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SMOOTHNESS IN n–FOLD HYPERSPACES

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Abstract. We prove that $\mathcal{C}^*$–smoothness of a homogeneous continuum implies its indecomposability. We define the analogue of $\mathcal{C}^*$–smoothness for n–fold hyperspaces and investigate its relation to $\mathcal{C}^*$–smoothness. We characterize the class of hereditarily indecomposable continua in terms of $\mathcal{C}_n^*$–smoothness.

1. Introduction

The notion of $\mathcal{C}^*$–smoothness was defined by Sam B. Nadler, Jr., in 1978 [6, (15.5)] and the notion of absolute $\mathcal{C}^*$–smoothness was defined by Grispolakis and Tymchatyn [3, p. 177]. We extend these concepts to n–fold hyperspaces.

In section 2, we study $\mathcal{C}_n^*$–smoothness and its relation with $\mathcal{C}^*$–smoothness. One of our main results characterize hereditarily indecomposable continua (Theorem 2.6). In section 3, we present results about points at which a continuum X is $\mathcal{C}_n^*$–smooth and we show a connection between $\mathcal{C}_n^*$–smoothness and indecomposability. In section 4, we give an affirmative answer to 15.21 of [6]; we also include a result characterizing when each subcontinuum of a continuum X is absolutely $\mathcal{C}^*$–smooth.

If (Y, d) is a metric space, then given $A \subset Y$ and $\varepsilon > 0$, the open ball about A of radius ε is denoted by $\mathcal{V}_\varepsilon(A)$, the interior of A is denoted by $\operatorname{int}(A)$, and the closure of A is denoted by $\overline{A}$.

A *continuum* is a compact connected metric space.

Given a continuum X and a positive integer n, we define its *n–fold hyperspace* as the set $\mathcal{C}_n(X)$ consisting of all nonempty closed subsets of X having

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at most n components. We consider the n–fold hyperspaces topologized with the *Hausdorff metric* [6, (0, 1)]. The Hausdorff metric will be denoted by $\mathcal{H}$. For a given continuum X, $\mathcal{F}_1(X)$ denotes the hyperspace of singletons of X.

For a continuum X and a positive integer n, an *order arc* is a one–to–one continuous function $\alpha\colon [0,1] \to \mathcal{C}_n(X)$ such that $\alpha(s) \subset \alpha(t)$ if $s < t$.

Throughout this paper, n denotes a positive integer. The new concepts in this paper are defined at appropriate places. Definitions of known concepts can be found in either [6] or [7].

2. $\mathcal{C}_n^*$–SMOOTHNESS

Recall that a continuum X is said to be *$\mathcal{C}^*$–smooth at a subcontinuum A of X,* provided that for any sequence $\{A_k\}_{k=1}^{\infty}$ of subcontinua of X converging to A, the sequence of hyperspaces $\{\mathcal{C}(A_k)\}_{k=1}^{\infty}$ converges to $\mathcal{C}(A)$; i. e., the map $\mathcal{C}^*\colon \mathcal{C}(X) \to \mathcal{C}(\mathcal{C}(X))$ is continuous at A. A continuum X *is $\mathcal{C}^*$–smooth* if it is $\mathcal{C}^*$–smooth at each element of $\mathcal{C}(X)$; i. e., $\mathcal{C}^*$ is continuous.

We generalize $\mathcal{C}^*$–smoothness to the n–fold hyperspaces as follows: *X is $\mathcal{C}_n^*$–smooth at $A \in \mathcal{C}_n(X)$,* provided that for any sequence $\{A_k\}_{k=1}^{\infty}$ of elements of $\mathcal{C}_n(X)$ converging to A, the sequence $\{\mathcal{C}_n(A_k)\}_{k=1}^{\infty}$ of hyperspaces converges to $\mathcal{C}_n(A)$; i. e., the map $\mathcal{C}_n^*\colon \mathcal{C}_n(X) \to 2^{2^X}$ is continuous at A. A continuum X *is $\mathcal{C}_n^*$–smooth* if it is $\mathcal{C}_n^*$–smooth at each element of $\mathcal{C}_n(X)$; i. e., $\mathcal{C}_n^*$ is continuous.

