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ON A CHERN-TYPE PROBLEM FOR SPACE-LIKE KAEHLER SUBMANIFOLDS

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Abstract. The purpose of this paper is to investigate a Chern type problem for *space-like* Kaehler submanifolds M in indefinite complex hyperbolic space $CH_p^{n+p}(c)$, $c < 0$. Accordingly, we give better estimations of the squared norm of the second fundamental form of M in $CH_p^{n+p}(c)$, $c < 0$.

1. Introduction

Let M be an n-dimensional complex submanifold of $(n+p)$-dimensional complex space form $M^{n+p}(c)$. Then in this family of submanifolds Chern [4], Chern, do Carmo and Kobayashi [5] pointed out that it is of interest to study the distribution of the value of squared norm $|\alpha|_2$ of the second fundamental form α of M as follows:

> *Problem. Let M be an n-dimensional complete Kaehler submanifold of an $(n+p)$-dimensional complex space form $M^{n+p}(c)$ of constant holomorphic curvature $c(< 0)$. Then does there exist a constant h in such a way that if it satisfies $h_2 > h$, then M is totally geodesic ?*

It is known that a complete space-like complex submanifold of an indefinite complex space form $M_p^{n+p}(c), c\geq 0$, $p\geq 1$, is totally geodesic in [1] and [2]. However, in the case $c < 0$, it was known that there exist many complete Einstein Kaehler space-like submanifolds in the indefinite complex hyperbolic space $CH_p^{n+p}(c)$, $c < 0$, $p\geq 1$, which are not totally geodesic (See [1, 8, 11]).

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From this point of view, for the case where $c < 0$ we have studied in [1] the classification problem of *space-like* complex submanifolds of $CH_p^{n+p}(c)$ with bounded scalar curvature.

On the other hand, Goldberg and Kobayashi [6] and Houh [7] (*resp.* Barros and Romero [3], and Montiel and Romero [9]) introduced the notion of totally real bisectional curvature on Kaehler (*resp.* indefinite Kaehler) manifolds. Such a curvature is so much closely related to the scalar curvature given in section 3. Then naturally, motivated by the result in [1] concerned with the scalar curvature, we also have investigated the classification problem with bounded totally real bisectional curvature in [6]. That is, we proved the following

Theorem A. *Let M be an $n(\geq 3)$-dimensional complete complex submanifold of $CH_p^{n+p}(c)$, $p > 0$, with totally real bisectional curvature $\geq b$. Then the following holds*

1) *b is smaller than or equal to $\frac{c}{4}$.*
2) *If $b = \frac{c}{4}$, then M is a complex space form $CH^n(\frac{c}{2})$, $p \geq \frac{n(n+1)}{2}$.*
3) *If $b = \frac{n(n+p+1)c}{2(n+2p)(n+1)}$, then M is a complex space form $CH^n(\frac{c}{2})$, $p = \frac{n(n+1)}{2}$.*

On the other hand, it is seen in Aiyama, Nakagawa and the present author [1] and Ki and the present author [8] that the squared norm

$$h_2 = |\alpha|_2 = -\sum_{i,j} h_{ij}^x \bar{h}_{ij}^x$$

of the second fundamental form α of M in $CH_p^{n+p}(c)$ satisfies

$$0 \geq |\alpha|_2 \geq n(n+1)\frac{c}{4} \tag{1.1}$$

the latter equality arising only when M is a complex space form of constant holomorphic sectional curvature $\frac{c}{2}$. However, by estimating the Laplacian of h_2, that is, $\triangle h_2$, we have obtained the same result as in Theorem A with bounded scalar curvature or with bounded totally real bisectional curvature, respectively.

Now in this paper let us investigate the above estimations of $h_2 = |\alpha|_2$, that is, a Chern type problem for *space-like* complex submanifolds M in $CH_p^{n+p}(c)$; more explicitly, for this we will estimate the Laplacian of the squared norm h_4, $h_4 = \sum_j \mu_j^2$, where μ_j denotes an eigenvalue of the Hermitian matrix $H = (h_{i\bar{j}}^2)$, which is given by $h_{i\bar{j}}^2 = -\sum_{x,k} h_{ik}^x \bar{h}_{kj}^x$. Here we are able to give better estimations than (1.1).

Now let us denote by $a(M)$ the infimum of *totally real bisectional curvatures* of M in $CH_p^{n+p}(c)$. Then we assert the following

THEOREM 1.1. *Let M be an $n = 3$ or $n = 4$-dimensional complete space-like complex submanifold of an $(n + p)$-dimensional indefinite complex hyperbolic space $CH_p^{n+p}(c)$ of constant holomorphic sectional curvature $c(> 0)$ and of index $2p(> 0)$. Then there are constants $a = a(n, p, c)$ and $h = h(n, p, a(M), c)$, $c < 0$, so that if $a(M) \geq a$ and the squared norm $|\alpha|_2 = -\sum_{x,i,j} h_{ij}^x \bar{h}_{ij}^x$ of the second fundamental form α of M satisfies $|\alpha|_2 \geq h$, then M is totally geodesic.*

THEOREM 1.2. *Let M be an $n \geq 5$-dimensional complete space-like complex submanifold of an $(n + p)$-dimensional indefinite complex hyperbolic space $CH_p^{n+p}(c)$ of constant holomorphc sectional curvature $c(< 0)$ and of index $2p(> 0)$ and $p \leq \frac{3(n^2-1)}{n^2-4n-2}$. Then there exists a constant $a = a(n, p, c)$ and a negative constant $h = h(n, p, a(M), c)$ so that if $a(M) \geq a$ and $|\alpha|_2 \geq h$, then M is totally geodesic.*

2. Local formulas

This section is concerned with recalling basic formulas on semi-definite Kaehler manifolds. Let M be a complex $m(\geq 2)$-dimensional semi-definite Kaehler manifold equipped with semi-definite Kaehler metric tensor g and almost complex structure J. For the semi-definite Kaehler structure $\{g, J\}$, it follows that J is integrable and the index of g is even, say $2t$ $(0 \leq t \leq m)$. In the case where t is contained in the range $0 < t < m$, M is called an *indefinite Kaehler manifold* and the structure $\{g, J\}$ is called an *indefinite Kaehler structure* and in particular, in the case where $t = 0$ or m, M is only called a *Kaehler manifold*, and then the structure $\{g, J\}$ is called a *Kaehler structure*.

