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THE PROPERTY *RNT* ON DENDROIDS

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Abstract. A continuum is said to be *retractable onto near trees* (abbreviating *RNT*) provided that for each $\varepsilon > 0$ there exists $\delta > 0$ such that if T is a tree in X and $H(T, X) < \delta$, then there exists an ε-retraction r of X onto T. In this paper it is proved that an arcwise connected continuum with the property *RNT* is a dendroid. We focus our study on dendroids. Dendrites are characterized as: (a) smooth dendroids with property *RNT*, (b) dendroids with the property *RNT* hereditarily. Finally, an example of a nonlocally connected dendroid with property *RNT* is shown.

1. Introduction

A *continuum* is a compact, connected metric space. A property of a continuum X is said to be *hereditary* provided that each subcontinuum of X has the property. In particular, a continuum X is said to be *hereditarily unicoherent* provided that the intersection of any two subcontinua of X is connected. A *dendroid* is an arcwise connected hereditarily unicoherent continuum. For a dendroid X and points $a, b \in X$, let ab denote the only arc in X joining a and b if $a \neq b$ or $ab = \{a\}$ if $a = b$. A *dendrite* is defined as a locally connected dendroid.

A *finite graph* is a finite union of arcs intersecting only at their end points. A *tree* is a finite graph that does not contain simple closed curves. Let X be a continuum with a metric d. A *retraction* from X onto a subcontinuum Y of X is defined as a mapping $r : X \to Y$ such that $r(y) = y$ for every $y \in Y$. An ε-*retraction* $r : X \to Y$ is a retraction such that every $x \in X$ satisfies $d(x, r(x)) < \varepsilon$.

Let $C(X)$ denote the hyperspace of subcontinua of X with the Hausdorff metric H.

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Given a continuum X with a metric d, a point p in X, a closed subset A of X and a positive number ε, let $B_\varepsilon(p) = \{q \in X : d(p,q) < \varepsilon\}$, $d(p,A) = \min\{d(p,a) : a \in A\}$, $N(\varepsilon, A) = \{q \in X :$ there exists a point $a \in A$ such that $d(q,a) < \varepsilon\}$ and $\mathrm{cl}_X(B)$ denote the closure of the subset B of X.

In the early sixties in Wrocław Higher Topology Seminar of the Polish Academy of Sciencies, Knaster saw dendroids as those continua that can be retracted onto their subdendrites or even onto their subtrees under small retractions (ε-retractions). The contemporary definition of dendroids (the one given above) was formulated in a more convenient way. It is still an open problem if for every dendroid X and every positive ε, there exist a subtree T of X and an ε-retraction from X onto T. Some partial positive answers to this question can be found in [2, 3, 4].

As we shall see later (Theorem 3.1) each dendrite X has the property that for each $\varepsilon > 0$ and for each subtree T of X which is close enough to X, there exists an ε-retraction from X onto T. This property motivates the following definition.

Definition 1.1. *A continuum X has the property RNT provided that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if T is a tree contained in X and $H(T,X) < \delta$, then there exists an ε-retraction $r : X \to T$.*

In this paper we prove the following results.

(1) An arcwise connected continuum with the property *RNT* is a dendroid (Theorem 2.5).
(2) Dendrites have the property *RNT* (Theorem 3.1).
(3) A dendroid has property *RNT* hereditarily if and only if it is a dendrite (Theorem 4.1).
(4) A dendroid is smooth and has the property *RNT* if and only if it is a dendrite (Theorem 5.1).
(5) There exists a nonlocally connected dendroid with the property *RNT*.

2. The property *RNT* in arcwise connected continua

The study of the property *RNT* was motivated by a question on dendroids. Arcwise connected continua have the property of containing trees close to X. Therefore it makes sense to study property *RNT* on these continua. The aim of this section is to prove that arcwise connected continua with the property *RNT* are dendroids.