The following lemma is easy to prove.

LEMMA 2.1. *Let X be a continuum and let $\{A_k\}_{k=1}^{\infty}$ be a sequence in $\mathcal{C}_n(X)$ converging to A. If $\lim_{k\to\infty} \mathcal{C}_n(A_k)$ exists for a given n, then $\lim_{k\to\infty} \mathcal{C}_n(A_k) \subset \mathcal{C}_n(A)$.*

Before we study the continuity of $\mathcal{C}_n^*$ on all of $\mathcal{C}_n(X)$, we prove a theorem about the continuity of $\mathcal{C}_n^*$ restricted to $\mathcal{C}(X)$.

THEOREM 2.2. *Let X be a continuum. If A is a subcontinuum of X, then the following statements are equivalent:*

1) *X is $\mathcal{C}^*$–smooth at A;*
2) *$\mathcal{C}_n^*|_{\mathcal{C}(X)}$ is continuous for all n;*
3) *$\mathcal{C}_n^*|_{\mathcal{C}(X)}$ is continuous for some n.*

PROOF. Assume 1), and let $n \geq 2$. We prove 2). Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of subcontinua of X converging to A. Let B be any element of $\mathcal{C}_n(A)$. Let $B_1, \dots, B_\ell$ ($\ell \leq n$) be the components of B. Hence each B_j is a subcontinuum of A, $j \in \{1, \dots, \ell\}$. Since X is $\mathcal{C}^*$–smooth at A, there exist subcontinua $B_k^1, \dots, B_k^\ell$ of A_k for each $k \in \mathbb{N}$ such that $\lim_{k\to\infty} B_k^j = B_j$ for each $j \in \{1, \dots, \ell\}$. Hence, $B_k = \cup_{j=1}^{\ell} B_k^j$ is an element of $\mathcal{C}_n(A_k)$ for each $k \in \mathbb{N}$, and $\lim_{k\to\infty} B_k = B$ [6, (1.48)]. Therefore, $\mathcal{C}_n(A) \subset \lim_{k\to\infty} \mathcal{C}_n(A_k)$.

By Lemma 2.1, we may conclude that $\lim_{k\to\infty} \mathcal{C}_n(A_k) = \mathcal{C}_n(A)$. Therefore, 2) is satisfied.

Assume 3). We prove 1). Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of subcontinua of X converging to A. Let B be a subcontinuum of A. Let $x_1, \ldots, x_{n-1}$ be $n-1$ distinct points in $A \setminus B$. Let $D = B \cup \{x_1, \ldots, x_{n-1}\}$. Since 3) is assumed for A, there exists $D_k \in \mathcal{C}_n(A_k)$ for each $k \in \mathbb{N}$ such that the sequence $\{D_k\}_{k=1}^{\infty}$ converges to D. Since D has n components, we may assume without loss of generality that D_k also has n components for any $k \in \mathbb{N}$. Since n is the maximum number of components we allow, there exists a component D_k^1 of D_k such that $\{D_k^1\}_{k=1}^{\infty}$ converges to B. Therefore, $\mathcal{C}(A) \subset \lim_{k\to\infty} \mathcal{C}(A_k)$. By Lemma 2.1, we conclude that $\lim_{k\to\infty} \mathcal{C}(A_k) = \mathcal{C}(A)$.

The fact that 2) implies 3) is obvious. □

In connection with Theorem 2.2, we note that a continuum X may be $\mathcal{C}^*$–smooth at X but not $\mathcal{C}_n^*$–smooth at X for any $n > 1$. This follows from Theorem 3.3 (using an arc).

Let X be a continuum. We say that X *is absolutely* $\mathcal{C}^*$*–smooth,* provided that for any continuum Z in which X can be embedded and for any sequence $\{A_k\}_{k=1}^{\infty}$ of elements of $\mathcal{C}(Z)$ converging to X, the sequence $\{\mathcal{C}(A_k)\}_{k=1}^{\infty}$ of hyperspaces converges to $\mathcal{C}(X)$.