Now we choose a local field

$$\{E_\alpha\} = \{E_A, E_{A^*}\} = \{E_1, \ldots, E_m, E_{1^*}, \ldots, E_{m^*}\}$$

of orthonormal frames on a neighborhood of M, where $E_{A^*} = JE_A$ and $A^* = m + A$. Here the indices $A, B, \ldots$ run from 1 to m and the indices $\alpha, \beta, \ldots$ run from 1 to $2m = m^*$. We set $U_A = (E_A - iE_{A^*})/\sqrt{2}$ and $\bar{U}_A = (E_A + iE_{A^*})/\sqrt{2}$, where i denotes the imaginary unit. Then $\{U_A\}$ constitutes a local field of unitary frames on the neighborhood of M. This is a complex linear frame which is orthonormal with respect to the semi-definite Kaehler metric, that is, $g(U_A, \bar{U}_B) = \varepsilon_A \delta_{AB}$, where

$$\varepsilon_A = 1 \text{ or } -1, \text{ according as } 1 \leq A \leq m - t \text{ or } m - t + 1 \leq A \leq m.$$

Let $\{\theta_\alpha\} = \{\theta_A, \theta_{A^*}\}$, $\{\theta_{\alpha\beta}\}$ and $\{\Theta_{\alpha\beta}\}$ be the canonical form, the connection form and the curvature form on M respectively, with respect to the local field $\{E_\alpha\} = \{E_A, E_{A^*}\}$ of orthonormal frames. Then we have the structure equations

$$
\begin{aligned}
& d\theta_\alpha + \sum_\beta \varepsilon_\beta \theta_{\alpha\beta} \wedge \theta_\beta = 0, \quad \theta_{\alpha\beta} - \theta_{\alpha^*\beta^*} = 0, \\
& \theta_{\alpha^*\beta} + \theta_{\alpha\beta^*} = 0, \quad \theta_{\alpha\beta} + \theta_{\beta\alpha} = 0, \quad \theta_{\alpha\beta^*} - \theta_{\beta\alpha^*} = 0, \\
& d\theta_{\alpha\beta} + \sum_\gamma \varepsilon_\gamma \theta_{\alpha\gamma} \wedge \theta_{\gamma\beta} = \Theta_{\alpha\beta}, \quad \Theta_{\alpha\beta} = -\frac{1}{2} \sum_{\gamma,\delta} \varepsilon_\gamma \varepsilon_\delta K_{\alpha\beta\gamma\delta} \theta_\gamma \wedge \theta_\delta,
\end{aligned} \tag{2.1}
$$

where $K_{\alpha\beta\gamma\delta}$ denotes the components the Riemannian curvature tensor R of M.

Now, let $\{\omega_A\}$ be the dual coframe field with respect to the local field $\{U_A\}$ of unitary frames on the neighborhood of M given by

$$\omega_A = (\theta_A + i\theta_{A^*})/\sqrt{2}.$$

Then $\{\omega_A\} = \{\omega_1, \dots, \omega_m\}$ consists of complex-valued 1-forms of type $(1,0)$ on M such that $\omega_A(U_B) = \varepsilon_A \delta_{AB}$ and $\{\omega_A, \bar\omega_A\} = \{\omega_1, \dots, \omega_m, \bar\omega_1, \dots, \bar\omega_m\}$ are linearly independent. The semi-definite Kaehler metric g of M can be expressed as $g = 2\sum_A \varepsilon_A \omega_A \otimes \bar\omega_A$. Associated with the frame field $\{U_A\}$, there exist complex-valued forms ω_{AB} given by

$$\omega_{AB} = \theta_{AB} + i\theta_{A^*B},$$

which are usually called *connection forms* on M such that they satisfy the structure equations of M ;

$$
\begin{aligned}
& d\omega_A + \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B = 0, \quad \omega_{AB} + \bar\omega_{BA} = 0, \\
& d\omega_{AB} + \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} = \Omega_{AB}, \\
& \Omega_{AB} = \sum_{C,D} \varepsilon_C \varepsilon_D R_{\bar A B C \bar D} \omega_C \wedge \bar\omega_D,
\end{aligned} \tag{2.2}
$$

where $\Omega = (\Omega_{AB})$, $\Omega_{AB} = \Theta_{AB} + i\Theta_{A^*B}$ (resp. $R_{\bar ABC\bar D}$) denotes the curvature form (resp. the components of the semi-definite Riemannian curvature tensor R) of M. So, by (2.1) and (2.2) we obtain

$$R_{\bar ABC\bar D} = -\{(K_{ABCD} + K_{A^*BC^*D}) + i(K_{A^*BCD} - K_{ABC^*D})\}. \tag{2.3}$$

Let M be an m-dimensional semi-definite Kaehler manifold of index $2t$ $(0 \le t \le m)$. A plane section P of the tangent space T_xM of M at any point x is said to be *non-degenerate* provided that $g_x|_P$ is non-degenerate. It is easily seen that P is non-degenerate if and only if it has a basis $\{X, Y\}$ such that

$$g(X,X)g(Y,Y) - g(X,Y)^2 \ne 0.$$

If the non-degenerate plane P is invariant by the complex structure J, it is said to be *holomorphic*. It is also trivial that the plane P is holomorphic if and only if it contains a vector X in P such that $g(X,X) \neq 0$. For the non-degenerate plane P spanned by X and Y in P, the sectional curvature $K(P)$ is usually defined by