Lemma 2.1. *Let X be an arcwise connected continuum, T a tree contained in X and F a finite subset of X. Then there exists a tree S such that $T \cup F \subset S \subset X$.*

Proof. We prove this lemma by induction on the number of elements of F. First, suppose that F has one element p. We may assume that $p \notin T$. Since X is arcwise connected, there exists an arc $A \subset X$, with end points p

and t such that $T \cap A = \{t\}$. Then $T_1 = T \cup A$ is the required tree. Suppose that the conclusion is true for finite subsets of X with exactly n elements, and take a subset F of X with $n+1$ elements. Let x be an element of F. By the induction hypothesis, there exists a tree R such that $(F - \{x\}) \cup T \subset R \subset X$. Applying the first step of the induction to R and the point x, we obtain the desired tree S. This completes the induction and the proof of the lemma. □

LEMMA 2.2. *Let X be an arcwise connected continuum and T a tree contained in X. Then for every $\delta > 0$ there exists a tree $T_\delta \subset X$ such that $T \subset T_\delta$ and $H(T_\delta, X) < \delta$.*

PROOF. By compactness of X, there exist a finite number of points $x_1, x_2, \ldots, x_n$ of X such that $X = B_\delta(x_1) \cup B_\delta(x_2) \cup \cdots \cup B_\delta(x_n)$. By Lemma 2.1, there exists a tree $T_\delta \subset X$ such that $T \subset T_\delta$ and $\{x_1, x_2, \ldots, x_n\} \subset T_\delta$. Clearly, $H(T_\delta, X) < \delta$. □

DEFINITION 2.3. *A lock L in a continuum X is a subcontinuum of X of the form $L = A \cup B$, where $A, B \in C(X)$, A is an arc with end points p and q, and $A \cap B = \{p, q\}$.*

LEMMA 2.4. *If an arcwise connected continuum is not hereditarily unicoherent, then it contains a lock.*

PROOF. Suppose that X is not hereditarily unicoherent. Then there exist $M, N \in C(X)$ such that $M \cap N$ is not connected. Take points $p' \in C_1$ and $q' \in C_2$, where C_1 and C_2 are different components of $M \cap N$. Since X is arcwise connected, there exists an arc $A' \subset X$ with end points p' and q'. Since $A' \not\subseteq M \cap N$, we may assume that A' is not contained in M. Let J be a component of $A' \setminus M$. So $A = \mathrm{cl}_X(J)$ is an arc that intersects M exactly at its end points. Therefore $L = M \cup A$ is a lock in X. □

THEOREM 2.5. *Each arcwise connected continuum with the property RNT is a dendroid.*

PROOF. Let X be an arcwise connected continuum with the property *RNT*. We only need to prove that X is hereditarily unicoherent.

Suppose that X is not hereditarily unicoherent. By Lemma 2.4 there exists a lock L in X. Let $L = A \cup B$, where $A, B \in C(X)$, A is an arc with end points p, q and $A \cap B = \{p, q\}$. Fix a point $s \in A \setminus B$. Then $\varepsilon = d(s, B)$ is a positive number. Take any positive number δ. By Lemma 2.2 there exists a tree T_δ such that $A \subset T_\delta \subset X$ and $H(X, T_\delta) < \delta$. Let $r : X \to T_\delta$ be any retraction. Then $r(B)$ is a subcontinuum of the tree T_δ that contains p and q. This implies that $A \subset r(B)$. Therefore there exists a point $b \in B$ such that $r(b) = s$. Hence $d(b, r(b)) \geq \varepsilon$. We have proved that r is not an ε-retraction. Thus, X does not have property *RNT*. This ends the proof of the theorem. □

3. The property RNT on dendrites

We know that arcwise connected continua with the property RNT are dendroids. It will be proved that among dendroids, dendrites always have the property RNT.