With a proof similar to the one given for Theorem 2.2, we have the following result:

THEOREM 2.3. *Let X be a continuum. Then the following statements are equivalent:*

1) *X is absolutely $\mathcal{C}^*$–smooth;*
2) *for any continuum Z in which X is embedded, $\mathcal{C}_n^*|_{\mathcal{C}(Z)}$ is continuous at X for all n;*
3) *for any continuum Z in which X is embedded, $\mathcal{C}_n^*|_{\mathcal{C}(Z)}$ is continuous at X for some n.*

Our next main result is Theorem 2.6 which shows that $\mathcal{C}_n^*$–smoothness characterizes hereditary indecomposability.

LEMMA 2.4. *Let X be a continuum, let A be an indecomposable subcontinuum of X, and let $\{B_m\}_{m=1}^{\infty}$ be a sequence of elements of $\mathcal{C}_n(X)$ converging to A. Then, there exists a subsequence $\{B_{m_k}\}_{k=1}^{\infty}$ of $\{B_m\}_{m=1}^{\infty}$ such that for each k, there exists a component D_k of B_{m_k} such that the sequence $\{D_k\}_{k=1}^{\infty}$ of continua converges to A.*

PROOF. Since A is an indecomposable continuum, A has uncountably many mutually disjoint composants [7, 11.15 and 11.17]. Let $a_1, \ldots, a_{n+1}$ be $n+1$ points in $n+1$ distinct composants of A. We may assume that $\mathcal{V}_{\frac{1}{\ell}}(a_i) \cap \mathcal{V}_{\frac{1}{\ell}}(a_j) = \emptyset$ if $i \neq j$ for each positive integer ℓ.

Since $\{B_m\}_{m=1}^{\infty}$ converges to A for each ℓ, there exists an integer m_ℓ such that $\mathcal{H}(A, B_{m_\ell}) < \frac{1}{\ell}$. Thus, $B_{m_\ell} \cap \mathcal{V}_{\frac{1}{\ell}}(a_j) \neq \emptyset$ for each $j \in \{1, \ldots, n+1\}$. Since

B_{m_ℓ} has at most n components, we have that at least one of the components of B_{m_ℓ} intersects two of the balls $\mathcal{V}_{\frac{1}{\ell}}(a_j)$, $j \in \{1, \dots, n+1\}$.

Since we only have $n+1$ balls, there exist $j_0, j_1 \in \{1, \dots, n+1\}$ such that for infinitely many indices k, B_{m_k} has a component D_k such that $D_k \cap \mathcal{V}_{\frac{1}{k}}(a_{j_0}) \neq \emptyset$ and $D_k \cap \mathcal{V}_{\frac{1}{k}}(a_{j_1}) \neq \emptyset$ for each k. Since $\mathcal{C}(X)$ is compact [6, (0.8)], we may assume without loss of generality that the sequence $\{D_k\}_{k=1}^{\infty}$ converges to a subcontinuum D of A. Since a_{j_0} and a_{j_1} belong to D and they are in distinct composants of A, we conclude that $D = A$. □

The converse of Lemma 2.4 is false (as can be seen from the argument in Example 3.4).

LEMMA 2.5. *Let X be a decomposable continuum, and let A and B be nondegenerate proper subcontinua of X such that $X = A \cup B$. Assume that there exist two order arcs $\alpha, \beta\colon [0,1] \to \mathcal{C}(X)$ with the following properties: $\alpha(0) \in \mathcal{F}_1(A)$, $\alpha(1) = A$, $\beta(0) \in \mathcal{F}_1(B)$, $\beta(1) = B$ and $(A \cap B) \cap (\alpha(t) \cup \beta(t)) = \emptyset$ for each $t \in [0,1)$. Then X is not $\mathcal{C}_n^*$–smooth at X for any $n > 1$.*

PROOF. Suppose X is $\mathcal{C}_n^*$–smooth at X. Let $\{t_m\}_{m=1}^{\infty}$ be an increasing sequence of numbers in $[0,1)$ converging to 1. For each positive integer m, let $D_m = \alpha(t_m) \cup \beta(t_m)$. For each $m \geq 1$, $(A \cap B) \cap (\alpha(t_m) \cup \beta(t_m)) = \emptyset$, hence $D_m \in \mathcal{C}_2(X) \setminus \mathcal{C}(X)$.