$$K(P) = K(X,Y) = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2} .$$

The holomorphic plane spanned by a space-like or time-like vector X and JX is said to be *space-like* or *time-like*, respectively. The sectional curvature $K(P)$ of the holomorphic plane P is called the *holomorphic sectional curvature*, which is denoted by $H(P)$. The semi-definite Kaehler manifold M is said to be *of constant holomorphic sectional curvature* if its holomorphic sectional curvatures $H(P)$ are constant for all holomorphic planes at all points of M. Then M is called a *semi-definite complex space form*, which is denoted by $M_t^m(c)$ provided that it is of constant holomorphic sectional curvature c, of complex dimension m and of index $2t(\geq 0)$. It is seen in Wolf [12] that the standard models of semi-definite complex space forms are the following three kinds : the semi-definite complex projective space $CP_t^m(c)$, the semi-definite complex Euclidean space C_t^m or the semi-definite complex hyperbolic space $CH_t^m(c)$, according as $c > 0$, $c = 0$ or $c < 0$. For any integer q $(0 \leq t \leq m)$ it is also seen by [12] that they are complete simply connected semi-definite complex space forms of dimension m and of index $2t$. The Riemannian curvature tensor $R_{\bar{A}BC\bar{D}}$ of $M_t^m(c)$ is given by

$$R_{\bar{A}BC\bar{D}} = \frac{c}{2}\varepsilon_B\varepsilon_C(\delta_{AB}\delta_{CD} + \delta_{AC}\delta_{BD}). \tag{2.4}$$

Now, let M be an m-dimensional semi-definite Kaehler manifold of an index $2t$ equipped with semi-definite Kaehler structure $\{g, J\}$. We can choose a local field of $\{E_\alpha\} = \{E_A, E_{A^*}\}$ of orthonormal frames on the neighborhood of M such that $g(E_A, E_B) = \varepsilon_A\delta_{AB}$. Let $\{U_A\}$ be a local field of unitary frames associated with the orthonormal frames $\{E_A, E_{A^*}\}$ on the neighborhood of M stated above in the first of this section. This is a complex linear frame, which is orthonormal with respect to the semi-definite Kaehler metric, that is, $g(U_A, \bar{U}_B) = \varepsilon_A\delta_{AB}$.

Given two holomorphic planes P and Q in T_xM at any point x in M, the holomorphic bisectional curvature $H(P,Q)$ determined by the two planes P and Q of M is defined by

$$H(P,Q) = \frac{g(R(X,JX)JY,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}, \tag{2.5}$$

where X (resp. Y) is a non-zero vector in P (resp. Q). In particular, the holomorphic bisectional curvature $H(P,Q)$ is said to be *space-like* or *time-like* if P and Q are both space-like or either P or Q is time-like. It is a simple matter to verify that the right hand side in (2.5) depends only on P and Q and so it is well defined. It may be denoted by $H(P,Q) = H(X,Y)$. It is easily seen that $H(P,P) = H(P) = H(X,X) =: H(X)$ is the holomorphic sectional curvature determined by the holomorphic plane P, where X is a non-zero vector in P.

We denote by P_A the holomorphic plane $[E_A, JE_A]$ spanned by E_A and $JE_A = E_{A^*}$. We set

$$H(P_A, P_B) = H_{AB}\ (A \neq B), \quad H(P_A, P_A) = H(P_A) = H_{AA} = H_A.$$

When two holomorphic planes P_A and P_B are orthogonal to each other, naturally we are able to define the *totally real bisectional curvature* $B_{AB} = H(P_A, P_B)(A \neq B)$ in such a way that (See also [3, 6, 7, 11])

$$B_{AB} = \frac{g(R(E_A, JE_A)JE_B, E_B)}{g(E_A, E_A)g(E_B, E_B)} = -\varepsilon_A \varepsilon_B K_{AA^*BB^*}\ (A \neq B).$$

Moreover, when two holomorphic planes P_A and P_B coincide with each other, the *holomorphic sectional curvature* is defined by

$$H_A = \frac{g(R(E_A, JE_A)JE_A, E_A)}{g(E_A, E_A)g(E_A, E_A)} = -K_{AA^*AA^*}.$$

Then by (2.3) it can be respectively given by

$$B_{AB} = \varepsilon_A \varepsilon_B R_{\bar{A}AB\bar{B}}\ (A \neq B), \text{ and } H_A = R_{\bar{A}AA\bar{A}}. \tag{2.6}$$

3. Space-like Kaehler submanifolds

This section is concerned with space-like complex submanifolds of an indefinite Kaehler manifold. First of all, the basic formulas for the theory of space-like complex submanifolds are prepared.

Let M' be an $(n+p)$-dimensional connected indefinite Kaehler manifold of index $2p$ with indefinite Kaehler structure (g', J'). Let M be an n-dimensional connected space-like complex submanifold of M' and let g be the induced Kaehler metric tensor of index $2p$ on M from g'. We can choose a local field $\{U_A\} = \{U_j, U_x\} = \{U_1, \ldots, U_{n+p}\}$ of unitary frames on a neighborhood of M' in such a way that, restricted to M, $U_1, \ldots, U_n$ are tangent to M and the others are normal to M. Here and in the sequel, the following convention on the range of indices is used throughout this paper, unless otherwise stated :

$$A, B, C, \ldots = 1, \ldots, n, n+1, \ldots, n+p\ ;$$
$$i, j, k, \ldots = 1, \ldots, n\ ; \quad x, y, z, \ldots = n+1, \ldots, n+p.$$

With respect to the frame field, let $\{\omega_A\} = \{\omega_j, \omega_y\}$ be its dual frame fields. Then the indefinite Kaehler metric tensor g' of M' is given by $g' = 2\sum_A \varepsilon_A \omega_A \otimes \bar{\omega}_A$, where $\{\varepsilon_A\} = \{\varepsilon_j, \varepsilon_y\}$. The connection forms on M' are denoted by $\{\omega_{AB}\}$. Then by virtue of (2.2) the canonical forms ω_A

and the connection forms ω_{AB} of the ambient space M' satisfy the structure equations

$$d\omega_A + \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B = 0, \quad \omega_{AB} + \bar{\omega}_{BA} = 0,$$
$$d\omega_{AB} + \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} = \Omega'_{AB}, \tag{3.1}$$
$$\Omega'_{AB} = \sum_{C,D} \varepsilon_C \varepsilon_D R'_{\bar{A}BC\bar{D}} \omega_C \wedge \bar{\omega}_D,$$

where Ω'_{AB} (resp. $R'_{\bar{A}BC\bar{D}}$) denotes the curvature form (resp. the components of the indefinite Riemannian curvature tensor R') of M'.