Theorem 3.1. *Each dendrite has property RNT.*

Proof. Given a subcontinuum Z of a dendrite X, let $r : X \to Z$ be the first point map, that is, given a point $x \in X$, let $r(x)$ be the unique point in Z such that $r(x)$ is a point of any arc in X from x to any point of Z. The existence and continuity of r is guaranteed by [7, Lemma 10.24, p. 175]. Note that r is a retraction. Let $\varepsilon > 0$. By compactness of X, there exists a finite cover $\mathcal{U}$ of X consisting of open arcwise connected subsets of X such that each element of $\mathcal{U}$ has diameter less than ε. Let δ be a Lebesgue number for $\mathcal{U}$. Take any subtree T of X such that $H(T, X) < \delta$. Let $r : X \to T$ be the first point map corresponding to T. Then for each point $x \in X$, there exists a point $t \in T$ such that $d(x, t) < \delta$. Thus there exists an element $U \in \mathcal{U}$ such that $x, t \in U$, and therefore $xt \subset U$. By definition of r we have $r(x) \in xt \subset U$. So, $d(x, r(x)) < \varepsilon$. Therefore, X has the property RNT. □

4. The hereditary property RNT

On Section 6 of the paper we show an example of a nonlocally connected dendroid with the property RNT. In this section we characterize dendrites as dendroids which have the property RNT hereditarily.

Theorem 4.1. *A dendroid has the property RNT hereditarily if and only if it is a dendrite.*

Proof. Suppose that dendroid X is a dendrite. Since every subcontinuum of X is a dendrite, the sufficiency follows from Theorem 3.1.

In order to prove the necessity, suppose that X has the property RNT hereditarily and X is not locally connected. By Theorem 13 of [6], X contains a semi-comb or a semi-broom ([6, Definition 11]).

It follows then that X contains:

(a) two points $a, b \in X$,
(b) two points $p \neq q$ in $A = ab$,
(c) a sequence of points $\{p_n\}_{n=1}^{\infty}$ in $X - A$, and
(d) a sequence of points $\{q_n\}_{n=1}^{\infty}$ in A (for the case of semi-broom ([6, Definition 11]), take $q_n = q$ for each n) such that:
  (i) $p_n \to p$, $q_n \to q$,
  (ii) $p_n q_n \cap A = \{q_n\}$ for each n, and
  (iii) $p_1 q_1 - \{q_1\}, p_2 q_2 - \{q_2\}, \ldots$ are pairwise disjoint.

Taking the natural order for ab, where $a < b$, we may assume that $a \leq q < p \leq b$. Since $q_n \to q$, we may also assume that there exists a point

$c \in ab$ such that $a \leq q_n \leq c < p$ for each $n \geq 1$. Fix a point $e \in cp - \{c, p\}$. Consider the subcontinuum $Y = ae \cup \mathrm{cl}_X(\bigcup\{p_n q_n : n \geq 1\})$ of X. Let $\varepsilon > 0$ be such that $d(p, ae) > \varepsilon$ and $\varepsilon < \min\{d(x, y) : x \in ac \text{ and } y \in ep\}$. Since we are assuming that X has the property RNT hereditarily, Y has the property RNT. Let $\delta > 0$ be as in the definition of the property RNT for Y and ε. Take $M \geq 1$ such that the tree $T = ae \cup (\bigcup\{p_n q_n : n \leq M\})$ has the property that $H(T, Y) < \delta$. Since $p_n \in Y$ for each n, $p \in Y$. Let $r : Y \to T$ be an ε-retraction. Since $d(p, r(p)) < \varepsilon$, $r(p) \notin ae$. Therefore, $r(p) \in p_m q_m$ for some $m \leq M$. Since $e \in T$, we have that $r(e) = e$. It follows that $r(ep)$ contains q_m. Thus there exists a point $s \in ep$ such that $r(s) = q_m$. Therefore $d(s, r(s)) = d(s, q_m) > \varepsilon$, since $s \in ep$ and $q_m \in ac$. This contradicts that r is an ε-retraction and proves that X is locally connected. ☐

5. The property *RNT* in smooth dendroids

Recall that a dendroid X is said to be *smooth* provided that there exists a point $v \in X$ such that for each sequence $\{x_n\}_{n=1}^{\infty}$ in X converging to a point $x \in X$, the sequence of arcs vx_n converges to vx (in $C(X)$) ([1, p. 298]). The point v is called an *initial point* of X.