Let R be a component of $A \cap B$. Let H and K be proper subcontinua of A and B, respectively, such that they properly contain R [7, 5.5]. Let $x_1, \dots, x_{n-1}$ be $n-1$ distinct points of $X \setminus (H \cup K)$. Let $L = \{x_1, \dots, x_{n-1}\} \cup (H \cup K)$. Let $\varepsilon > 0$ be such that the following hold:

$$\mathcal{V}_{2\varepsilon}(x_i) \cap \mathcal{V}_{2\varepsilon}(x_j) = \emptyset \text{ if and only if } i \neq j,$$

$$\{x_1, \dots, x_{n-1}\} \cap \mathcal{V}_{2\varepsilon}(H \cup K) = \emptyset,$$

$$\cup_{j=1}^{n-1} \mathcal{V}_{2\varepsilon}(x_j) \cap (H \cup K) = \emptyset,$$

$$H \setminus \mathcal{V}_{2\varepsilon}(K) \neq \emptyset, \text{ and } K \setminus \mathcal{V}_{2\varepsilon}(H) \neq \emptyset.$$

Since X is $\mathcal{C}_n^*$–smooth at X, there exists a positive integer m_0 such that if $m \geq m_0$, then there exists $E_m \in \mathcal{C}_n(D_m)$ such that $\mathcal{H}(E_m, L) < \varepsilon$. Let $m' \geq m_0$. Then, $E_{m'} \subset \mathcal{V}_\varepsilon(L) = \left(\cup_{j=1}^{n-1} \mathcal{V}_\varepsilon(x_j)\right) \cup \mathcal{V}_\varepsilon(H \cup K)$, $E_{m'} \cap \mathcal{V}_\varepsilon(x_j) \neq \emptyset$ for each $j \in \{1, \dots, n-1\}$, and $E_{m'} \cap \mathcal{V}_\varepsilon(H \cup K) \neq \emptyset$. Hence, $E_{m'}$ has exactly n components. Let $G_1, \dots, G_n$ be the components of $E_{m'}$. Since the ε–balls about each $x_1, \dots, x_{n-1}$ and $H \cup K$ are pairwise disjoint, we may assume without loss of generality that $G_j \subset \mathcal{V}_\varepsilon(x_j)$ for each $j \in \{1, \dots, n-1\}$ and $G_n \subset \mathcal{V}_\varepsilon(H \cup K)$. Since G_n is a subcontinuum of $D_{m'}$, G_n is contained either in $\alpha(t_{m'})$ or in $\beta(t_{m'})$. Suppose that G_n is contained in $\alpha(t_{m'})$. Let $x \in K \setminus \mathcal{V}_{2\varepsilon}(H)$. Then for each point z of $E_{m'}$, $d(y, z) \geq \varepsilon$. This is a contradiction; therefore, X is not $\mathcal{C}_n^*$–smooth at X. □

The following result characterizes the class of continua for which the map $\mathcal{C}_n^*$ is continuous for $n > 1$.

THEOREM 2.6. *A continuum X is $\mathcal{C}_n^*$–smooth for some $n > 1$ if and only if X is hereditarily indecomposable.*

PROOF. If X is hereditarily indecomposable then, X is $\mathcal{C}_n^*$–smooth by Lemma 2.4 and [6, (1.207.8)].

Suppose that X is $\mathcal{C}_n^*$–smooth for some integer $n > 1$. Then condition (3) of Theorem 2.3 is satisfied. Hence X is $\mathcal{C}^*$–smooth by Theorem 2.3. Since X is $\mathcal{C}^*$–smooth, X is hereditarily unicoherent [2, (3.4)].