Since we assume M is a space-like complex submanifold in an indefinite Kaehler manifold M', hereafter it will be denoted by $\epsilon_j = 1$ and $\epsilon_y = -1$. Restricting the above forms to the submanifold M, we have

$$\omega_x = 0,$$

and the induced Kaehler metric g of M is given by $g = 2\Sigma\omega_j \otimes \bar{\omega}_j$. Then $\{E_j\}$ is a local unitary frame field with respect to this metric and $\{\omega_j\}$ is a local dual field of $\{E_j\}$, which consists of complex-valued 1-forms of type (1,0) on M. Moreover $\omega_1, ..., \omega_n, \bar{\omega}_1, ..., \bar{\omega}_n$ are lineary independent, and they are said to be cannonical 1-forms on M. It follows from the above formula and the Cartan lemma that the exterior derivatives of $\omega_x = 0$ give rise to

$$\omega_{xi} = \sum h^x_{ij} \omega_j, h^x_{ij} = h^x_{ji}.$$

The quadratic form $\sum h^x_{ij} \omega_i \otimes \omega_j \otimes E_x$ with values in the normal bundle is called the *second fundamental form* of the submanifold M. Similarly, from the structure equation of M' it follows that the structure equations for M are given by

$$d\omega_i + \sum \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \bar{\omega}_{ji} = 0, \tag{3.2}$$

$$d\omega_{ij} + \sum \omega_{ik} \wedge \omega_{jk} = \Omega_{ij}, \tag{3.3}$$
$$\Omega_{ij} = \sum R_{\bar{i}jk\bar{l}} \omega_k \wedge \bar{\omega}_l,$$

where we have put $\epsilon_j = \epsilon_k = \epsilon_l = 1$ and Ω_{ij}(resp. $R_{\bar{i}jk\bar{l}}$) denotes the Riemannian curvature form (resp. the components of the Riemannian curvature tensor R) on M. Moreover, the following relationships are defined:

$$d\omega_{xy} - \sum \omega_{xz} \wedge \omega_{zy} = \Omega_{xy}, \quad \Omega_{xy} = \sum R_{\bar{x}yk\bar{l}} \omega_k \wedge \bar{\omega}_l,$$

where we also have put $\epsilon_x = -1$ and $\epsilon_k = \epsilon_l = 1$.

For the Riemannian curvature tensors R and R' of M and M' in (3.3) and (2.2) respectively, the equation of Gauss gives rise to

$$R_{\bar{i}jk\bar{l}} = R'_{\bar{i}jk\bar{l}} + \sum h^x_{jk} \bar{h}^x_{il}, \tag{3.4}$$

where we have put $\epsilon_x = -1$. The components of the Ricci tensor S and the scalar curvature r of M are given by

$$S_{i\bar{j}} = \sum R'_{\bar{j}ik\bar{k}} + \sum h^x_{ir}\bar{h}^x_{rj}. \tag{3.5}$$

$$r = 2\sum S_{j\bar{j}} = 2\sum R'_{\bar{j}jk\bar{k}} - 2h_2, \tag{3.6}$$

where $h^2_{i\bar{j}} = -\sum h^x_{ik}\bar{h}^x_{kj}$ and $h_2 = \sum h^2_{k\bar{k}} = -\sum h^x_{ik}\bar{h}^x_{ik}$.

Now the components h^x_{ijk} and $h^x_{ij\bar{k}}$ of the covariant derivative of the second fundamental form of M are given by

$$\sum (h^x_{ijk}\omega_k + h_{ij\bar{k}}\bar{\omega}_k) = dh^x_{ij} - \sum (h^x_{kj}\omega_{ki} + h^x_{ik}\omega_{kj}) + \sum h^y_{ij}\omega_{xy}.$$

Then substituting dh^x_{ij} into the exterior derivative of ω_{xi} and using (3.2) and (3.3), we have

$$h^x_{ijk} = h^x_{jik} = h^x_{ikj}, \quad h^x_{ij\bar{k}} = -R'_{\bar{x}ij\bar{k}}. \tag{3.7}$$

Similarly the components h^x_{ijkl} and $h^x_{ijk\bar{l}}$ of the covariant derivative of h_{ijk} can be defined by

$$\begin{aligned}\sum (h^x_{ijkl}\omega_l + h^x_{ijk\bar{l}}\bar{\omega}_l) =& dh^x_{ijk} - \sum (h^x_{ijk}\omega_{li} + h^x_{ijk}\omega_{lj} + h^x_{ijk}\omega_{lk}) \\ & - \sum h^y_{ijk}\omega_{xy},\end{aligned}$$

and the simple calculation gives rise to

$$\begin{aligned} h^x_{ijkl} &= h^x_{ijlk}, \\ h^x_{ijk\bar{l}} - h^x_{ij\bar{l}k} &= \sum (R_{\bar{l}ki\bar{r}}h^x_{rj} + R_{\bar{l}kj\bar{r}}h^x_{ir}) + \sum R_{\bar{x}yk\bar{l}}h^y_{ij}, \end{aligned} \tag{3.8}$$

where we have put $\epsilon_r = 1$ and $\epsilon_y = -1$.

