GLASNIK MATEMATIČKI
Vol. 37(57)(2002), 347 – 363

THE HYPERSPACE $C_2(X)$ FOR A FINITE GRAPH X IS UNIQUE

Alejandro Illanes
Universidad Nacional Autónoma de México, México

Abstract. Let X be a metric continuum. Let $C_2(X)$ be the hyperspace of X consisting of all the nonempty and with at most two components closed subsets of X, with the Hausdorff metric. In this paper we prove that if X is a finite graph and Y is a metric continuum such that $C_2(X)$ is homeomorphic to $C_2(Y)$, then X is homeomorphic to Y.

1. Introduction

All the concepts not defined here will be taken as in the book [24]. A *continuum* is a nonempty compact and connected metric space. For a continuum X and a positive integer n, consider the following hyperspaces:

$$\begin{aligned} 2^X &= \{A \subset X : A \text{ is closed and nonempty}\}, \\ C(X) &= \{A \in 2^X : A \text{ is connected}\}, \\ F_n(X) &= \{A \in 2^X : A \text{ contains at most } n \text{ points}\} \text{ and} \\ C_n(X) &= \{A \in 2^X : A \text{ has at most } n \text{ components}\}. \end{aligned}$$

All the hyperspaces are considered with the Hausdorff metric H.

Let $\mathcal{H}(X)$ denote one of the hyperspaces 2^X, $C(X)$, $F_n(X)$ or $C_n(X)$. We say that a continuum X has *unique hyperspace* $\mathcal{H}(X)$ provided that the following implication holds: if Y is a continuum and $\mathcal{H}(X)$ is homeomorphic to $\mathcal{H}(Y)$, then X is homeomorphic to Y.

The topic of this paper is inserted in the following general problem.

Problem. Find conditions on the continuum X in order that X has unique hyperspace $\mathcal{H}(X)$.

2000 *Mathematics Subject Classification.* Primary 54B20.
Key words and phrases. Continuum, finite graph, hyperspaces, unique hyperspace.

Generalizing previous results by Duda, Nadler, Nadler and Eberhart and Macías, Acosta proved that the following continua have unique hyperspace $C(X)$:

(a) Finite graphs different from an arc and the simple closed curve ([7, 9.1]),
(b) Hereditarily indecomposable continua ([23, 0.60]),
(c) Smooth fans ([10, Corollary 3.3]),
(d) Indecomposable continua such that all their proper nondegenerate subcontinua are arcs ([18]).

Acosta also proved that the metric, with nondegenerate remainder, compactifications of the ray $[0,\infty)$ have unique hyperspace $C(X)$ ([1]).

Macías has shown that hereditarily indecomposable continua have unique hyperspace 2^X ([19]). Castañeda has proved that finite graphs have unique hyperspace $F_2(X)$ and $F_3(X)$ ([6]). Related results to the subject of this paper can be found in [2, 3, 4, 5, 13, 14, 15] and [21, Theorem 6.1].

In this paper we prove the following theorem:

Theorem. *Finite graphs have unique hyperspace $C_2(X)$.*

2. The hyperspaces $C_2([0,1])$ and $C_2(S^1)$

A *map* is a continuous function. The unit circle in the euclidean plane E^2 is denoted by S^1. Given a subset A of the real line E^1 and two real numbers s and t let $s+tA=\{s+ta\in E^1 : a\in A\}$. A connected space Y is said to be *unicoherent* if $A\cap B$ is connected for every pair of closed connected subsets A and B such that $Y=A\cup B$. The exponential map $e:E^1\to S^1$ is defined by $e(t)=(\cos(t),\sin(t))$. Given a connected space Y, a map $f:Y\to S^1$ is said to be *inessential* provided that there is a map $g:Y\to E^1$ such that $f=e\circ g$. It is known that if Y is a connected, locally connected metric space, then Y is unicoherent if and only if each map $f:Y\to S^1$ is inessential ([11, Theorems 2 and 3]).

Lemma 2.1. *$[0,1]^4-\{z\}$ is unicoherent for each $z\in[0,1]^4$.*

Proof. Let $z\in[0,1]^4$. It is easy to show that $[0,1]^4-\{z\}$ can be put as the union of two contractible closed subsets A and B such that $A\cap B$ is connected. Let $f:[0,1]^4-\{z\}\to S^1$ be a map. Then $f|A$ and $f|B$ are homotopic to a constant. By the main result of [22], $f|A$ and $f|B$ are inessential. Thus, there exist maps $g_A:A\to E^1$ and $g_B:B\to E^1$ such that $f|A=e\circ g_A$ and $f|B=e\circ g_b$. Fix a point $p\in A\cap B$. Adding an integer multiple of π to g_A it is possible to assume that $g_A(p)=g_B(p)$. Since $A\cap B$ is connected, $g_A|A\cap B=g_B|A\cap B$. Thus there exists a map $g:[0,1]^4-\{z\}\to E^1$ which extends both maps g_A and g_B. This shows that $f=e\circ g$, so f is inessential. By ([11, Theorems 2 and 3]) we can conclude that $[0,1]^4-\{z\}$ is unicoherent. □

The following lemma and its proof was privately communicated to the author by R. Schori.

Lemma 2.2. (R. Schori) $C_2([0,1])$ *is homeomorphic to* $[0,1]^4$.

Proof. Let $D^1 = \{A \in C_2([0,1]) : 1 \in A\}$ and $D^1_0 = \{A \in C_2([0,1]) : 0,1 \in A\}$. In order to prove the lemma, we are going to show that $C_2([0,1])$ is homeomorphic to $\mathrm{Cone}(D^1)$, D^1 is homeomorphic to $\mathrm{Cone}\,(D^1_0)$ and D^1_0 is homeomorphic to $[0,1]^2$.

Let $f : \mathrm{Cone}\,(D^1) \to C_2([0,1])$ be given by $f(A,t) = (1-t)A$. Since $f(A,1) = \{0\}$ for each $A \in C_2([0,1])$, f is a well defined map. In order to see that f is one-to-one, suppose that $f(A,t) = f(B,s)$. Since $1 \in A \cap B$, $1-t = \max(1-t)A$ and $1-s = \max(1-s)B$. This implies that $t = s$. In the case that $t = s = 1$, (A,t) represents the same element as (B,s) in $\mathrm{Cone}(D^1)$. Hence, we may assume that $t < 1$. Since $(1-t)A = (1-t)B$, it follows that $A = B$. Therefore, f is one-to-one. Given $A \in C_2([0,1]) - \{\{0\}\}$, let $t = 1 - \max A$. Then, $\max(\frac{1}{1-t}A) = 1$. Thus $\frac{1}{1-t}A \in D^1$ and $A = f(\frac{1}{1-t}A, t)$. This completes the proof that f is bijective. Therefore, $\mathrm{Cone}(D^1)$ is homeomorphic to $C_2([0,1])$.

Now, let $g : \mathrm{Cone}(D^1_0) \to D^1$ be given by $g(A,t) = t + (1-t)A$. Proceeding as before, it can be seen that g is a homeomorphism. Therefore, D^1 is homeomorphic to $\mathrm{Cone}(D^1_0)$.

Let $T = \{(a,b) \in E^2 : 0 \leq a \leq b \leq 1\}$ and let S be the space obtained by identifying the diagonal $\Delta = \{(a,b) \in T : a = b\}$ of T to a point. Note that S is homemomorphic to $[0,1]^2$. Let $h : T \to D^1_0$ be given by $h(a,b) = [0,a] \cup [b,1]$. Then h is continuous and $h(a,b) = h(c,d)$ if and only if $(a,b) = (c,d)$ or $a = b$ and $c = d$. This implies that there is a homeomorphism between S and D^1_0. This completes the proof of the lemma. □

Lemma 2.3. $C_2([0,1])$ *is not homeomorphic to* $C_2(S^1)$.

Proof. By Lemma 2.2, $C_2([0,1])$ is homeomorphic to $[0,1]^4$. By Lemma 2.1, $C_2([0,1]) - \{A\}$ is unicoherent for each $A \in C_2([0,1])$. In order to prove this lemma we are going to show that $C_2(S^1) - \{S^1\}$ is not unicoherent. This will be made by defining an essential map $f : C_2(S^1) - \{S^1\} \to S^1$.

Let $A \in C_2(S^1) - \{S^1\}$. We consider two cases:

Case 1. If A is connected, then A is a subarc of S^1. Let z_A be the middle point of the arc $S^1 - A$ and let $f(A) = (z_A)^2$.

Case 2. If A is not connected, then the complement of A in S^1 consists of two open subarcs I and J. Let p and q be the respective middle points of I and J, and let r and s be the respective lengths of I and J. Let α and β be real numbers such that $|\alpha - \beta| \leq \pi$, $p = e(\alpha)$ and $q = e(\beta)$. Thus, let

$$f(A) = \begin{cases} e(2(\alpha r + \beta s - s\pi)/(r+s)), & \text{if } \alpha < \beta, \\ e(2(\alpha r + \beta s - r\pi)/(r+s)), & \text{if } \beta < \alpha. \end{cases}$$

In this case, it is easy to check that $f(A)$ does not depend on the choice of α and β, even when $|\alpha - \beta| = \pi$.