Suppose X is decomposable. Then there exist two proper subcontinua A and B of X such that $X = A \cup B$.

Let $a \in A \setminus B$ and $b \in B \setminus A$. Let $\alpha, \beta \colon [0,1] \to \mathcal{C}(X)$ be order arcs such that $\alpha(0) = \{a\}$, $\alpha(1) = A$, $\beta(0) = \{b\}$ and $\beta(1) = B$. Let t_0 and s_0 be points of $[0,1]$ such that $\alpha(t_0) \cap \beta(s_0) \neq \emptyset$ and such that for each $t < t_0$ and each $s < s_0$, $\alpha(t) \cap \beta(s) = \emptyset$. Note that $t_0 > 0$ and $s_0 > 0$. Let $\{t_k\}_{k=1}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ be increasing sequences in $[0,1)$ converging to t_0 and s_0, respectively.

Let $Y = \alpha(t_0) \cup \beta(s_0)$. Then Y is a subcontinuum of X. Then, by Lemma 2.5, X is not $\mathcal{C}_n^*$–smooth at Y, a contradiction. Therefore, X is indecomposable.

A similar argument shows that each subcontinuum of X is indecomposable. □

3. POINTS OF $\mathcal{C}_n^*$–SMOOTHNESS

We now present some results about the points at which a continuum X is $\mathcal{C}_n^*$–smooth.

THEOREM 3.1. *Let X be a continuum and let A be an element of $\mathcal{C}_n(X)$ for some $n > 1$. If X is $\mathcal{C}_n^*$–smooth at A, then X is $\mathcal{C}^*$–smooth at each component of A.*

PROOF. Let A be an element of $\mathcal{C}_n(X)$ and suppose X is $\mathcal{C}_n^*$–smooth at A. Observe that if A is connected, then X is $\mathcal{C}^*$–smooth at A by Theorem 2.2.

Suppose A has at least two components. Let $A_1, \ldots, A_k$ be the components of A. We show that X is $\mathcal{C}^*$–smooth at A_1. Let $\{K_m\}_{m=1}^{\infty}$ be a sequence of subcontinua of X converging to A_1. Without loss of generality, we may assume that $K_m \cap \left(\cup_{j=3}^{k} A_j\right) = \emptyset$. Let L be a subcontinuum of A_1.

Let $\alpha \colon [0,1] \to \mathcal{C}(X)$ be an order arc such that $\alpha(0) \in \mathcal{F}_1(A_2)$ and $\alpha(1) = A_2$. Let $\{t_m\}_{m=1}^{\infty}$ be an increasing sequence of numbers in $[0,1)$ converging to 1. For each m, let $p_m^{(1)}, \ldots, p_m^{(n-k)}$ be $n-k$ distinct points in $A_2 \setminus \alpha(t_m)$.

For each positive integer m, let

$$F_m = K_m \cup \alpha(t_m) \cup \left(\cup_{j=3}^{k} A_j\right) \cup \{p_m^{(1)}, \ldots, p_m^{(n-k)}\}.$$

Then $\lim_{m\to\infty} F_m = A$. Since X is $\mathcal{C}_n^*$–smooth at A, there exists an element D_m of $\mathcal{C}_n(F_m)$ such that

$$\lim_{m\to\infty} D_m = L \cup \alpha(t_1) \cup \left(\cup_{j=3}^{k} A_j\right) \cup \{p_1^{(1)}, \ldots, p_1^{(n-k)}\}.$$

For each positive integer m, let $L_m = D_m \cap K_m$. Then, L_m is a subcontinuum of K_m and $\lim_{m\to\infty} L_m = L$. Therefore, X is $\mathcal{C}^*$–smooth at A_1. Similarly, X is $\mathcal{C}^*$–smooth at the other components of A. ∎

We note that the converse of the Theorem 3.1 is false as can be seen from Theorem 3.3 (since if $X = [0,1]$ then X is not $\mathcal{C}_n^*$–smooth at any subcontinuum for each $n > 1$, by Theorem 3.3).