Now we consider a space-like complex submanifold M in an indefinite complex space form $M^{n+p}_p(c)$ of constant holomorphic sectional curvature c. Then from (2.4),(3.4),(3.5),(3.6), (3.7) and (3.8) it follows that

$$R_{\bar{i}jk\bar{l}} = \frac{c}{2}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl}) + \sum h^x_{jk}\bar{h}^x_{il}, \tag{3.9}$$

$$S_{i\bar{j}} = (n+1)\frac{c}{2}\delta_{ij} - h^2_{i\bar{j}}, \tag{3.10}$$

$$r = n(n+1)c - 2h_2, \tag{3.11}$$

$$\begin{aligned} h^x_{ijk\bar{l}} =& \frac{c}{2}(h^x_{ij}\delta_{kl} + h^x_{jk}\delta_{il} + h^x_{ki}\delta_{jl}) \\ & + \sum (h^x_{ri}h^y_{jk} + h^x_{rj}h^y_{ki} + h^x_{rk}h^y_{ij})\bar{h}^y_{rl}, \end{aligned} \tag{3.12}$$

where we have put $\epsilon_i = 1$ and $\epsilon_x = -1$, because its tangent (*resp.* normal) space of M in $M^{n+p}_p(c)$ is space-like (*resp.* time-like).

In order to prove our theorems, we introduce a generalized maximum principal due to Omori [10] and Yau [13], which has been widely used in the proof of geometric problems in complete Riemannian manifolds.

THEOREM 3.1. *Let M be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below on M. Let F be a C^2-function bounded from below on M, then for any $\epsilon > 0$, there exists a point p such that*

$$|\nabla F(p)| < \epsilon, \quad \triangle F(p) > -\epsilon \quad \textit{and} \quad \inf F + \epsilon > F(p).$$

4. Proofs of Theorems 1.1 and 1.2

Let M be an $n(\geq 3)$-dimensional space-like complex submanifold of an indefinite complex hyperbolic space $M' = CH_p^{n+p}(c)$, $c < 0$, $p \geq 1$.

Since M is space-like, the normal space at any point of M can be regarded as a time-like space. So hereafter unless otherwise stated, let us put the sign of the codimension index by $\epsilon_x = -1$.

Now let us denote by

$$h_4 = \sum h^2_{i\bar{j}} h^2_{j\bar{i}} \text{ and } A_2 = \sum A^x_y A^y_x,$$

where $A^x_y = \sum h^x_{ij} \bar{h}^y_{ij}$. Then the matrix $(h^2_{j\bar{k}})$ given above is a negative semi-definite Hermitian one, whose eigenvalues μ_j are non-positive real valued functions on M. The matrix $A = (A^x_y)$ is also by definition positive semi-definite Hermitian one and its eigenvalues μ_x are non-negative real valued functions. Then it can be easily seen that

$$-\sum_x \mu_x = -TrA = h_2, \quad h_2 = \sum_j \mu_j = \sum_j h^2_{j\bar{j}}, \quad \text{and } h_6 = \sum_j \mu_j^3.$$

Then it follows that

$$\begin{aligned} h_2^2 &\geq h_4 = \sum_j \mu_j^2 \geq \frac{h_2^2}{n}, \\ h_2^2 &\geq A_2 = \sum_x \mu_x^2 \geq \frac{h_2^2}{p}, \end{aligned} \tag{4.1}$$

where the first equality in the first inequalities above holds if and only if the rank of the matrices H and A is at most one, respectively and the second equality in the second inequalities above, which are derived from the Cauchy-Schwarz inequality, holds if and only if $\mu_i = \mu_j$ for any indices i and j and $\mu_x = \mu_y$ for any indices x and y. Moreover, we have

$$h_2 h_4 / n \geq h_6 \geq h_2 h_4, \tag{4.2}$$

where the first equality holds if and only if $\mu_i = \mu_j$ for any indices i and j.

In fact, it is easily seen that in the second inequality, equality holds if and only if the rank of the matrix is at most one. Concerning the first equality, we have

$$0 = nh_6 - h_2 h_4 = n\sum_j \mu_j^3 - \sum_j \mu_j \sum_k \mu_k^2$$
$$= \sum_{j<k} (\mu_j - \mu_k)^2 (\mu_j + \mu_k) \leq 0,$$

because eigenvalues μ_j are non-positive for any j. From this we have our assertion.

Firstly, let us take a differentiation to $h_2 = -\sum h_{ij}^x \bar{h}_{ij}^x$ and use the fact $h_{ij\bar{k}}^x = 0$. Then it follows

$$\begin{aligned}(h_2)_{k\bar{l}} =& - \sum_{x,i,j} h_{ijk}^x \bar{h}_{ijl}^x \\ & - \sum_{x,i,j} \{c(h_{ij}^x \delta_{kl} + h_{jk}^x \delta_{il} + h_{ki}^x \delta_{jl})/2 \\ & + \sum_{y,m} (h_{mi}^x h_{jk}^y + h_{mj}^x h_{ki}^y + h_{km}^x h_{ij}^y) \bar{h}_{ml}^y \} \bar{h}_{ij}^x,\end{aligned} \tag{4.3}$$

where we have put $\epsilon_x = \epsilon_y = -1$, because the normal space is time-like.

Next the Laplacian of the squared norm h_4 of the Hermitian matrix $H = (h_{i\bar{j}}^2)$, which is given by $h_4 = \sum h_{i\bar{j}}^2 h_{j\bar{i}}^2$, is estimated. By using (3.12) and also the fact $h_{ij\bar{k}}^x = 0$, we have

$$\begin{aligned}\triangle h_4 =& - 2\sum_{x,i,j} \big[\{ (n+2) c h_{ij}^x /2 \\ & - \sum_{k,l,m} (h_{mi}^x h_{j\bar{m}}^2 + h_{jm}^x h_{i\bar{m}}^2 - \sum_y A_y^x h_{ij}^y) \} \bar{h}_{kj}^x h_{k\bar{i}}^2 \\ & + \sum_{k,m} h_{ijm}^x \bar{h}_{jkm}^x h_{k\bar{i}}^2 - \sum_{y,k,l,m} h_{ijm}^x \bar{h}_{jk}^x h_{kl}^y \bar{h}_{iml}^y \big],\end{aligned} \tag{4.4}$$

where we also have put $\epsilon_x = \epsilon_y = -1$.