It is also easy to show that f is continuous in each one of the two sets where f is defined. Finally, to conclude that f is continuous, we only need to check that if A is a connnected element of $C_2(S^1)$ and $\{A_n\}_{n=1}^{\infty}$ is a sequence of disconnected elements of $C_2(S^1)$ such that $A_n \to A$, then $f(A_n) \to f(A)$.

Let z_A be the middle point of the arc $S^1 - A$. Suppose that $z_A = e(\alpha)$ for some real number α. For each $n \geq 1$, let I_n, J_n, p_n, q_n, r_n and s_n be as before for the set A_n. Since $A_n \to A$, we may assume that $\mathrm{cl}(I_n) \to \mathrm{cl}(S^1 - A)$, $p_n \to z_A$, $r_n \to$ (the length of the subcontinuum $\mathrm{cl}(S^1 - A)), \alpha_n \to \alpha$ and $s_n \to 0$. Thus, $f(A_n) \to e(2\alpha) = (z_A)^2 = f(A)$. Therefore, f is continuous.

Let $\mathcal{S} = \{A \in C(S^1) : A \text{ is connected and length}(A) = \pi\}$. Let $g : \mathcal{S} \to S^1$ be given by $g(A) =$ the middle point of $(S^1 - A)$. Then g is a homeomorphism. Since $f \circ g^{-1} : S^1 \to S^1$ is the map $f \circ g^{-1}(z) = z^2$, this map is essential. Therefore, f is essential. This completes the proof of the theorem. $\square$

3. Dimension in $C_2(X)$ for a finite graph X

From now on, then letter X will denote a finite (connected) graph. Then in X are defined segments (edges) and vertices. The *vertices* of X are the end points of the segments of X. We are interested in having as few subgraphs as possible, so we assume that each vertex of X is either an end point of X or a ramification point of X. Since this convention is not applicable to a simple closed curve, we assume that X is not a simple closed curve, unless we say the contrary. With this restriction two end points of a segment of X may coincide and such a "segment" is a simple closed curve. We also assume that the metric d in X is the metric of arc length and each segment of X has length equal to one. For each segment J of X, we identify J with a closed interval $[(0)_J, (1)_J]$. Notice that it is possible that $(0)_J = (1)_J$, in this case J will be named a *loop* of X, the elements of J are denoted by $(s)_J$, where $s \in [0, 1]$, we write simply s if it causes no confusion. Thus, in X there are only three kind of segments, namely: loops, segments that contains end points and segments joining ramification points. The set of ramification points of X is denoted by $R(X)$.

By a *subgraph* of X we mean a *connected* subgraph of X (the empty set and the sets consisting of exactly one vertex of X are considered subgraphs of X). A *fine subgraph* is a subgraph S of X such that S is acyclic and S does not contain end points of X. The *order* of a point p in X is denoted by $\mathrm{ord}(p)$ or $\mathrm{ord}_X(p)$ when it is necessary to make explicit the graph X. Two different vertices p and q of X are said to be *adjacent* provided that there is a segment J of X such that p and q are the end points of J.

If $A \in 2^X$ and $\epsilon > 0$, let $D_X(\epsilon, A) = \{p \in X :$ there exists a point $a \in A$ such that $d(a, p) \leq \epsilon\}$, $N_X(\epsilon, A) = \{p \in X :$ there exists a point $a \in A$ such that $d(a,p) < \epsilon\}$, and if $\mathcal{H}(X)$ is one hyperspace of X, let $B_{\mathcal{H}(X)}(\epsilon, A) = \{B \in \mathcal{H}(X) : H(A,B) < \epsilon\}$.

For a nonempty fine subgraph S of X, let $\mathcal{M}_S = \{B \in C(X) : S \subset B \subset D_X(1,S)$ and $B \cap N_X(1,S)$ is connected$\}$. In the case that S is of the form $\{p\}$, we write $\mathcal{M}_p$ instead $\mathcal{M}_{\{p\}}$. For the case that we take $S = \emptyset$ and J is a segment of X, let $\mathcal{M}_\emptyset^J = \{A \in C(X) : A \subset J\}$. We only write $\mathcal{M}_\emptyset$ instead of $\mathcal{M}_\emptyset^J$ when it is not necessary to mention the segment J.

Given a nonempty fine subgraph S of X, let $I_1, \ldots, I_r$ be the segments of X such that, for each i, I_i intersects S at exactly one of the end points of I_i (then I_i is an arc) and let $J_1^*, \ldots, J_s^*$ be the segments of X such that, for each j, J_j^* intersects S at exactly the two end points of J_j^* (here the two end points of J_j^* can agree). Then $D_X(1,S) = S \cup I_1 \cup \cdots \cup I_r \cup J_1^* \cup \cdots \cup J_s^*$. This is called the *canonical representation* of $D_X(1,S)$.

In [7, 8, 9], R. Duda, made a very detalied study of the hyperspace $C(X)$ for a finite graph X. He showed that $C(X)$ is a polyhedron by showing that the sets of the form $\mathcal{M}_S$ give a nice decomposition of $C(X)$. In the following lemma, we summarize some of the known results about the dimension of $C(X)$. Some other results concerning hyperspaces of finite graphs can be found in [16, Section 65].

LEMMA 3.1. (see [7, 5.2] and [17, Lemma 1.4]) *Let S be a nonempty fine subgraph of X and let $D_X(1,S) = S \cup I_1 \cup \cdots \cup I_r \cup J_1^* \cup \cdots \cup J_s^*$ be its canonical representation, then:*

(a) *$\mathcal{M}_\emptyset^J$ is homeomorphic to $[0,1]^2$ for each segment J of X,*
(b) *the elements in $\mathcal{M}_S$ are exactly those subcontinua A of X that can be represented in the form $A = S \cup (\bigcup\{[(0)_{I_i}, (a)_{I_i}] : 1 \le i \le r\}) \cup (\bigcup\{[(0)_{J_j^*}, (c)_{J_j^*}] \cup [(d)_{J_j^*}, (1)_{J_j^*}] : 1 \le j \le s\})$, where $0 \le a_i \le 1$ for each i and $0 \le c_j \le d_j \le 1$ for each j,*
(c) *$\mathcal{M}_S$ is homeomorphic to $[0,1]^{r+2s}$,*
(d) *if $A \in C(X)$, then $\dim_A(C(X)) = \dim(\mathcal{M}_T)$, where T is a fine subgraph of X, $T \subset A$ and T is maximal with respect to the inclusion,*
(e) *if $A, B \in C(X)$ and $A \subset B$, then $\dim_A(C(X)) \le \dim_B(C(X))$,*
(f) *if T and R are fine subgraphs of X and $T \subset R \neq T$, then $\dim(\mathcal{M}_T) < \dim(\mathcal{M}_R)$*

LEMMA 3.2. *Let X be a finite graph and let $A \in C_2(X)$. Then:*

(a) *if A has two components B and C, then $\dim_A(C_2(X)) = \dim_B(C(X)) + \dim_C(C(X))$,*
(b) *if A does not contain ramification points of X, then $\dim_A(C_2(X)) = 4$,*
(c) *if $A \in C(X)$, then $\dim_A(C_2(X)) \ge \dim_A(C(X)) + 2$ and there exist $B \in C_2(X)$ and a map $\alpha : [0,1] \to C_2(X)$ such that B is disconnected, $\dim_{\alpha(t)}(C_2(X)) = \dim_A(C(X)) + 2$ for each $t > 0$, $\alpha(0) = A$ and*

$\alpha(1) = B$. Furthermore, if A contains more than one ramification point of X, then B contains more than one ramification point of X,

(d) *let p be a ramification point of X such that ord$(p) = n$, let $\mathcal{K}_p = \{A \in C_2(X) : p \in A$ and p is the only ramification point of X that belongs to $A\}$. Then $C_2(X)$ has dimension equal to $n+2$ at each one of the elements of $\mathcal{K}_p$,*

(e) *suppose that p is a ramification point of X and ord$(p) = n$, let Let C_p be the component of $X - (R(X) - \{p\})$ that contains p, let $\mathcal{D}_p = \{A \in C_2(X) : p \in A \subset C_p\}$. Then $C_2(X)$ has dimension equal to $n+2$ at each one of the elements of $\mathcal{D}_p$ and $\mathcal{D}_p$ can be separated by a closed (in $\mathcal{D}_p$) subset of dimension less than or equal to n,*

(f) *if A contains more than one ramification point of X, then there exists a map $\alpha : [0,1] \to C_2(X)$ such that $\alpha(0) = A$, $\dim_{\alpha(s)}(C_2(X)) \geq \dim_{\alpha(t)}(C_2(X))$ for every $s \leq t$, $\alpha(1)$ contains exactly one ramification point and $\dim_A(C_2(X)) > \dim_{\alpha(1)}(C_2(X))$.*

Proof. (a) Suppose that B and C are the components of A. Let $\epsilon = \min\{d(b,c) : b \in B$ and $c \in C\}$. Given $A_1 \in B_{C_2(X)}(\epsilon/2, A)$, there exist $B_1 \in B_{C(X)}(\epsilon/2, B)$ and $C_1 \in B_{C(X)}(\epsilon/2, C)$ such that B_1 and C_1 are the components of A_1. Clearly, the map $A_1 \to (B_1, C_1)$ is a homeomorphism from $B_{C(X)}(\epsilon/2, A)$ onto $B_{C(X)}(\epsilon/2, B) \times B_{C(X)}(\epsilon/2, C) \subset C(X) \times C(X)$. Since $C(X)$ is a polyhedron ([7]), $\dim_{(B,C)}(C(X) \times C(X)) = \dim_B(C(X)) + \dim_C(C(X))$. Therefore, $\dim_A(C_2(X)) = \dim_B(C(X)) + \dim_C(C(X))$.