Theorem 5.1. *A dendroid is smooth and has the property RNT if and only if it is a dendrite.*

Proof. By [1, Corollary 4, p. 298] dendrites are smooth dendroids. Then the sufficiency follows from Theorem 3.1.

In order to prove the necessity suppose that X is a nonlocally connected smooth dendroid with an initial point v. By [1, Theorem 1], X is locally connected at v. We are going to show that X does not have the property RNT. Since X is not locally connected, there exist (see [6, Lemma 12]):

(1) three open subsets V, W and U of X,
(2) two different points $p, q \in \mathrm{cl}_X(V)$,
(3) two sequences $\{p_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ of points of $\mathrm{cl}_X(V)$,
(4) a sequence of distinct components $C_0, C_1, C_2, \ldots$ of U, such that
  (a) $\mathrm{cl}_X(V) \subset W \subset \mathrm{cl}_X(W) \subset U$,
  (b) $pq \subset \mathrm{cl}_X(V) \cap C_0$,
  (c) $p_n \to p$, $q_n \to q$,
  (d) $p_n q_n \subset \mathrm{cl}_X(V) \cap C_n$ for each $n \geq 1$,
  (e) $v \notin U$.

For each $n \geq 1$, let E_n be the component of $\mathrm{cl}_X(V) \cap C_n$ that contains $p_n q_n$ and let D_n be the component of $\mathrm{cl}_X(W)$ that contains E_n. Denote by c_n the only point in E_n such that $vc_n \cap E_n = \{c_n\}$. Taking a subsequence if necesary, we may assume that $c_n \to c$ for some $c \in X$ and $E_n \to K$ for some continuum K of X. Since $p \neq q$, we may also assume that $c \neq q$ and $c_n \neq q_n$ for each $n \geq 1$. Note that $p, q, c \in K \subset \mathrm{cl}_X(V)$. Let E_0 be the component of $\mathrm{cl}_X(V)$ that contains p. Then $c \in E_0$ and $E_0 \subset C_0$. Given $n \geq 1$, by the choice of c_n, we infer that $vc_n \subset vq_n$. By the smoothness of X at v, $vc_n \to vc$ and $vq_n \to vq$. Thus $vc \subset vq$. Since $c \neq q$, $q \notin vc$.

Take $\varepsilon > 0$ such that $\varepsilon < \min\{d(x,y) : x \in \mathrm{cl}_X(V)$ and $y \notin W\}$ and $3\varepsilon < d(q, vc)$. We show that X does not satisfy the definition of property RNT for this ε. Given $\delta > 0$, by Lemma 2.2 there exists a tree $T \subset X$ such that $H(T,X) < \delta$ and $v \in T$.

We claim that T intersects only finitely many sets D_n. Let $T_1, \ldots, T_m$ be subtrees of T such that each T_i has diameter less than $\min\{d(x,y) : x \in \mathrm{cl}_X(W)$ and $y \notin U\}$ and $T = T_1 \cup \cdots \cup T_m$. If T_i intersects D_n, then $T_i \subset U$ and therefore $T_i \subset C_n$. So, each T_i can only intersect one set of the form D_n. Thus T intersects at most m of the sets D_n. Let $N \geq 1$ be such that $T \cap D_n = \emptyset$ for each $n \geq N$.

Since v is an initial point of X, $vc_n \to vc$. Thus we may assume that $vc_n \subset N(\varepsilon, vc)$ and $d(q, q_n) < \varepsilon$ for each $n \geq N$.