Since the matrix $H = (h_{i\bar{j}}^2)$ and the matrix $A = (A_y^x)$ are negative semi-definite Hermitian ones with eigenvalues μ_j and μ_x respectively, we choose a local field $\{e_A\} = \{e_j, e_x\}$ of unitary frames such that $h_{i\bar{j}}^2 = \mu_i \delta_{ij}$ and $A_y^x = \mu_x \delta_{xy}$. The third term of the right side of (4.4) is given by

$$\begin{aligned}&-\sum_{x,i,j,k,m} h_{ijm}^x \bar{h}_{jkm}^x h_{k\bar{i}}^2 \\ &\qquad = -\sum_{x,i,j,k,m} \mu_k h_{ijm}^x \bar{h}_{jkm}^x \delta_{ki} \\ &\qquad = -\sum_{x,i,j,k} \mu_k h_{ijk}^x \bar{h}_{ijk}^x \geq 0,\end{aligned}$$

because of $\epsilon_x = -1$, since the normal space is time-like and $\mu_k \leq 0$, where the equality holds if and only if the second fundamental form is parallel. That is, $h_{ijk}^x = 0$ for any indices i, j, k and x.

In fact, in this case the equality holds of the above formula if and only if

$$\mu_k h_{ijk}^x \bar{h}_{ijk}^x = 0$$

for any indices i, j, k and x. Suppose that there is a point p at which $\mu_1(p) = 0$. We denote by M_0 the non-empty subset of points p on M at which $\mu_1(p) = 0$. Then on M_0 we have $\mu_1 = \sum_{x,j}\epsilon_j h^x_{ij}\bar{h}^x_{ij} = 0$ which means that $h^x_{ij} = 0$, because of $\epsilon_j = 1$. On the subset $M - M_0$ it is trivial that h^x_{ijk} vanishes identically for any indices j, k and x. Suppose that the interior $\mathrm{Int}M_0$ of the set M_0 is not empty. Then h^x_{ij} vanishes identically on $\mathrm{Int}M_0$. So this implies h^x_{ijk} vanishes identically for any indices j, k and x on $\mathrm{Int}M_0$. Then $\mathrm{Int}M_0 \cup (M - M_0)$ is a dense subset of M. So by the continuity it vanishes identically on the whole M.

On the other hand, in the case where the interior of the set M_0 is empty, again by the continuity it vanishes identically on the whole M. Thus the second fundamental form α of M is parallel.

Next the last term means that the squared norm of the tensor -$\sum_{x,l} h^x_{il}\bar{h}^x_{ljk}$ is non-negative. Then by using this frame to (4.4) we have

$$\begin{aligned}\triangle h_4 \geq &(n+2)ch_4 - 2h_6 + 2\sum\nolimits_{x,i,j}\mu_i\mu_j h^x_{ij}\bar{h}^x_{ij}\\ &- 2\sum\nolimits_{x,y,i,j}\mu_x\mu_y\delta_{xy}h^x_{ij}\bar{h}^y_{ij},\end{aligned}$$

where we have put $\epsilon_x = \epsilon_y = -1$. Because of $\epsilon_x = -1$, it turns out to be

$$\begin{aligned}\triangle h_4 \geq &(n+2)ch_4 - 2h_6 + 2\sum\nolimits_{x,i,j}\mu_i\mu_j h^x_{ij}\bar{h}^x_{ij}\\ &- 2\sum\nolimits_{x,y,i,j}\mu_x\mu_j\delta_{xy}h^x_{ij}\bar{h}^y_{ij}.\end{aligned} \tag{4.5}$$

From the equation of Gauss (3.4) we have

$$R_{\bar{j}jk\bar{k}} = \frac{c}{2} + \sum\nolimits_x h^x_{jk}\bar{h}^x_{jk} \geq \frac{c}{2},\ j \neq k \tag{4.6}$$

where we have used the fact that $\epsilon_x = -1$, because the normal space of M is time-like. Thus from (4.6) we see that for any totally real bisectional plane section $[u, v]$ satisfies $B(u, v) \geq \frac{c}{2}$.

Let $a(M)$ be the infimum of the set B of totally real bisectional curvatures of M. As in Theorem A 1) we have proved that $a(M) \leq \frac{c}{4}$ and the equality holds if and only if M is a complex hyperbolic space $CH^n(\frac{c}{2})$, $p \geq n(n+1)/2$.

Lemma 4.1. *Let M be an $n(\geq 3)$-dimensional complete space-like complex submanifold of an $(n+p)$-dimensional indefinite complex hyperbolic space $CH^{n+p}_p(c)$ of constant holomorphic sectional curvature c and of index $2p(>0)$. If M is not totally geodesic, then the squared norm $|\alpha|_2 = h_2$ of the second fundamental form α of M satisfies*

$$\begin{aligned}\inf h_2 < &\frac{\sqrt{n}}{2(n^2+2)}\big[\big\{3np + 2(n^2-1)\big\}c\\ &- 2\big\{(n^2+2n-2)p + n^2 - 1\big)\big\}a(M)\big].\end{aligned} \tag{4.7}$$

REMARK 4.2. We denote by $h(n, p, a(M), c)$ the right side of the inequality (4.7). In [1], Aiyama, Nakagawa and the present author proved that under the same situation as in Lemma 4.1 we have

$$0 \geq h_2 \geq n(n+1)c/4, \quad c < 0, \tag{4.8}$$

where the second equality holds if and only if M is a complex space form $M^n(\frac{c}{2})$ and $p = n(n+1)/2$ (See [1], Theorem 3.2 (2)). It can be easily seen by the simple calculation that

$$h(n, p, a(M), c) > n(n+1)\frac{c}{4}. \tag{4.9}$$

PROOF OF LEMMA 4.1. First of all, the third term of the right side of (4.5) will be estimated. It is seen that

$$\begin{aligned}\text{the third term} &= 2\sum_{x,i,j}\mu_i\mu_j h^x_{ij}\bar{h}^x_{ij}\\ &= 2(\sum_{x,i}\mu_i^2 h^x_{ij}\bar{h}^x_{ij} + \sum_{x,i\neq j}\mu_i\mu_j h^x_{ij}\bar{h}^x_{ij}).\end{aligned}$$

Since $a(M)$ is the infimum of the set B of totally real bisectional curvatures, we have $R_{\bar{i}ij\bar{j}} \geq a(M)$ for any distinct indices i and j, from which together with (4.1) it follows that

$$-\sum_x h^x_{ij}\bar{h}^x_{ij} \leq \frac{c}{2} - a(M) \text{ for any } i, j (i \neq j), \tag{4.10}$$

where we have put $\epsilon_x = -1$.