(b) Suppose that A does not contain ramification points. Then each component of A is contained in the interior of a segment of X. We analyze two cases: the first case is when A is contained in one segment J of X. In this case, there exists and arc J_1 such that $A \subset \text{int}_X(J_1) \subset J_1 \subset J$. Thus $C_2(J_1)$ is a closed neighborhood of A in $C_2(X)$. By Lemma 2.2, $C_2(J_1)$ is homeomorphic to $[0,1]^4$. Thus, $\dim_A(C_2(X)) = 4$. The second case is when A intersects two segments J and L of X. In this case, each one of the components A_1 and A_2 of A is a subcontinuum of X without ramification points. This implies that each A_i is a subarc of the interior of some segment of X. This implies that $\dim_{A_i}(C(X)) = 2$. According to (a), we conclude that $\dim_A(C(X)) = 4$.

(c) Let $n = \dim_A(C(X))$. By Lemma 3.1 (d), there exists a fine subgraph S of X such that $A \in \mathcal{M}_S$, and $\dim(\mathcal{M}_S) = n$ (then $\mathcal{M}_S$ is homeomorphic to $[0,1]^n$, by Lemma 3.1 (c)). We analyze four cases:

Case 1. A contains a cycle and $S \neq \emptyset$. Then there exists a segment J of X such that $J \subset A$, J is not a segment of S and $(A - J) \cup \{(0)_J, (1)_J\}$ is connected. We claim that the two end points of J belong to S. Since $A \subset D_X(1, S)$, then at least one end point of J belongs to S. Suppose to the contrary that J contains another end point q that does not belong to S. Since $(A - J) \cup \{(0)_J, (1)_J\}$ is connected, q is a ramification point of X. Let $T = S \cup J$. Then T is a fine subgraph of X and $A \subset D_X(1, T)$. By Lemma 3.1 (f), $\dim(\mathcal{M}_S) < \dim(\mathcal{M}_T)$. Since $A - A \cap N_X(1, S)$ is a finite set and A is nondegenerate, $A \cap N_X(1, S)$ is dense in A and $A \cap N_X(1, S) \subset A \cap N_X(1, T) \subset A$, then $A \cap N_X(1, T)$ is connected. Thus $A \in \mathcal{M}_T$. This implies that $\dim_A(C(X)) \geq \dim(\mathcal{M}_T) > n$ which is a contradiction. Therefore, both end points of J belong to S. Let $L = [(\frac{1}{3})_J, (\frac{2}{3})_J]$. It is easy to prove that the set $\mathcal{M} = \{D \in \mathcal{M}_S : D \cap J \subset [(0)_J, (\frac{1}{3})_J] \cup [(\frac{2}{3})_J, (1)_J]\}$ is homeomorphic to $[0, 1]^n$. Let $z = (\frac{1}{2})_J$. Let $\mathcal{K} = \{E \in C([(\frac{1}{3})_J, (\frac{2}{3})_J]) : z \in E\}$. Then $\mathcal{K}$ is homeomorphic to $[0, 1]^2$. Let $f : \mathcal{M} \times K \to C_2(X)$ be given by $f(D, E) = D \cup E$. Clearly, f is a one-to-one map and $f((A - J) \cup [(0)_J, (\frac{1}{3})_J] \cup [(\frac{2}{3})_J, (1)_J], L) = A$. Thus there is a $(n + 2)$-cell in $C_2(X)$ that contains A. Hence $\dim_A(C_2(X)) \geq n + 2$.

Let $A_0 = (A - J) \cup [(0)_J, (\frac{1}{3})_J] \cup [(\frac{2}{3})_J, (1)_J]$ and let $\alpha : [0, 1] \to C_2(X)$ be given by $\alpha(t) = A_0 \cup [(t\frac{1}{2} + (1 - t)(\frac{1}{3}))_J, (t\frac{1}{2} + (1 - t)(\frac{2}{3}))_J]$. Then $\alpha(0) = A$, for each $t > 0$, $\alpha(t)$ is disconnected and, by (a), $\dim_{\alpha(t)}(C_2(X)) = \dim_{A_0}(C(X)) + 2$. Notice that $A_0 \in \mathcal{M}_S$, this implies that $n \leq \dim_{A_0}(C(X))$. Since $A_0 \subset A$, we conclude that $\dim_{A_0}(C(X)) = n$ (Lemma 3.1 (e)). Therefore, $\dim_{\alpha(t)}(C_2(X)) = n + 2$ for each $t > 0$.

Let $B = \alpha(1)$.

Case 2. A does not contain cycles, $S \neq \emptyset$ and $A \neq S$. Then there exist a segment J of X and a number $t_0 > 0$ such that if $L = [(0)_J, (t_0)_J]$ and $A_1 = (A - L) \cup \{(0)_J\}$, then A_1 is a subcontinuum of X, $A = A_1 \cup L$ and $L \cap S = \{(0)_J\}$. If $(1)_J \in S$, there exists $t_1 \in [0, 1]$ such that $A = (A - J) \cup L \cup [(t_1)_J, (1)_J]$ and $t_0 < t_1$. Let

$$\mathcal{M}_1 = \begin{cases} \{D \in \mathcal{M}_S : D \cap J \subset [(0)_J, (\frac{t_0}{2})_J] \cup [(\frac{t_0 + t_1}{2})_J, (1)_J]\}, & \text{if } (1)_J \in S, \\ \{D \in \mathcal{M}_S : D \cap J \subset [(0)_J, (\frac{t_0}{2})_J]\}, & \text{if } (1)_J \notin S. \end{cases}$$

It is easy to check that $\mathcal{M}_1$ is homeomorphic to $[0, 1]^n$. Let $\mathcal{K} = \{E \in C([(\frac{t_0}{2})_J, (t_0)_J]) : (\frac{3t_0}{4})_J \in E\}$. Then $\mathcal{K}$ is homeomorphic to $[0, 1]^2$. Let $f : \mathcal{M}_1 \times \mathcal{K} \to C_2(X)$ be given by $f(D, E) = D \cup E$. Let $A_0 = A - ((\frac{t_0}{2})_J, (t_0)_J]$. Clearly, f is a one-to-one map and $f(A_0, [(\frac{t_0}{2})_J, (t_0)_J]) = A$. Thus there is a $(n + 2)$-cell in $C_2(X)$ that contains A. Hence $\dim_A(C_2(X)) \geq n + 2$.

Let $\alpha : [0, 1] \to C_2(X)$ be given by $\alpha(t) = A_0 \cup [(t(\frac{3t_0}{4}) + (1 - t)\frac{t_0}{2})_J, (t(\frac{3t_0}{4}) + (1 - t)t_0)_J]$.

Then $\alpha(0) = A$, for each $t > 0$, $\alpha(t)$ is disconnected and $\dim_{\alpha(t)}(C_2(X)) = \dim_{A_0}(C(X)) + 2$, by (a). Notice that $A_0 \in \mathcal{M}_1$, this implies that $n \leq \dim_{A_0}(C(X))$. Since $A_0 \subset A$, we conclude that $\dim_{A_0}(C(X)) = n$ (Lemma 3.1 (e)). Therefore, $\dim_{\alpha(t)}(C_2(X)) = n + 2$ for each $t > 0$.

Let $B = \alpha(1)$.

Case 3. $A = S$ (then A does not contain cycles and $S \neq \emptyset$). Let p be an end point of S if S is not degenerate, and let p be such that $S = \{p\}$, if

S is degenerate. Choose a segment J of X such that $p=(0)_J$ and J is not a segment of S. Let

$$\mathcal{M}_1=\begin{cases}\{D\in\mathcal{M}_S : D\cap J\subset\{p\}\cup[(\frac{2}{3})_J,(1)_J]\}, & \text{if } (1)_J\in S,\\ \{D\in\mathcal{M}_S : D\cap J=\{p\}\}, & \text{if } (1)_J\notin S.\end{cases}$$

It is easy to check that $\mathcal{M}_1$ is homeomorphic to $[0,1]^{n-1}$. Let $\mathcal{S}=\{(a,b)\in E^2 : 0\leq a\leq b\leq 1\}$ and let $\mathcal{T}$ be the cone over $\mathcal{S}$. Then $\mathcal{T}$ is a 3-cell. Let $f:\mathcal{M}_1\times\mathcal{T}\to C_2(X)$ be given by

$$\begin{aligned}&f(D,((a,b),t))=\\&\quad= D\cup[(0)_J,((1-t)/6)_J]\cup[((2+a)(1-t)/6)_J,((2+b)(1-t)/6)_J].\end{aligned}$$

Clearly, f is a well defined one-to-one map and $f(A,((0,0),1))=A$. Since $\mathcal{M}\times T$ is homeomorphic to $[0,1]^{n+2}$, we conclude that $\dim_A(C_2(X))\geq n+2$.

