementations) are possible.

At least two cases of embeddability of one PWO–type in another are of interest.

PROBLEM 4.1. *Let L_1, L_2 be two PWO–types with antichains of Co μ–subsets T_1, T_2 respectively. If $\mathrm{Card}\, T_1 = \mathrm{Card}\, T_2$, find necessary and sufficient conditions for L_1 to be embeddable in L_2. Consider also the case without the restriction.*

THEOREM 4.15. *Two posets S_1, S_2 with PWO perspectives V_1, V_2 respectively, with isomorphic PWO–poset types need not necessarily be isomorphic to each other even if $\mathrm{Card}\, S_1 = \mathrm{Card}\, S_2$ and $\mathrm{Card}\, V_1 = \mathrm{Card}\, V_2$.*

PROOF. Counterexamples are easy. $\square$

For posets, associatable invariants include the dimension, rank, collections of intervals, collections of convex intervals, lattice of antichains, cardinalities of sets of atoms and coatoms, invariants derivable via CW, PWO–types and height. For PWO–posets all these remain applicable, but it suffices to restrict to a smaller subcollection, but the type of decomposition into partial ordinals becomes useful. For V–PWO–posets too all these are applicable. All these invariants do not strongly relate to products on the underlying set. This generally results in irregular characterisation of V from the mentioned invariants, even when as many as five of them are specified. In the context of V–PWO–posets these will be therefore be referred to as partial invariants. For difference V–PWO–posets, it is necessary to make use of CT_c, CT and CF also.

Based on the nature of sets of invariants we can classify them into cardinal, gross and restricted invariants. These will respectively correspond to the component invariants being cardinal numbers, cardinals and structures and restricted versions thereof. An example of a cardinal invariant system is $\langle \mathrm{Card}(S), \dim(S), \mathrm{ht}(S), \mathrm{Card}(\mathrm{At}(S)), \mathrm{Card}(\mathrm{mac}(S)), \mathrm{Card}(\mathrm{cat}(S))\rangle$ where $\mathrm{At}(S)$, $\mathrm{ht}(S)$, $\mathrm{mac}(S)$ and $\mathrm{cat}(S)$ correspond respectively to the set of atoms, height, set of maximal antichain and set of coatoms of S.

If we include $\mathrm{Int}(S)$ and $\mathrm{PWI}(S)$ in the above we have an example of a gross invariant system. But if we use a forgetful version of $\mathrm{Int}(S)$ then we have a restricted invariant system.

Interesting partial invariant systems for difference V–PWO–posets include $\langle \mathrm{Card}(S), \mathrm{CF}(S), \mathrm{PWO}(S), \mathrm{C}\widehat{\mathrm{T}}_c, \mathrm{C}\widehat{\mathrm{W}}, \mathrm{C}\widehat{\mathrm{T}}, \mathrm{C}\widehat{\mathrm{F}}\rangle$, $\langle \mathrm{PWO}(S), \mathrm{C}\widehat{\mathrm{T}}_c, \mathrm{C}\widehat{\mathrm{W}}, \mathrm{C}\widehat{\mathrm{T}}, \mathrm{C}\widehat{\mathrm{F}}, \mathrm{CNV}(S)\rangle$ and $\langle \dim(S), \mathrm{At}(S), \mathrm{CNV}(S), \mathrm{PWI}(S), \mathrm{PWO}(S), \mathrm{C}\widehat{\mathrm{W}}, \mathrm{mac}(S)\rangle$ ($\mathrm{C}\widehat{\mathrm{T}}, \mathrm{C}\widehat{\mathrm{W}}$ and other operators with symbols $\widehat{\ }$ mean the associated collections of closed sets in 2^S). Some gross invariant systems have been considered for difference V–PWO–posets in [5] by the present author.

The results obtained therein have connections with partial ordinals. Further work in the above are naturally motivated. The best forms of invariants for the context are apparently those which use special products.

PROBLEM 4.2. *Let S be a V–difference PWO–poset. Find generalised product processes $\mathcal{H}$ for which special products of the form $S^\omega|\mathcal{H}$ coincide with the perspective V.*

CONCLUSION. In this original research paper, we have generalised the notion of PWO to V–PWO, considered the relation between PWO and difference operations, formulated notions of invariants, considered the relation with the different types of intervals and have proved interesting results on all of them. We continue with different applications and extensions in a subsequent paper.

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