On the other hand, the scalar curvature r on M satisfies

$$\begin{aligned}r = 2\sum_j S_{j\bar{j}} &= 2\sum_{i,j} R_{\bar{i}ij\bar{j}} = 2(\sum_i R_{\bar{i}ii\bar{i}} + \sum_{i\neq j} R_{\bar{i}ij\bar{j}})\\ &\geq 2\sum_i R_i + 2n(n-1)a(M),\end{aligned}$$

where $R_i = R_{\bar{i}ii\bar{i}}$. Since we have $R_i + R_j \geq 4a(M)$ for any $i, j (i \neq j)$ from [6], we have

$$\begin{aligned}&(n-2)R_i + \sum_i R_i \geq 4(n-1)a(M),\\ &2na(M) \leq \sum_j R_j \leq \frac{r}{2} - n(n-1)a(M),\end{aligned}$$

from which it follows that

$$(n-2)R_j \geq 4(n-1)a(M) - \sum_j R_j \geq (n-1)(n+4)a(M) - \frac{r}{2}$$

and hence we have

$$-\sum_x h^x_{jj}\bar{h}^x_{jj} = c - R_j \leq \{(n-1)(n+4)(c - 2a(M)) - 2h_2\}/2(n-2). \tag{4.11}$$

By using (4.10) and (4.11), the estimation of the third term is given by

$$\begin{aligned}\text{the third term} \geq &-\sum_j \mu_j^2\{(n-1)(n+4)(c-2a(M)) - 2h_2\}/2(n-2)\\ &-\sum_{i\neq j}\mu_i\mu_j(\frac{c}{2} - a(M)).\end{aligned}$$

From this together with the property $\sum_{i\neq j}\mu_i\mu_j = \sum_i \mu_i(h_2-\mu_i) = h_2^2 - h_4$ it follows

$$\text{the third term} \geq \big[\{(n^2+2n-2)h_4 + (n-2)h_2^2\}(2a(M)-c) + 2h_2h_4\big]/2(n-2). \tag{4.12}$$

Next, we will estimate the last term of (4.5) from below. First the eigenvalue μ_j of the matrix H is estimated. We have

$$\mu_j = -\sum_{k,x} h^x_{jk}\bar{h}^x_{jk} = -h^x_{jj}\bar{h}^x_{jj} - \sum_{k\neq j} h^x_{jk}\bar{h}^x_{jk},$$

from which together with (4.10) and (4.11) it follows that

$$\begin{aligned}\mu_j &\leq \{(n-1)(n+4)(c-2a(M)) - 2h_2\}/2(n-2) + \sum_{k\neq j}(c-2a(M))/2\\ &= \{(n-1)(n+1)(c-2a(M)) - h_2\}/(n-2).\end{aligned}$$

On the other hand, since the eigenvalue μ_x of the matrix A is non-negative, we see $\sum_x \mu_x h^x_{ij}\bar{h}^x_{ij} \geq 0$, and hence we have

$$\begin{aligned}\text{the fourth term} &\geq -2\{(n-1)(n+1)(c-2a(M)) - h_2\}\sum_x \mu_x h^x_{ij}\bar{h}^x_{ij}/(n-2)\\ &= 2\{(n-1)(n+1)(2a(M)-c) + h_2\}\sum_x \mu_x^2/p(n-2),\end{aligned}$$

where we have used (4.1) and the property $a(M) \geq \frac{c}{2}$. Therefore we have

$$\text{the fourth term} \geq 2\{(n-1)(n+1)(2a(M)-c) + h_2\}h_2^2/p(n-2). \tag{4.13}$$

By (4.5), (4.12) and (4.13) we obtain

$$\begin{aligned}(n-2)\triangle h_4 \geq &\{(n^2-4)c + (n^2+2n-2)(2a(M)-c)\}h_4 - 2(n-2)h_6\\ &+ 2h_2h_4 + \{(n-2) + 2(n^2-1)/p\}(2a(M)-c)h_2^2 + 2h_2^3/p.\end{aligned}$$

From this it follows

$$\begin{aligned}(n-2)\triangle h_4 \geq &2\{(n^2+2n-2)a(M) - (n+1)c\}h_4\\ &- 2(n-2)h_6 + 2h_2h_4\\ &+ \{(n-2) + 2(n^2-1)/p\}(2a(M)-c)h_2^2 + 2h_2^3/p.\end{aligned} \tag{4.14}$$

Since it is seen that $h_6 \leq h_2h_4/n$ and $h_2^2 \geq h_4 \geq h_2^2/n$ by (4.1) and (4.2), the inequality (4.14) is reestimated as

$$\begin{aligned}(n-2)\triangle h_4 \geq h_4\big[&2\big\{(n^2+2n-2) + 2(n^2-1)/p\big\}a(M)\\ &- \big\{3n + 2(n^2-1)/p\big\}c - 2(n^2+2p)\frac{\sqrt{h_4}}{p\sqrt{n}}\big].\end{aligned} \tag{4.15}$$

We denote by f the function h_4. Then the right side of the above inequality can be regarded as the function of f, say F. So we have the following Liouville-type inequality

$$\triangle f \geq F(f)/(n-2). \tag{4.16}$$

Now, let $b(M)$ be the supremum of the set B of totally real bisectional curvatures on M. Since M is a complete space-like complex submanifold of $CH_p^{n+p}(c)$, $c<0$, by (4.1) and (4.8) it follows that

$$h_2 \geq n(n+1)\frac{c}{4},$$

which implies that the function h_2 is bounded. It shows that the function f is also bounded, because it satisfies $0 \leq f \leq h_2^2$.