Let $\alpha:[0,1]\to C_2(X)$ be given by $\alpha(s)=f(A,((0,s),1-s))$. Then $\alpha(0)=A$, for each $t>0$, $\alpha(s)$ is disconnected and $\dim_{\alpha(s)}(C_2(X))=\dim_{A\cup[(0)_J,(s/6)_J]}(C(X))+2$, by (a). Notice that $A\cup[(0)_J,(s/6)_J]\in\mathcal{M}_S$, this implies that $n\leq\dim_{A\cup[(0)_J,(s/6)_J]}(C(X))$. Given a fine subgraph T of X such that $T\subset A\cup[(0)_J,(s/6)_J]$, we have that $T\subset S$. By Lemma 3.1 (f), $\dim(\mathcal{M}_T)\leq\dim(\mathcal{M}_S)=n$. Therefore, $\dim_{A\cup[(0)_J,(s/6)_J]}(C(X))=n$. Therefore, $\dim_{\alpha(s)}(C_2(X))=n+2$ for each $s>0$.

Let $B=\alpha(1)$.

Case 4. $S=\emptyset$. In this case $n=2$ and there exists a segment J of X such that $\mathcal{M}_S=\{D\in C(X) : D\subset J\}$. Since $\dim_A(C(X))=2$, A does not contain any ramification point of X and $A=[(a)_J,(b)_J]$ for some $0\leq a\leq b\leq 1$. By (b), $\dim_A(C_2(X))=4=\dim_A(C(X))+2$. If $A\neq J$, we may assume that $b<1$. In this subcase, let $\alpha:[0,1]\to C_2(X)$ be given by $\alpha(t)=A\cup\{(t(\frac{b+1}{2})+(1-t)b)_J\}$ and let $B=\alpha(1)$. If $A=J$, then X is an arc (remember that we are assuming that X is not a simple closed curve). In this subcase, let $\alpha:[0,1]\to C_2(X)$ be given by $\alpha(t)=\{(\frac{t}{2})_J\}\cup[(t)_J,(1)_J]$ and let $B=\alpha(1)$.

This completes the proof of (c).

(d) Let $A\in\mathcal{K}_p$ be such that A is disconnected, let $A=B\cup C$, where B and C are the components of A. Suppose that $p\in B$. Then C does not contain ramification points of X. On the other hand, the only two acyclic subgraphs of X that can be contained in B are $\emptyset$ and $\{p\}$. If q is a vertex of X such that $q\in B$ and $d(p,q)=1$, then q is not a ramification point of X, so q is an end point of X. This implies that $B\in\mathcal{M}_p$. Hence, $\dim_B(C(X))=\dim(\mathcal{M}_p)=n$ (By Lemma 3.1 (c) and (d)). Therefore, $\dim_A(C_2(X))=\dim_B(C(X))+\dim_C(C(X))=n+2$. Notice that $\mathcal{K}_p\cap C(X)\subset\mathcal{M}_p$, so $\dim(\mathcal{K}_p\cap C(X))=n$. Thus $\mathcal{K}_p$ can be put as the union of one closed (in $\mathcal{K}_p$) set $\mathcal{K}_p\cap C(X)$ and an open (in $\mathcal{K}_p$) set $\mathcal{K}_p-C(X)$, where both sets are of dimension less than or equal to $n+2$. Since open sets in metric spaces are F_σ, Theorem III 2 of Chapter III of [12] implies that $\dim(\mathcal{K}_p) = n+2$.

Let $\mathcal{K} = \{A \in C_2(X) : A$ does not have any ramification point of X different from $p\} = \mathcal{K}_p \cup \{A \in C_2(X) : A$ does not have any ramification point of $X\}$. Then $\mathcal{K}$ is an open subset of $C_2(X)$ such that it is the union of an $(n+2)$-dimensional closed (in $\mathcal{K}$) subset and a 4-dimensional open subset. This implies that $\dim_A(\mathcal{K}) \leq n+2$ for each $A \in \mathcal{K}$. Since $\mathcal{K}$ is open in $C_2(X)$, $\dim_A(C_2(X)) \leq n+2$ for each $A \in \mathcal{K}_p$.

Given $A \in \mathcal{K}_p$, $\dim_A(C_2(X)) \leq n+2$. In the case that A is disconnected, we have shown that $\dim_A(C_2(X)) = n+2$. In the case that A is connected, by (c) $\dim_A(C_2(X)) \geq \dim_A(C(X)) + 2$. Therefore, $\dim_A(C_2(X)) = n+2$.

(e) Since $\mathcal{D}_p \subset \mathcal{K}_p$, by (d), $C_2(X)$ has dimension equal to $n+2$ at each one of the elements of $\mathcal{D}_p$. Let $J_1, \ldots, J_s$ be the components of $C_p - \{p\}$. Since p is a ramification point, $s \geq 2$. For each $j \in \{1, \ldots, s\}$, let $\mathcal{C}_j = \{A \in \mathcal{D}_p : A$ has one component contained in $J_j\}$. Since each J_j is open in X, it follows that each $\mathcal{C}_j$ is open in $\mathcal{D}_p$. Thus $\mathcal{D}_p - (\mathcal{D}_p \cap C(X)) = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_s$ is a decomposition of $\mathcal{D}_p - (\mathcal{D}_p \cap C(X))$. On the other hand, $\mathcal{D}_p \cap C(X)$ is contained in $\mathcal{M}_p$. Since $\dim(\mathcal{M}_p) = n$, the proof of (e) is complete.

(f) First, we are going to see that it is enough to consider the case when A is disconnected. Thus, assume that A is connected. By (c), there exist $A_0 \in C_2(X)$ and a map $\beta : [0,1] \to C_2(X)$ such that A_0 is disconnected, A_0 has more that one ramification point of X and $\dim_{\beta(t)}(C_2(X)) = \dim_A(C(X)) + 2 \leq \dim_A(C_2(X))$ for each $t > 0$, $\beta(0) = A$ and $\beta(1) = A_0$. Hence, if we can prove (f) for the set A_0, combining the resulting map with β, we would have the corresponding map for A. Therefore, we may assume that A is disconnected.

Let B and C be the components of A. We analyze two cases:

Case 1. B has two different ramification points p and q of X. Since B is connected, we may assume that p and q are the end points of a segment J of X such that $J \subset B$. Fix a point $z \in C$ and let y be a point of X such that there is an arc λ joining z and y with $\lambda \cap B = \emptyset$ and $\lambda - \{z\}$ does not contain vertices of X. By [16, Theorem 15.3], there exists a map $\beta : [0,1] \to C(X)$ such that $\beta(0) = B$, $\beta(\frac{1}{2}) = J$, $\beta(1) = \{p\}$ and $\beta(t) \subset \beta(s)$ if $s \leq t$. Using [16, Theorem 15.3] again, there exists a map $\gamma : [0,1] \to C(X)$ such that $\gamma(0) = C$, $\beta(\frac{1}{2}) = \{z\}$, $\gamma(1) = \{y\}$, $\beta(t) \subset \beta(s)$ if $s \leq t \leq \frac{1}{2}$ and $\gamma(s) \subset \lambda - \{z\}$ for each $s \in (\frac{1}{2}, 1]$. Let $\alpha : [0,1] \to C_2(X)$ be given by $\alpha(s) = \beta(s) \cup \gamma(s)$. Notice that $\alpha(s)$ is disconnected for each s. If $s \leq t$, by (a) and Lemma 3.1 (e), $\dim_{\alpha(s)}(C_2(X)) = \dim_{\beta(s)}(C(X)) + \dim_{\gamma(s)}(C(X)) \geq \dim_{\beta(t)}(C(X)) + \dim_{\gamma(t)}(C(X)) = \dim_{\alpha(t)}(C_2(X))$. Since $\dim_{\alpha(\frac{1}{2})}(C_2(X)) = \dim_J(C(X)) + \dim_{\{z\}}(C(X)) > \dim_{\{p\}}(C(X)) + \dim_{\{y\}}(C(X)) = \dim_{\alpha(1)}(C_2(X))$, we have completed the proof of (f) for this case.