Note that $c_N q_N \subset E_N \cap D_N \subset \mathrm{cl}_X(V) \cap D_N$ and $c_N q_N \cap T = \emptyset$. Let $T_0 = T \cup vc_N$. Then T_0 is a tree, $q_N \notin T_0$ and $H(T_0, X) < \delta$.

Suppose that there is an ε-retraction $r : X \to T_0$. If $r(q_N) \in vc_N$, by the choice of N, there is a point $y \in vc$ such that $d(r(q_N), y) < \varepsilon$. Then $d(q,y) \leq d(q,q_N) + d(q_N, r(q_N)) + d(r(q_N), y) < 3\varepsilon$. This contradicts the choice of ε and proves that $r(q_N) \notin vc_N$. Since $r(q_N) \in B(\varepsilon, q_N) \cap T \subset W \cap T$, we have that there exists a component D of $\mathrm{cl}_X(W)$ such that $r(q_N) \in D$ and $D \neq D_N$. Since r is a retraction, $r(c_N) = c_N$. Then the set $r(c_N q_N)$ intersects two different components of $\mathrm{cl}_X(W)$. Thus there exists $z \in c_N q_N$ such that $r(z) \notin W$. Since $z \in \mathrm{cl}_X(V)$, by the choice of ε, $d(z, r(z)) \geq \varepsilon$. This contradiction proves that there is not any ε-retraction from X onto T_0. Therefore X does not have the property RNT. This completes the proof of the theorem. □

6. A nonlocally connected dendroid with the property RNT

In [5] A. Illanes constructed an example of a nonlocally connected dendroid X with the property that, for each subcontinuum A of X, there exists a retraction $r : X \to A$. The dendroid is constructed as a subset of the Euclidean plane $\mathbb{R}^2$. In this section we modify Illanes' dendroid to obtain a nonlocally connected dendroid Y with the property RNT.

Let us recall the construction of X.

Given two points p and q in the Euclidean plane $\mathbb{R}^2$, denote by $\langle p, q\rangle$ the convex segment joining them, if $p \neq q$, and put $\langle p, q\rangle = \{p\}$, if $p = q$. For a point $p = (x,y) \in \mathbb{R}^2$, define $p' = (-x, y)$. Given a subset B in $\mathbb{R}^2$, define $B' = \{p' \in \mathbb{R}^2 : p \in B\}$. The origin in $\mathbb{R}^2$ is denoted by Θ. Let $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ be the respective projections on the first and second coordinates. We will define, inductively, a sequence $A_0, \ldots, A_n, \ldots$ of subsets of $\mathbb{R}^2$ such that, for each $n > 0$, A_n is a polygonal line joining Θ to a point $a_n = (u_n, v_n)$. Let $a_0 = \Theta$.

Let $A_0 = \{\Theta\}$, $A_1 = \left\langle \Theta, (1, -\frac{1}{4}) \right\rangle$. Suppose that $A_0, \ldots, A_n$ have been defined and $n \geqslant 1$. Define $b_n = a_n + a'_{n-1}$ and $A_{n+1} = A_n \cup (a_n + A'_{n-1}) \cup (b_n + A_n)$. See Figure 1 in [5, p. 803].

For each $n > 1$, define $C_n = (0, \frac{1}{2^{n-2}}) + \{(\frac{x}{n}, \frac{-y}{2^n v_n}) \in \mathbb{R}^2 : (x, y) \in A_n\}$ and $B_n = C_n \cup \left\langle (0, \frac{1}{2^{n-2}}), \Theta \right\rangle \cup \langle \Theta, (1, 0) \rangle$. Given points p and q in B_n, $\langle\langle p, q \rangle\rangle$ will denote the subarc in B_n joining p and q, if $p \neq q$ and $\langle\langle p, q \rangle\rangle = \{p\}$, if $p = q$.