The following Lemma is easy to establish, but we include a proof for completeness.

LEMMA 3.2. *Let C be a closed subset of a space Z. Let $A = \overline{Z \setminus C}$ and $B = \overline{Z \setminus A}$. Then $A = \overline{Z \setminus B}$.*

PROOF. Since A is closed in Z, $\overline{\mathrm{int}(A)} \subset A$; thus, since

$$A = \overline{Z \setminus C} = \overline{\mathrm{int}(Z \setminus C)} \subset \overline{\mathrm{int}(\overline{Z \setminus C})} = \overline{\mathrm{int}(A)},$$

we have that $A = \overline{\mathrm{int}(A)}$. Therefore, since $\mathrm{int}(A) = Z \setminus \overline{(Z \setminus A)} = Z \setminus B$, $A = \overline{Z \setminus B}$. ∎

THEOREM 3.3. *If X is an irreducible continuum such that X is $\mathcal{C}_n^*$–smooth at X for some $n > 1$, then X is indecomposable.*

PROOF. Assume that a and b are points about which X is irreducible. Suppose X is decomposable. Let C be a nondegenerate proper subcontinuum of X, with nonempty interior, containing b. Let $A = \overline{X \setminus C}$ and $B = \overline{X \setminus A}$.

Then A and B are subcontinua of X [7, 11.6] containing a and b, respectively. Note that $A = \overline{X \setminus B}$, by Lemma 3.2. Since $A \cap B = \mathrm{Bd}(A) = \mathrm{Bd}(B)$ and since $B = \overline{X \setminus A}$ (and $A = \overline{X \setminus B}$), A (and B, respectively) is irreducible between a (and b, respectively) and any point of $A \cap B$ [7, 11.42].

Let $\alpha, \beta\colon [0,1] \to \mathcal{C}(X)$ be order arcs such that $\alpha(0) = \{a\}$, $\alpha(1) = A$, $\beta(0) = \{b\}$ and $\beta(1) = B$.

Notice that for any $t \in [0,1)$, $(A \cap B) \cap \alpha(t) = \emptyset$ and $(A \cap B) \cap \beta(t) = \emptyset$. By Lemma 2.5, X is not $\mathcal{C}_n^*$–smooth at X, a contradiction. Therefore, X is indecomposable. ∎

EXAMPLE 3.4. We give a nonirreducible continuum which is $\mathcal{C}_2^*$–smooth at X. Let X be the cone over the Cantor middle–thirds set. Let v be the vertex of X. By [2, (4.9)], it is easy to see that X is $\mathcal{C}^*$–smooth.

To see X is $\mathcal{C}_2^*$–smooth at X, let $\{A_m\}_{m=1}^{\infty}$ be a sequence of elements of $\mathcal{C}_2(X)$ converging to X, and let B be an element of $\mathcal{C}_2(X)$.

Since $\{A_m\}_{m=1}^{\infty}$ converges to X, it is easy to see that there are components A_m^1 of A_m such that $\{A_m^1\}_{m=1}^{\infty}$ converges to X. Therefore, since X is $\mathcal{C}^*$–smooth for each m, there exists $B_m \in \mathcal{C}_2(A_m^1) \subset \mathcal{C}_2(A_m)$ such that $\lim_{m\to\infty} B_m = B$.

For $n = 1$, the following definition agrees with the notion of absolutely $\mathcal{C}^*$–smoothness.

Let X be a continuum. We say that X *is absolutely* $\mathcal{C}_n^*$*–smooth,* provided that for any continuum Z in which X can be embedded and for any sequence $\{A_k\}_{k=1}^{\infty}$ of elements of $\mathcal{C}_n(Z)$ converging to X, the sequence $\{\mathcal{C}_n(A_k)\}_{k=1}^{\infty}$ of hyperspaces converges to $\mathcal{C}_n(X)$.

Corollary 3.5. *If X is an irreducible continuum which is absolutely $\mathcal{C}_n^*$–smooth, for some $n > 1$, then X is indecomposable.*

Note that the converse of Corollary 3.5 is not true. A modification of the Knaster buckethandle continuum, obtained by replacing a point with a (nowhere dense) simple triod, results in an indecomposable continuum which is not $\mathcal{C}^*$–smooth.