Case 2. B contains a ramification point p of X and C contains a ramification point q of X. Let y be a point of X such that there is an arc λ joining q and y with $\lambda \cap B = \emptyset$ and $\lambda - \{q\}$ does not contain vertices of X. By [16, Theorem 15.3], there exists a map $\beta : [0,1] \to C(X)$ such that $\beta(0) = B$, $\beta(1) = \{p\}$ and $\beta(t) \subset \beta(s)$ if $s \leq t$. Using [16, Theorem 15.3] again, there exists a map $\gamma : [0,1] \to C(X)$ such that $\gamma(0) = C$, $\gamma(\frac{1}{2}) = \{q\}$, $\gamma(1) = \{y\}$, $\gamma(t) \subset \gamma(s)$ if $s \leq t \leq \frac{1}{2}$ and $\gamma(s) \subset \lambda - \{q\}$ for each $s \in (\frac{1}{2}, 1]$. Let $\alpha : [0,1] \to C_2(X)$ be given by $\alpha(s) = \beta(s) \cup \gamma(s)$. Notice that $\alpha(s)$ is disconnected for each s. If $s \leq t$, by (a) and Lemma 3.1 (e), $\dim_{\alpha(s)}(C_2(X)) = \dim_{\beta(s)}(C(X)) + \dim_{\gamma(s)}(C(X)) \geq \dim_{\beta(t)}(C(X)) + \dim_{\gamma(t)}(C(X)) = \dim_{\alpha(t)}(C_2(X))$. Since $\dim_{\alpha(\frac{1}{2})}(C_2(X)) = \dim_{\beta(\frac{1}{2})}(C(X)) + \dim_{\{q\}}(C(X)) > \dim_{\{p\}}(C(X)) + \dim_{\{y\}}(C(X)) = \dim_{\alpha(1)}(C_2(X))$, we have completed the proof of (f) for this case. □

4. The main result

Theorem 4.1. *Finite graphs have unique hyperspace $C_2(X)$.*

Proof. Let X and Y be continua such that X is a finite graph (here, the continuum X could be a simple closed curve) and $C_2(X)$ is homeomorphic to $C_2(Y)$. By [20, Theorem 6.1], Y is locally connected. If Y is not a finite graph, then $\dim(C(Y)) = \infty$ ([23, Theorem 5.1]). Thus $\dim(C_2(X)) = \dim(C_2(Y)) = \infty$. By ([7]), $\dim(C(X))$ is finite. Let $n = \dim(C(X))$. By Lemma 3.2 (a), $\dim_A(C_2(X)) \leq 2n$ for each $A \in C_2(X) - C(X)$. Thus $C_2(X)$ is the union of the finite dimensional open (then F_σ) set $C_2(X) - C(X)$ and the finite dimensional closed set $C(X)$. By Theorem III 2, Chapter III of [12], $C_2(X)$ is finite dimensional. This contradiction proves that Y is a finite graph.

Let $h : C_2(X) \to C_2(Y)$ be a homeomorphism.

If X does not contain ramification points, $\dim_A(C_2(X)) = 4$ for each $A \in C_2(X)$, by Lemma 3.2 (b). This implies that $\dim_B(C_2(Y)) = 4$ for each $B \in C_2(Y)$. If Y contains a ramification point q such that $\mathrm{ord}(q) = n \geq 3$, by Lemma 3.2 (d), $\dim_{\{q\}}(C_2(Y)) = n + 2 \geq 5$. This contradiction proves that $C_2(Y)$ does not contain ramification points. Thus each of the continua X and Y is either homeomorphic to $[0,1]$ or to S^1. By Lemma 2.3, X is homeomorphic to Y.

Therefore, we may assume that each of the continua X and Y contains ramification points.

Claim 4.2. If $A \in C_2(X)$ and A contains exactly one ramification point p of X, then $B = h(A)$ contains exactly one ramification point of Y.

Notice that by Lemma 3.2 (d) $\dim_A(C_2(X)) \geq 5$. This implies that $\dim_B(C_2(Y)) \geq 5$. By Lemma 3.2 (b), B contains ramification points of Y.

Suppose that B contains more than one ramification point of Y. By Lemma 3.2 (f), there exists a map $\beta : [0,1] \to C_2(Y)$ such that $\beta(0) = B$,

$\dim_{\beta(s)}(C_2(Y)) \geq \dim_{\beta(t)}(C_2(Y))$ for every $s \leq t$, $\beta(1)$ contains exactly one ramification point of Y and $\dim_B(C_2(Y)) > \dim_{\beta(1)}(C_2(Y))$. Let $\alpha : [0,1] \to C_2(X)$ be given by $\alpha(s) = h^{-1}(\beta(s))$. Then $\alpha(0) = A$, $\dim_{\alpha(s)}(C_2(X)) \geq \dim_{\alpha(t)}(C_2(X))$ for every $s \leq t$, $\dim_A(C_2(X)) = \dim_B(C_2(Y)) > \dim_{\beta(1)}(C_2(Y)) = \dim_{\alpha(1)}(C_2(X))$ and $\dim_{\alpha(1)}(C_2(X)) = \dim_{\beta(1)}(C_2(X)) \geq 5$ (Lemma 3.2 (d)), so $\alpha(1)$ contains at least one ramification point of X (Lemma 3.2 (b)).

Let $n = \operatorname{ord}(p)$, let $\mathcal{K}_p = \{D \in C_2(X) : p \in D$ and p is the only ramification point of X that belongs to $D\}$. By Lemma 3.2 (d), $C_2(X)$ has dimension equal to $n+2$ at each one of the elements of $\mathcal{K}_p$. In particular, $\dim_A(C_2(X)) = n+2$. Since $\dim_A(C_2(X)) > \dim_{\alpha(1)}(C_2(X))$, $\alpha(1) \notin \mathcal{K}_p$. Since, for each $s \in [0,1]$, $\dim_{\alpha(s)}(C_2(X)) \geq \dim_{\alpha(1)}(C_2(X)) \geq 5$, by Lemma 3.2 (b), we conclude that $\alpha(s)$ contains ramification points of X for each $s \in [0,1]$. Let $\mathcal{K} = \{D \in C_2(X) : D$ contains ramification points of $X\}$. Then $\alpha(s) \in \mathcal{K}$ for each $s \in [0,1]$. Notice that $\mathcal{K}_p$ is an open subset of $\mathcal{K}$. Let $s_0 = \min\{s \in [0,1] : \alpha(s) \notin \mathcal{K}_p\}$. Since $A = \alpha(0) \in \mathcal{K}_p$, $s_0 > 0$. Notice that $\alpha(s_0) \notin \mathcal{K}_p$. Fix a sequence $\{s_n\}_{n=1}^{\infty}$ of elements in $[0, s_0)$ such that $s_n \to s_0$. Since $p \in \alpha(s_n)$ for each $n \geq 1$, $p \in \alpha(s_0)$. Thus, there exists another ramification point u of X such that $u \in \alpha(s_0)$.

In the case that $\alpha(s_0)$ has a component E such that $p, u \in E$, let S be a maximal fine subgraph X that is contained in E and contains p and u. Then $E \in \mathcal{M}_S$. By Lemmas 3.1 and 3.2, $\dim_A(C_2(X)) \geq \dim_{\alpha(s_0)}(C_2(X)) \geq \dim_E(C(X)) + 2 \geq \dim \mathcal{M}_S + 2 > \dim \mathcal{M}_p + 2 = n+2 = \dim_A(C_2(X))$. This contradiction shows that no component of $\alpha(s_0)$ can contain both points p and u. In particular, $\alpha(s_0)$ is disconnected.

Suppose that E and F are the components of $\alpha(s_0)$. By the paragraph above, we may assume that $p \in E$ and $u \in F$. By Lemma 3.2 (a), $\dim_A(C_2(X)) \geq \dim_{\alpha(s_0)}(C_2(X)) = \dim_E(C(X)) + \dim_F(C(X)) \geq \dim \mathcal{M}_p + \dim \mathcal{M}_u > n+2 = \dim_A(C_2(X))$. This contradiction completes the proof that B contains exactly one ramification point of Y.

By symmetry, we conclude that if $A \in C_2(X)$, then A contains exactly one ramification point of X if and only if $h(A)$ contains exactly one ramification point of Y.

Given a ramification point p of X, let $\mathcal{K}_p = \{A \in C_2(X) : p \in A$ and p is the only ramification point of X that belongs to $A\}$. For a ramification point q of Y, define the corresponding set $\mathcal{K}'_q$ in $C_2(Y)$. Let C_p be the component of $X - (R(X) - \{p\})$ that contains p, let $\mathcal{D}_p = \{A \in C_2(X) : p \in A \subset C_p\}$. Let $J_1, \ldots, J_r$ be the segments of X which do not contain p. For each $i \in \{1, \ldots, r\}$, let $\mathcal{D}_i = \{A \in C_2(X) : p \in A,\ A \subset C_p \cup \operatorname{int}_X(J_i),\ A \cap C_p \neq \emptyset$ and $A \cap \operatorname{int}_X(J_i) \neq \emptyset\}$.

Claim 4.3. The components of $\mathcal{K}_p$ are the sets $\mathcal{D}_p, \mathcal{D}_1, \ldots, \mathcal{D}_r$.