On the other hand, the Ricci tensor S is given by

$$S_{i\bar{j}} = \{(n+1)\frac{c}{2} - \mu_i\}\delta_{ij}$$

from (3.10). Since the eigenvalue μ_i is non-positive, the Ricci curvature is bounded from below by a constant $(n+1)\frac{c}{2}$. Accordingly, we can apply Theorem 3.1 to the function f, so for any sequence $\{\epsilon_m\}$ which converges to zero as m tends to infinity, there exists a point sequence $\{p_m\}$ on M such that

$$|\nabla f(p_m)| < \epsilon_m,\ \triangle f(p_m) < \epsilon_m,\ \sup f - \epsilon_m < f(p_m). \tag{4.17}$$

From (4.16) and (4.17) we have

$$\epsilon_m \ > \ \triangle f(p_m) \geq F(f(p_m))/(n-2),$$

which implies, taking into account (4.17), that we have $\lim_{m\to\infty} f(p_m) = \sup f$, and hence we get

$$0 \geq F(\sup f).$$

That is, we have

$$\sup f = 0 \tag{4.18}$$

or

$$\begin{aligned}\sqrt{\sup f} \geq \frac{-\sqrt{n}}{2(n^2+2p)}\big[\big\{3np + 2(n^2-1)\big\}c \\ - 2\big\{(n^2+2n-2)p + n^2 - 1\big\}a(M)\big].\end{aligned} \tag{4.19}$$

Let us denote by $-h = -h(n,p,c,a(M))$ the right side of the above inequality.

Suppose that (4.18) holds. Then $f \equiv 0$, because f is non-negative, and therefore M is totally geodesic.

Suppose that M is not totally geodesic. Then the inequality (4.19) holds. Since $\sup h_4 \leq \sup(-h_2)^2 = (-\inf h_2)^2$, we have

$$\inf h_2 \leq \ h = h(n,p,c,a(M)). \tag{4.20}$$

Now suppose that the equality of (4.20) holds. Then from the proof of (4.15) and (4.20) we have $nh_6 = h_2h_4$ and $h_4 = h_2^2$. The first equation holds if and only if the eigenvalues μ_j of the matrix $H = (h^2_{i\bar{j}})$ are all equal and the second one holds if and only if the rank of H is at most one. Then both of these implies that H is the zero matrix. Namely, M must be totally geodesic, a contradiction. Then finally we have the inequality (4.7). It completes the proof of Lemma 4.1. ∎

From the above definition (4.7) of the constant h and a simple calculation we assert the following

LEMMA 4.3. *Under the same situation as in Lemma 4.1 we have*

$$h(n, p, a(M), c) > \frac{n(n+1)}{4}c. \tag{4.21}$$

We define a constant $a = a(n, p, c)$ depending on n, p and c by

$$a = a(n, p, c) = \frac{\{3np + 2(n^2 - 1)\}c}{2\{(n^2 + 2n - 2)p + n^2 - 1\}}. \tag{4.22}$$

By (4.19) and (4.22) the following property is trivial.

LEMMA 4.4. *Under the same situation as in Lemma 4.1, if $a(M) \geq a$ (resp. $> a$), then we have $h = h(n, p, a(M), c) \leq 0$ (resp. < 0).*

REMARK 4.5. Let M be fully immersed in $CH_p^{n+p}(c)$, i.e. there is no integer q $(0 < q < p)$ such that M is immersed in $CH_q^{n+q}(c)$. If M is totally geodesic, then $p = 1$. In this case we see $a(n, 1, c) = \frac{2n^2+3n-2}{2(2n^2+2n-3)}c < \frac{c}{2}$ for any n.

On the other hand, we have already remarked that Ki and the present author [8] have proved that the infimum $a(M)$ of the set of totally real bisectional curvature of space-like submanifolds M in $CH_p^{n+p}(c)$ is not greater than $\frac{c}{4}$, that is $a(M) \leq \frac{c}{4}$. So under our assumptions we shoud have $a \leq \frac{c}{4}$. Moreover

LEMMA 4.6. *Under the same situation as in Lemma 4.1 we have*

1) *If $p = 1$, then $a < \frac{c}{2}$,*
2) *If $p \geq 2$ and $n = 3$ or 4, then $\frac{c}{2} \leq a \leq \frac{c}{4}$,*
3) *If $n \geq 5$ and $p \leq \frac{3(n^2-1)}{n^2-4n-2}$, then $a \leq \frac{c}{4}$.*

PROOF. From Remark 4.5 we know 1). By putting $n = 3$ or $n = 4$ in (4.22) it can be easily seen that 2) holds for any $p \geq 2$. Now let us prove the assertion 3). As we have explained above, the constant $a = a(n, p, c)$ in (4.22) can not be greater than $\frac{c}{4}$. From this we should have

$$\frac{3np + 2(n^2 - 1)}{(n^2 + 2n - 2)p + n^2 - 1} \geq \frac{1}{2}.$$

So, we conclude 3), which ends the proof of Lemma 4.6. ∎

Now we are in a position to prove our main theorems in the introduction. Suppose that M is not totally geodesic. By the assumption of dimension and the assertions 1) and 2) in Lemma 4.6 there is a constant $a = a(n, p, c)$ depending on n, p and c such that $a = a(n, p, c) \leq \frac{c}{4}$. By (4.21) there exists a constant

$h = h(n, p, a(M), c)$ depending on $n, p, a(M)$ and c such that $h > n(n+1)\frac{c}{4}$. By Lemma 4.4, we know that if $a(M) \geq a$, then $h = h(n, p, a(M), c) \leq 0$.

On the other hand, by (4.20) and Lemma 4.1 we have $\inf h_2 < h \leq 0$ since we have supposed that M is not totally geodesic. But from the assumption of Theorem 1.1 we know $h_2 = |\alpha|_2 \geq h$, which makes a contradiction. It completes the proof of Theorem 1.1.

In the case where $n \geq 5$, the condition of the codimension implies $p \leq \frac{3(n-1)}{n^2-4n-2}$. Accordingly, by Lemma 4.1 and Lemma 4.6 we also complete the proof of Theorem 1.2.

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