Theorem 3.6. *Let X be a continuum and let A be an element of $\mathcal{C}_n(X)$ with exactly n components, $n > 1$. Then X is $\mathcal{C}_n^*$–smooth at A if and only if X is $\mathcal{C}^*$–smooth at each component of A.*

Proof. The only if part is true by Theorem 3.1.

Let A be an element of $\mathcal{C}_n(X)$ with n components $A_1, \ldots, A_n$. Suppose X is $\mathcal{C}^*$–smooth at each A_j for each $j \in \{1, \ldots, n\}$.

Let $\{B_k\}_{k=1}^{\infty}$ be a sequence of elements of $\mathcal{C}_n(X)$ converging to A. Since A has n components, without loss of generality, we may assume that B_k has n components, $B_k^1, \ldots, B_k^n$, for each positive integer k. In fact, we may suppose that $\lim_{k\to\infty} B_k^j = A_j$ for each $j \in \{1, \ldots, n\}$.

Let C be an element of $\mathcal{C}_n(A)$. Let $A_{j_1}, \ldots, A_{j_\ell}$ be the components of A intersecting C, i. e., $C = \cup_{i=1}^{\ell}(A_{j_i} \cap C)$. Let $C_{j_i} = A_{j_i} \cap C$ for each $i \in \{1, \ldots, \ell\}$. Since X is $\mathcal{C}^*$–smooth at A_{j_i}, there exists a subcontinuum $D_k^{j_i}$ of $B_k^{j_i}$ for each $i \in \{1, \ldots, \ell\}$ such that $\lim_{k\to\infty} D_k^{j_i} = C_{j_i}$. For k, let $D_k = \cup_{i=1}^{\ell} D_k^{j_i}$. Hence, $D_k \in \mathcal{C}_n(B_k)$ and $\lim_{k\to\infty} D_k = C$. Therefore, X is $\mathcal{C}_n^*$–smooth at A by Lemma 2.1. ∎

Theorem 3.7. *Let X be a continuum. If A is an element of $\mathcal{C}_n(X)$ for some $n > 1$ such that all the components of A are indecomposable and X is $\mathcal{C}^*$–smooth at each component of A, then X is $\mathcal{C}_n^*$–smooth at A.*

Proof. Let A be an element of $\mathcal{C}_n(X)$. Let $A_1, \ldots, A_\ell$ ($\ell \leq n$) be the components of A. Suppose A_j is an indecomposable continuum and X is $\mathcal{C}^*$–smooth at A_j for each $j \in \{1, \ldots, \ell\}$.

Let $\{B_k\}_{k=1}^{\infty}$ be a sequence of elements of $\mathcal{C}_n(X)$ converging to A. Let C be an element of $\mathcal{C}_n(A)$. Let $A_{j_1}, \ldots, A_{j_s}$ be the components of A intersecting

C, i. e., $C = \cup_{i=1}^{s}(A_{j_i} \cap C)$. Let $C_{j_i}^{1}, \dots, C_{j_i}^{\ell_{j_i}}$ be the components of $A_{j_i} \cap C$ for each $i \in \{1, \dots, s\}$.

In what follows, k is any positive integer and $i \in \{1, \dots, s\}$. By Example 3.4, there are components $B_k^{j_i}$ of B_k such that $\lim_{k\to\infty} B_k^{j_i} = A_{j_i}$. Since X is $\mathcal{C}^*$–smooth at each A_{j_i}, there are subcontinua $D_{k,1}^{j_i}, \dots, D_{k,\ell_{j_i}}^{j_i}$ of $B_k^{j_i}$ such that $\lim_{k\to\infty} D_{k,m}^{j_i} = C_{j_i}^{m}$ for each $m \in \{1, \dots, \ell_{j_i}\}$. Let $D_k^{j_i} = \cup_{m=1}^{\ell_{j_i}} D_{k,m}^{j_i}$ and let $D_k = \cup_{i=1}^{s} D_k^{j_i}$. Then, $D_k \in \mathcal{C}_n(B_k)$ and $\lim_{k\to\infty} D_k = C$. Therefore, X is $\mathcal{C}_n^*$–smooth at A by Lemma 2.1. □