Let $A \in \mathcal{K}_p$. If A is connected, then A is a connected subset of $X - (R(X) - \{p\})$ that contains p. Thus $A \subset C_p$, so $A \in \mathcal{D}_p$. Now, suppose that A is not connected and A is not contained in C_p. Let B and C be the components of A, where $p \in B$. Then $B \subset C_p$. This implies that C is not contained in C_p and C does no contain any ramification point of X. Thus C is contained in the interior of some segment J of X. If $p \in J$, then $C \subset \text{int }_X(J) \subset C_p$ which is a contradiction. Hence, $p \notin J$ and $J = J_i$ for some $i \in \{1, \ldots, r\}$. Therefore, $A \in \mathcal{D}_i$. We have proved that $\mathcal{K}_p \subset \mathcal{D}_p \cup \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_r$.

Since the unique ramification point in the set $C_p \cup \text{int}_X(J_1) \cup \cdots \cup \text{int}_X(J_r)$ is p, it follows that $\mathcal{D}_p \cup \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_r \subset \mathcal{K}_p$. Therefore, $\mathcal{K}_p = \mathcal{D}_p \cup \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_r$.

Since each one of the sets C_p, $\text{int}_X(J_1), \ldots, \text{int}_X(J_r)$ is open in X, each one of the sets $\mathcal{D}_p, \mathcal{D}_1, \ldots, \mathcal{D}_r$ is open in $\mathcal{K}_p$.

Let $i \in \{1, \ldots, r\}$. Since $C_p \cap \text{int}_X(J_i) = \emptyset$, $\mathcal{D}_p \cap \mathcal{D}_i = \emptyset$. If $j \neq i$, then $(C_p \cup \text{int}_X(J_i)) \cap \text{int}_X(J_j) = \emptyset$. This implies that $\mathcal{D}_i \cap \mathcal{D}_j = \emptyset$. Therefore, the sets $\mathcal{D}_p, \mathcal{D}_1, \ldots, \mathcal{D}_r$ are pairwise disjoint.

For each $i \in \{1, \ldots, r\}$, choose a point $p_i \in \text{int}_X(J_i)$.

Notice that $\{p, p_i\} \in \mathcal{D}_i$. Since $\{p\} \in \mathcal{D}_p$, we conclude that the sets $\mathcal{D}_p, \mathcal{D}_1, \ldots, \mathcal{D}_r$ are nonempty.

Let $i \in \{1, \ldots, r\}$. Let $A \in \mathcal{D}_i$. Since A intersects the disjoint open sets C_p and $\text{int}_X(J_i)$ and it is contained in their union, A is disconnected. Let B and C be the components of A. Suppose that $p \in B$. Then $B \subset C_p$ and $C \subset \text{int}_X(J_i)$. Let $\alpha : [0,1] \to C(X)$ be a map such that $\alpha(0) = \{p_i\}$, $\alpha(1) = C$ and $\alpha(s) \subset \text{int}_X(J_i)$ for each $s \in [0,1]$, such a map α can be constructed by using an arc joining p_i to a point c of C inside $\text{int}_X(J_i)$ and then taking an order arc ([16, Theorem 15.3]) from $\{c\}$ to C. Let $\beta : [0,1] \to C(X)$ be an order arc from $\{p\}$ to B. Then the map $\gamma : [0,1] \to C_2(X)$ given by $\gamma(s) = \alpha(s) \cup \beta(s)$ is a path that joins the set $\{p, p_i\}$ to A in the set $\mathcal{D}_i$. Therefore, $\mathcal{D}_i$ is connected.

In a similar way it can be proved that any element in $\mathcal{D}_p$ can be joined by a path in $\mathcal{D}_p$ to the element $\{p\}$.

This completes the proof that the components of $\mathcal{K}_p$ are the sets $\mathcal{D}_p$, $\mathcal{D}_1, \ldots, \mathcal{D}_r$.

Claim 4.4. If $i \in \{1, \ldots, r\}$ and $\text{ord}(p) = n$, then $\mathcal{D}_i$ contains an open (in $C_2(X)$) dense (in $\mathcal{D}_i$) subset $\mathcal{U}$ which is homeomorphic to $(0,1)^{n+2}$.

It can be shown that there exists a homeomorphism $f : [0,1]^n \to \mathcal{M}_p$ such that $f((0,1)^n)$ is an open subset of $C(X)$ and, for each $z \in (0,1)^n$, $f(z)$ does not contain any vertex of X different from p, then $f(z) \subset C_p$. Let $\mathcal{E} = [0,1]^2$, if J_i is not a loop and let $\mathcal{E} = [0,1]^2/\{(0,0),(1,0)\}$ (the continuum obtained from $[0,1]^2$, by identifying $(0,0)$ and $(1,0)$), if J_i is a loop. Let $\mathcal{W} = \{A \in C(J_i) : A - \{(0)_{J_i}, (1)_{J_i}\} \text{ is connected}\}$. Notice that if J_i is not a loop, then $\mathcal{W} = C(J_i)$, so $\mathcal{W}$ is homeomorphic to $[0,1]^2 = \mathcal{E}$, and if J_i

is a loop, then $\mathcal{W}$ is homeomorphic to $[0,1]^2/\{(0,0),(1,0)\} = \mathcal{E}$. Therefore, it is possible to define a homeomorphism $g : \mathcal{E} \to \mathcal{W}$ such that $g((0,1)^2) = \mathcal{E}_i$, where $\mathcal{E}_i = \{E \in C(X) : E \subset J_i - \{(0)_{J_i}, (1)_{J_i}\}$ and E is nondegenerate$\}$. Notice that $\mathcal{E}_i$ is open in $C(X)$. Let $\mathcal{U} = \{D \cup E : D \in f((0,1)^n)$ and $E \in \mathcal{E}_i\}$. Then $\mathcal{U}$ is an open subset of $C_2(X)$. Let $\psi_1 : [0,1]^n \times (0,1)^2 \to \mathcal{C}_2(X)$ be given by $\psi_1(s,t) = f(s) \cup g(t)$. Then ψ_1 is one-to-one and continuous. Let $\psi = \psi_1|(0,1)^n \times (0,1)^2 : (0,1)^n \times (0,1)^2 \to \mathcal{U}$. In order to check that ψ^{-1} is continuous, take and element $D \cup E \in \mathcal{U}$, where $D = f(s) \in f((0,1)^n)$ and $E \in \mathcal{E}_i$. Let E_0 be a subarc of J_i such that $E \subset \mathrm{int}_X(E_0)$. Then $\mathcal{G} = \{G \in C(E_0) : \mathrm{diameter}(G) \geq (\mathrm{diameter}(E_0))/2\}$ is a closed neighborhood of E. Let $\phi = \psi_1|[0,1]^n \times g^{-1}(\mathcal{G}) : [0,1]^n \times g^{-1}(\mathcal{G}) \to C_2(X)$. Then ϕ is a one-to-one continuous function. Thus ϕ is a homeomorphism on its image which is a closed neighborhood of $D \cup E$. Hence $\phi^{-1}|\phi([0,1]^n \times g^{-1}(\mathcal{G})) \cap \mathcal{U} = \psi^{-1}|\psi_1([0,1]^n \times g^{-1}(\mathcal{G})) \cap \mathcal{U}$ is continuous. Since $\phi(([0,1]^n \times g^{-1}(\mathcal{G})) \cap \mathcal{U}$ is a neighborhood of $D \cup E$, we conclude that ψ^{-1} is continuous. Therefore, ψ is a homeomorphism.

Let $A \in \mathcal{D}_i$, let B and D be the components of A, where $p \in B$. Then $B \subset C_p$ and $D \subset \mathrm{int}_X(J_i)$. Since B is connected, $p \in B \subset C_p$, then $B \subset D_X(1, \{p\})$ and $B \cap N_X(1, \{p\})$ is equal to B minus a (possibly empty) subset of end points of X. Thus $B \cap N_X(1, \{p\})$ is connected. Hence, $B \in \mathcal{M}_p$ thus $B \in \mathrm{cl}_{C(X)}(f((0,1)^n)$. On the other hand $D \in \mathcal{E}_i$ or D is a degenerate subset of $\mathrm{int}_X(J_i)$. In any case $D \in \mathrm{cl}_{C(X)}(\mathcal{E}_i)$. This implies that $A \in \mathrm{cl}_{C_2(X)}(\mathcal{U})$. Therefore, the proof of Claim 4.4 is complete.

Claim 4.5. For each ramification point p of X, there exists a (unique) ramification point q of Y such that $h(\mathcal{D}_p) = \mathcal{D}'_q$. Symmetrically, for each ramification point q of Y, there exists a (unique) ramification point p of X such that $h(\mathcal{D}_p) = \mathcal{D}'_q$. Furthermore, $\mathrm{ord}_X(p) = \mathrm{ord}_Y(q)$. Therefore, there is a bijection between $R(X)$ and $R(Y)$.