Finally, define $X = \bigcup\{B_n : n \geq 2\} = \langle \Theta, (0, 1) \rangle \cup \langle \Theta, (1, 0) \rangle \cup (\bigcup\{C_n : n \geq 2\})$. Clearly, X is a nonlocally connected dendroid. See Figure 4 in [5, p. 806].

Define $Y = X \cup \langle (1, 0), (2, 0) \rangle$. We show that Y has the property *RNT*. In order to do this, we will use the following assertion proved in [5, p. 801].

ASSERTION 6.1. *For each $q \in C_n$, there exists a retraction (which depends on q) $\phi_{n,q} : B_n \to \langle\langle q, (1, 0) \rangle\rangle$, such that $|\pi_1(p) - \pi_1(\phi_{n,q}(p))| \leq \frac{3}{n}$.*

Let $\varepsilon > 0$. Let $N \geq 1$ be such that $\frac{3}{N} + \frac{1}{2^{N-2}} < \varepsilon$. For each $2 \leq n \leq N$, let c_n be the end point of the arc C_n which is different from $(0, \frac{1}{2^{n-2}})$ and let $z_n \in C_n \setminus \{c_n\}$ be such that the diameter of the subarc $z_n c_n$ is less than ε. Let $\delta_n > 0$ be such that $B_{\delta_n}(c_n) \subset z_n c_n$. Let $\delta_0 > 0$ be such that $B_{\delta_0}((2, 0)) \cap X = \emptyset$ and let $\delta = \min\{\varepsilon, \delta_0, \delta_1, \ldots, \delta_N\}$.

Take any tree T of Y such that $H(T, Y) < \delta$. Notice that, for each $n \geq 2$, $T \cap C_n \neq \emptyset$, so there exists $t_n \in T$ such that $C_n \cap T = \left\langle \left\langle (0, \frac{1}{2^{n-2}}), t_n \right\rangle \right\rangle$ and if $n \leq N$, then $T \cap B_{\delta_n}(c_n) \neq \emptyset$, so $t_n \in z_n c_n$. Let $t_0 \in T$ be such that $T \cap \langle\langle \Theta, (2, 0) \rangle\rangle = \langle\langle \Theta, t_0 \rangle\rangle$. Then $t_0 \in B_{\delta_0}((2, 0))$.

Define $r : Y \to T$ by

$$r(y) = \begin{cases} y, & \text{if } y \in T, \\ t_n, & \text{if } y \in C_n \setminus T \text{ and } 2 \leq n \leq N, \\ \phi_{n,t_n}(y), & \text{if } y \in C_n \setminus T \text{ and } N < n\ , \\ t_0, & \text{if } y \in \langle t_0, (2, 0) \rangle\,, \end{cases}$$

where ϕ_{n,t_n} is chosen as in Assertion 6.1.

The continuity of r follows from Assertion 6.1. Therefore r is a retraction.

In order to show that r is an ε-retraction, take a point $y \in Y \setminus T$. If $y \in C_n \setminus T$ and $2 \leq n \leq N$, $d(y, r(y)) = d(y, t_n) < \varepsilon$, since $t_n, y \in z_n c_n$ and the diameter of $z_n c_n$ is less than ε. The case that $y \in \langle t_0, (2, 0) \rangle$ is similar. Finally, if $y \in C_n \setminus T$ and $N < n$, since $\phi_{n,t_n}(y) \in B_n$, $|\pi_2(y) - \pi_2(r(y))| = |\pi_2(y) - \pi_2(\phi_{n,t_n}(y))| \leq \frac{1}{2^{n-2}}$ and, by the Assertion 6.1, we know that $|\pi_1(y) - \pi_1(r(y))| = |\pi_1(y) - \pi_1(\phi_{n,t_n}(y))| \leq \frac{3}{n}$. This implies that $d(y, r(y)) < \varepsilon$. Therefore r is an ε-retraction. This completes the proof that Y has the property *RNT*.

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