Corollary 3.8. *Let X be a continuum and let $n > 1$. If A is an element of $\mathcal{C}_n(X)$ such that all the components of A are hereditarily indecomposable, then X is $\mathcal{C}_n^*$–smooth at A.*

Proof. This result follows from Theorem 3.7 and the fact that hereditarily indecomposable continua are absolutely $\mathcal{C}^*$–smooth continua ([6, (14.14.1)] and [3, 3.2]). □

4. $\mathcal{C}^*$–smoothness

We answer in the affirmative question 15.21 of [6].

Theorem 4.1. *If X is a $\mathcal{C}^*$–smooth homogeneous continuum, then X is indecomposable. Moreover, if X is a $\mathcal{C}^*$–smooth homogeneous plane continuum, then X is hereditarily indecomposable.*

Proof. Let X be a $\mathcal{C}^*$–smooth homogeneous continuum. Then X is hereditarily unicoherent [2, (3.4)]. By [5, Theorem 1], X is indecomposable.

If X is a $\mathcal{C}^*$–smooth homogeneous plane continuum, we have that X is indecomposable. Since any indecomposable homogeneous plane continuum is hereditarily indecomposable [4, Theorem 1], X is hereditarily indecomposable. □

Let us recall that a continuum X is said to be *absolutely $\mathcal{C}^*$–smooth* provided that whenever X is embedded in a continuum Z, X is a point of $\mathcal{C}^*$–smoothness of Z.

Note that absolute $\mathcal{C}^*$–smoothness of a continuum X does not say anything about the $\mathcal{C}^*$–smoothness of Z at proper subcontinua of X. For this reason, we consider the following notion: a continuum X is *strongly absolutely $\mathcal{C}^*$–smooth* provided that whenever X is embedded in a continuum Z, each subcontinuum of X is a point of $\mathcal{C}^*$–smoothness of Z. We show the following result:

Theorem 4.2. *A continuum X is strongly absolutely $\mathcal{C}^*$–smooth if and only if each subcontinuum of X is absolutely $\mathcal{C}^*$–smooth.*

Proof. If each subcontinuum of X is absolutely $\mathcal{C}^*$–smooth, then X is clearly strongly absolutely $\mathcal{C}^*$–smooth.

Next, suppose there exists a subcontinuum A of X such that it is not absolutely $\mathcal{C}^*$–smooth. Then there exist a continuum Y and an embedding $h\colon A \to Y$ such that $A' = h(A)$ is not a point of $\mathcal{C}^*$–smoothness of Y. Hence, there exists a sequence of $\{Y_m\}_{m=1}^{\infty}$ of subcontinua of Y converging to A such that the sequence $\{\mathcal{C}(Y_m)\}_{m=1}^{\infty}$ of hyperspaces does not converge to $\mathcal{C}(A')$.

Let $Z = X \cup_h Y$ be the adjunction space of X and Y under h [1, p. 127]. Let $q\colon X \cup Y \to Z$ be the quotient map. Since h is an embedding, it is easy to see that $q|_X : X \to Z$ is an embedding of X into Z; also, $q|_Y : Y \to Z$ is an embedding of Y into Z [1, p. 128]. Therefore, $\{q(Y_m)\}_{m=1}^{\infty}$ is a sequence of subcontinua of Z converging to $q(A') = q(A)$ such that the sequence $\{\mathcal{C}(q(Y_m))\}_{m=1}^{\infty}$ of hyperspaces does not converge to $\mathcal{C}(q(A))$. Thus, X is not absolutely $\mathcal{C}^*$–smooth. □

As a consequence of the previous theorem and [3, 3.2], we note that a continuum X is strongly $\mathcal{C}^*$–smooth if and only if X has the covering property hereditarily.

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