Let $n = \mathrm{ord}_X(p)$. Let $B = h(\{p\})$. Then, by Claim 4.2, B has exactly one ramification point q of Y, with $\mathrm{ord}_Y(q) = n'$. By Lemma 3.2 (d), $\dim_{\{p\}}(C_2(X)) = n + 2$ and $\dim_B(C_2(Y)) = n' + 2$. Thus $n = n'$. Let C'_q be the component of $Y - (R(Y) - \{q\})$ that contains q, let $\mathcal{D}'_q = \{D \in C_2(Y) : q \in D \subset C'_q\}$. Let $L_1, \ldots, L_s$ be the segments of Y that do not contain q. For each $j \in \{1, \ldots, s\}$, let $\mathcal{D}'_j = \{D \in C_2(Y) : q \in D,\ D \subset C'_q \cup\ \mathrm{int}_Y(L_j)$, $D \cap C'_q \neq \emptyset$ and $D \cap\ \mathrm{int}_Y(L_j) \neq \emptyset\}$.

Let $R(Y) = \{q_1, q_2, \ldots, q_m\}$, where $q_1 = q$. By Claim 4.3, $h(\mathcal{D}_p)$ is a connected subset of $\mathcal{K}'_{q_1} \cup \ldots \cup \mathcal{K}'_{q_s}$, the sets $h(\mathcal{D}_p) \cap \mathcal{K}'_{q_j}$ are pairwise disjoint and open in $h(\mathcal{D}_p)$ and $h(\{p\}) \in \mathcal{K}'_q$. Thus $h(\mathcal{D}_p) \subset \mathcal{K}'_q$. By Claim 4.3, the components of $\mathcal{K}'_q$ are $\mathcal{D}'_q$, $\mathcal{D}'_1, \ldots, \mathcal{D}'_s$. Hence, $h(\mathcal{D}_p) \subset \mathcal{D}'_t$ for some $t \in \{q, 1, \ldots, s\}$. With a similar reasoning as before, we conclude that $h^{-1}(\mathcal{D}'_t) \subset \mathcal{D}_p$. Therefore, $h(\mathcal{D}_p) = \mathcal{D}'_t$.

Suppose that $t \neq q$. By Claim 4.4, $\mathcal{D}'_t$ contains an open (in $C_2(Y)$) dense (in $\mathcal{D}'_t$) subset $\mathcal{U}'$ which is homeomorphic to $(0,1)^{n+2}$. Let $\mathcal{U} = h^{-1}(\mathcal{U}') \subset \mathcal{D}_p$. Then $\mathcal{U}$ is open in $C_2(X)$, $\mathcal{U}$ is dense in $\mathcal{D}_p$ and $\mathcal{U}$ is homeomorphic to $(0,1)^{n+2}$. By Lemma 3.2 (e) there exists a closed (in $\mathcal{D}_p$) subset $\mathcal{H}$ of $\mathcal{D}_p$ such that $\dim(\mathcal{H}) = n$ and $\mathcal{D}_p - \mathcal{H} = V \cup W$, where $\mathcal{V}$ and $\mathcal{W}$ are nonempty disjoint open (in $\mathcal{D}_p$) subsets of $\mathcal{D}_p$. Then $\mathcal{U} - (U \cap H) = (U \cap V) \cup (U \cap W)$ is a disconnection of $\mathcal{U} - (U \cap H)$. Therefore $\mathcal{U}$, and then $(0,1)^{n+2}$, can be separated by a subset of dimension less than or equal to n. This contradicts Theorem IV 4 of [12] and proves that $t = q$.

Therefore, $h(\mathcal{D}_p) = \mathcal{D}'_q$. This proves Claim 4.5.

For each ramification point p of X, put $k(p) = q$, where q is the unique ramification point of Y such that $h(\mathcal{D}_p) = \mathcal{D}'_q$. Then k is a bijection between $R(X)$ and $R(Y)$.

Claim 4.6. Let $p, x \in R(X)$. Then p and x are adjacent vertices of X if and only if $k(p)$ and $k(x)$ are adjacent vertices of Y.

By symmetry, we only need to prove the necessity of Claim 4.6. If p and x are adjacent vertices of X, then $\{p,x\} \in \mathrm{cl}_{C_2(X)}(\mathcal{D}_p) \cap \mathrm{cl}_{C_2(X)}(\mathcal{D}_x)$. Thus $F = h(\{p,x\}) \in \mathrm{cl}_{C_2(Y)}(\mathcal{D}'_{k(p)}) \cap \mathrm{cl}_{C_2(Y)}(\mathcal{D}'_{k(x)})$. Then there exists a sequence $\{F_i\}_{i=1}^{\infty}$ such that $F_i \to F$ and $F_i \in \mathcal{D}'_{k(p)}$ for each $i \geq 1$. Since $k(p) \in F_i \subset C'_{k(p)} \subset D_Y(1, \{k(p)\})$ for each $i \geq 1$, $k(p) \in F \subset D_Y(1, \{k(p)\})$. Similarly, $k(x) \in F \subset D_Y(1, \{k(x)\})$. Thus $k(p) \in D_Y(1, \{k(x)\})$. This implies that $d_Y(k(p), k(x)) = 1$. Therefore, $k(p)$ and $k(x)$ are adjacent vertices of Y.

Let $\mathcal{R} = \{A \in C_2(X) : \dim_A(C_2(X)) = 4\}$ and $\mathcal{R}' = \{B \in C_2(Y) : \dim_B(C_2(Y)) = 4\}$. Notice that $h(\mathcal{R}) = \mathcal{R}'$. Given two segments J and L of X, let $\mathcal{R}(\{J,L\}) = \{A \in C_2(X) : A \subset \mathrm{int}_X(J) \cup \mathrm{int}_X(L)$, $A \cap J \neq \emptyset$ and $A \cap L \neq \emptyset\}$. Notice that in the case that $J \neq L$, $A \in \mathcal{R}(\{J,L\})$ implies that A is disconnected.

Claim 4.7. The components of $\mathcal{R}$ are the sets of the form $\mathcal{R}(\{J,L\})$ and if $\mathcal{R}(\{J,L\}) = \mathcal{R}(\{J_1,L_1\})$, then $\{J,L\} = \{J_1,L_1\}$.

Combining (b), (d) and (f) of Lemma 3.2, it follows that $\mathcal{R} = \{A \in C_2(X) : A$ does not have ramification points of $X\}$. This implies that $\mathcal{R} = \bigcup\{\mathcal{R}(\{J,L\}) : J$ and L are segments of $X\}$. It is easy to prove that each set of the form $\mathcal{R}(\{J,L\})$ is open in $C_2(X)$. Now suppose that there exists an element $A \in \mathcal{R}(\{J,L\}) \cap \mathcal{R}(\{J_1,L_1\})$. Since $\emptyset \neq A \cap J \subset \mathrm{int}_X(J_1) \cup \mathrm{int}_X(L_1)$. This implies that $J \cap \mathrm{int}_X(J_1) \neq \emptyset$ or $J \cap \mathrm{int}_X(L_1) \neq \emptyset$. Thus $J = J_1$ or $J = L_1$. Hence $J \in \{J_1, L_1\}$. Using similar arguments, it follows that $\{J,L\} = \{J_1,L_1\}$. In particular the sets of the form $\mathcal{R}(\{J,L\})$ are pairwise disjoint. It is easy to check that each set of the form $\mathcal{R}(\{J,L\})$ is connected. This completes the proof of Claim 4.7.

Claim 4.8. Let p be a ramification point of X and let J and L be segments of X, then $\mathcal{D}_p \cap \operatorname{cl}_{C_2(X)}\mathcal{R}(\{J,L\}) \neq \emptyset$ if and only if $p \in J \cap L$.

Suppose that $A \in \mathcal{D}_p \cap \operatorname{cl}_{C_2(X)}\mathcal{R}(\{J,L\})$. Then $\emptyset \neq A \cap J \subset C_p \cap J$. This implies that $p \in J$. Similarly, $p \in L$.

Now, suppose that $p \in J \cap L$. Fix a point $x \in \operatorname{int}_X(J)$ and choose a sequence $\{p_n\}_{n=1}^{\infty}$ in $\operatorname{int}_X(L)$ such that $p_n \to p$. Then $\{p,x\} \in \mathcal{D}_p$, $\{p_n,x\} \to \{p,x\}$ (in $C_2(X)$) and $\{p_n,x\} \in \mathcal{R}(\{J,L\})$ for each $n \geq 1$. Hence $\{p,x\} \in \mathcal{D}_p \cap \operatorname{cl}_{C_2(X)}\mathcal{R}(\{J,L\})$.

Claim 4.9. Let p and p_1 be adjacent ramification points of X. Suppose that the number of segments of X joining p and p_1 is equal to S. Then the number of segments of Y joining $k(p)$ and $k(p_1)$ is equal to S.

Let J and L be segments of X. Then J and L join p and p_1 if and only if $p, p_1 \in J \cap L$. By Claim 4.8, this is equivalent to $\mathcal{D}_p \cap \operatorname{cl}_{C_2(X)}\mathcal{R}(\{J,L\}) \neq \emptyset$ and $\mathcal{D}_{p_1} \cap \operatorname{cl}_{C_2(X)}\mathcal{R}(\{J,L\}) \neq \emptyset$. This implies that the number of components of $\mathcal{R}$ whose closure intersects both sets $\mathcal{D}_p$ and $\mathcal{D}_{p_1}$ is $S + \binom{S}{2}$.

Let S' be the number of segments of Y joining $k(p)$ and $k(p_1)$. Since h is a homeomorphism, $h(\mathcal{R}) = \mathcal{R}', h(\mathcal{D}_p) = \mathcal{D}'_{k(p)}$ and $h(\mathcal{D}_{p_1}) = \mathcal{D}'_{k(p_1)}$, we conclude that $S + \binom{S}{2} = S' + \binom{S'}{2}$. This implies that $S = S'$ and completes the proof of Claim 4.9.

Claim 4.10. Let p be a ramification point of X such that $\operatorname{ord}(p) = n$. Suppose that the number of loops of X (resp., Y) containing p (resp., $k(p)$) is N (resp., N'), the number of end points of X (resp., Y) adjacent to p (resp., $k(p)$) is M (resp., M') and the number of segments in X (resp., Y) that joins p (resp., $k(p)$) to another ramification point of X (resp., Y) is R (resp., R'). Then $N = N'$, $M = M'$ and $R = R'$.

Let $q = k(p)$. We know that $\operatorname{ord}_Y(q) = n$. By Claims 4.6 and 4.9, $R = R'$. Notice that $n = 2N + M + R$ and $n = 2N' + M' + R'$. Thus $2N + M = 2N' + M'$. On the other hand, the number of components of $\mathcal{R}$ whose closure intersects $\mathcal{D}_p$ is $N + M + R + \binom{N+M+R}{2}$ and the number of components of $\mathcal{R}'$ whose closure intersects $\mathcal{D}'_{k(p)}$ is $N'+M'+R'+\binom{N'+M'+R'}{2}$. This implies that $N + M + R + \binom{N+M+R}{2} = N' + M' + R' + \binom{N'+M'+R'}{2}$. Thus $N + M = N' + M'$. Therefore, $N = N'$ and $M = M'$.

We are ready to prove that the graphs X and Y are equivalent as graphs and then they are homeomorphic continua.

Given two different adjacent ramification points p and x in X, let $\mathcal{A}(p,x) = \{J : J$ is a segment of X and J joins p and $x\}$ and let $\mathcal{A}'(p,x) = \{L : L$ is a segment of Y and L joins $k(p)$ and $k(x)\}$. By Claim 4.9, we can choose a bijection $k(p,x)$ from $\mathcal{A}(p,x)$ onto $\mathcal{A}'(p,x)$. Given a ramification point p of X, let $\mathcal{B}(p) = \{J : J$ is a loop of X and $p \in J\}$, $\mathcal{B}'(p) = \{L : L$ is a loop of Y and $k(p) \in L\}$, $\mathcal{C}(p) = \{J : J$ is a segment of X that joins p and an end point of $X\}$ and $\mathcal{C}'(p) = \{L : L$ is a segment of Y that joins $k(p)$ and an end point of $Y\}$. By Claim 4.10, it is possible to choose bijections $k_1(p) : \mathcal{B}(p) \to \mathcal{B}'(p)$ and $k_2(p) : \mathcal{C}(p) \to \mathcal{C}'(p)$.

Let $\mathcal{S}(X)$ (resp., $\mathcal{S}(Y)$) be the set of segments of X (resp., Y). Since varying the points p and x we obtain disjoint sets $\mathcal{A}(p,x)$, $\mathcal{B}(p)$ and $\mathcal{C}(p)$ and the union of all of them is $\mathcal{S}(X)$, we can define a common extension $K : \mathcal{S}(X) \to \mathcal{S}(Y)$ of all the functions of the form $k(p,x)$, $k_0(p)$ and $k_1(p)$, and K is a bijection.

Let $\mathcal{V}(X)$ (resp., $\mathcal{V}(Y)$) be the set of vertices of X (resp., Y). Now, we extend the function k (defined on the ramification points of X) to $\mathcal{V}(X)$. Given an end point x of X, there exists a segment J of X that joins x and a ramification point p of X. Then $K(J)$ contains exactly one end point y of Y. Then define $k(x) = y$. Hence k is a bijection.

Therefore, we have defined a bijection $K : \mathcal{S}(X) \to \mathcal{S}(Y)$ and a bijection $k : \mathcal{V}(X) \to \mathcal{V}(Y)$ such that $p \in J$ if and only if $k(p) \in K(J)$.

This proves that the graphs X and Y are isomorphic as graphs. Therefore, X is homeomorphic to Y. □

5. Questions

Question 5.1. Is $C_n([0,1])$ not homeomorphic to $C_n(S^1)$ for $n \geq 3$?

Question 5.2. Do finite graphs have unique hyperspace $C_n(X)$ for $n \geq 3$?

Question 5.3. ([6]) Do finite graphs have unique hyperspace $F_n(X)$ for $n \geq 4$?

Question 5.4. Do hereditarily indecomposable continua have unique hyperspace $F_2(X)$? And the same question for $F_n(X)$, with $n \geq 3$?

By Lemma 2.2, we can say that a model for $C_2([0,1])$ is $[0,1]^4$. This is the motivation for the following problem.

Problem 5.5. Find models for $C_3([0,1])$ and $C_2(S^1)$.

Acknowledgements.
The author wishes to thank the referee for his/her careful revision.

Added in Proof. Question 5.2 has been recently answered, in the positive, by the author and Question 5.3 has been answered, in the positive, by the author and E. Castañeda. With respect to Problem 5.5, the author has shown that $C_2(S^1)$ is homeomorphic to the cone over the solid torus.

References

[1] G. Acosta, Continua with unique hyperspace, Lecture notes in pure and applied mathematics 230, 33-49, Marcel Dekker, Inc., 2002.

[2] G. Acosta, *On compactifications of the real line and unique hyperspace*, Topology Proc. **25** (2000), 1–25.

[3] G. Acosta, *Continua with almost unique hyperspace*, Topology Appl. **117** (2002), 175–189.

[4] G. Acosta, *On smooth fans and unique hyperspace*, to appear in Houston J. Math.

[5] G. Acosta, J. J. Charatonik and A. Illanes, *Irreducible continua of type λ with almost unique hyperspace*, Rocky Mountain J. Math. **31** (2001), 745–772.

[6] E. Castañeda, *The second and third symmetric products for finite graphs are unique*, preprint.

[7] R. Duda, *On the hyperspace of subcontinua of a finite graph, I*, Fund. Math. **62** (1968), 265–286.

[8] R. Duda, *On the hyperspace of subcontinua of a finite graph, II*, Fund. Math. **63** (1968), 225–255.

[9] R. Duda, *Correction to the paper "On the hyperspace of subcontinua of a finite graph, I"*, Fund. Math. **69** (1970), 207–211.

[10] C. Eberhart and S. B. Nadler, Jr., *Hyperspaces of cones and fans*, Proc. Amer. Math. Soc. **77** (1979), 279–288.

[11] S. Eilenberg, *Transformations continues en circonférence et la topologie du plan*, Fund. Math. **26** (1936), 61–112.

[12] W. Hurewicz and H. Wallman, Dimension Theory, Princeton University Press, Princeton, New Jersey, 1948.

[13] A. Illanes, *Chainable continua are not C-determined*, Topology Appl. **98** (1999), 211–216.

[14] A. Illanes, *Fans are not C-determined*, Colloq. Math. **81** (1999), 299–308.

[15] A. Illanes, *Dendrites with unique hyperspace $F_2(X)$*, JP Jour. Geometry and Topology **2** No. 1 (2002), 75–96.

[16] A. Illanes and S. B. Nadler, Jr., Hyperspaces, Fundamentals and Recent Advances, Monographs and Textbooks in Pure and Applied Math., Vol. 216, Marcel Dekker, Inc., New York, N.Y., 1978.

[17] A. Illanes and R. Torres, *The dimension of Whitney leveles of a finite graph*, Glasnik Mat. **32** (1997), 263–273.

[18] S. Macías, *On C-determined continua*, Glasnik Mat. **32** (1997), 259–262.

[19] S. Macías, *Hereditarily indecomposable continua have unique hyperspace 2^X*, Bol. Soc. Mat. Mexicana **5** (1999), 415–418.

[20] S. Macías, *On the hyperspaces $C_n(X)$ of a continuum X*, Topology Appl. **109** (2001), 237–256.

[21] S. Macías, *On the hyperspaces $C_n(X)$ of a continuum X II*, Topology Proc. **25** (2000), 255–276.

[22] S. Mardešić, *Equivalence of singular and Čech homology for ANR's. Application to unicoherence.* Fund. Math. **46** (1958), 29–45.

[23] S. B. Nadler, Jr., Hyperspaces of sets, Monographs and Textbooks in Pure and Applied Math. Vol. 49, Marcel Dekker, Inc., New York, N.Y., 1978.

[24] S. B. Nadler, Jr., Continuum Theory, Monographs and Textbooks in Pure and Applied Math. Vol. 158, Marcel Dekker, Inc., New York, N.Y., 1992.

Instituto de Matemáticas
Circuito Exterior
Cd. Universitaria
México, 04510, México
E-mail address: illanes@matem.unam.mx

Received: 12.10.2000.
Revised: 07.11.2001.