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SPANS OF VARIOUS TWO CELLS, SURFACES, AND SIMPLE CLOSED CURVES

Thelma West

University of Louisiana - Lafayette, USA

Abstract. We define various classes of spaces in this paper. We define a class of surfaces which are formed by the rotation of a particular type of arc through the y-axis. We define a corresponding class of solids, two cells, and related simple closed curves. We determine all the spans of objects in these classes of spaces. We also determine bounds for spans of objects related to spaces in these classes.

1. Introduction

The concept of the span, which can be thought of as a continuous type analogue of the diameter, was introduced in [2]. Later variations of the span of a space, were introduced (cf. [3] and [4]). Generally, it is difficult to calculate the spans of even simple geometric objects. Nor is it easy to determine the relationships of the various spans of a given space.

Our main interest is in continua, that is metric spaces which are compact and connected. We define various classes of continua and we calculate all of the spans of the continua in these classes. We also find bounds for spans of continua which are related to continua in these classes. In particular, for the class of simple closed curves, we show that if X is an element of this class and Y is a plane separating continuum contained in the closure of the unbounded component of $R^2 - X$, then $\sigma(Y) \geq \sigma(X)$. So the question in [1, problem 173, p. 391] is answered in the affirmative for this class of simple closed curves.

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2. Preliminaries

If X is a non-empty metric space, we define the *span of* X, $\sigma(X)$, to be the least upper bound of the set of real numbers α which satisfy the following condition: there exists a connected space C and continuous mappings $g, f : C \to X$ such that

$$(\sigma) \qquad g(C) = f(C)$$

and $\alpha \leq \operatorname{dist}[g(c), f(c)]$ for $c \in C$.

The definition does not require X to be connected, but to simplify our discussion we will now consider X to be connected. The surjective span $\sigma^*(X)$, the semispan $\sigma_0(X)$, and the surjective semispan $\sigma_0^*(X)$ are defined as above, except we change conditions (σ) to the following:

$$(\sigma^*) \qquad g(C) = f(C) = X,$$

$$(\sigma_0) \qquad g(C) \subseteq f(C),$$

$$(\sigma_0^*) \qquad g(C) \subseteq f(C) = X,$$

Equivalently (see [2], p. 209), the span $\sigma(X)$ is the least upper bound of numbers α for which there exist connected subsets C_α of the product $X \times X$ such that

$$(\sigma)' \qquad p_1(C_\alpha) = p_2(C_\alpha)$$

and $\alpha \leq \operatorname{dist}(x, y)$ for $(x, y) \in C_\alpha$, where p_1 and p_2 denote the projections of $X \times X$ onto X, i.e., $p_1(x, y) = x$ and $p_2(x, y) = y$ for $x, y \in X$. Again, we will now consider X to be connected. The surjective span $\sigma^*(X)$, the semispan $\sigma_0(X)$, and the surjective semispan $\sigma_0^*(X)$ are defined as above, except we change conditions $(\sigma)'$ to the following (see [4]):

$$(\sigma^*)' \qquad p_1(C_\alpha) = p_2(C_\alpha) = X,$$

$$(\sigma_0)' \qquad p_1(C_\alpha) \subseteq p_2(C_\alpha),$$

$$(\sigma_0^*)' \qquad p_1(C_\alpha) \subseteq p_2(C_\alpha) = X.$$

We note that, for a compact space X, C in the first set of definitions and C_α in the second set can be considered to be closed. The following inequalities follow immediately from the definitions.

$$\begin{array}{ccccccccc} 0 & \leq & \sigma^*(X) & \leq & \sigma(X) & \leq & \sigma_0(X) & \leq & \operatorname{diam} X, \\ 0 & \leq & \sigma^*(X) & \leq & \sigma_0^*(X) & \leq & \sigma_0(X) & \leq & \operatorname{diam} X. \end{array}$$

It can easily be shown that, if J is an arc then $\sigma(J) = \sigma_0(J) = \sigma^*(J) = \sigma_0^*(J) = 0$. A simple consequence of this is that when X is a simple closed curve, $\sigma(X) = \sigma^*(X)$ and $\sigma_0(X) = \sigma_0^*(X)$.

We utilize the following theorem from [2, section 7].

THEOREM 2.1. *If Y is a closed subset of the Hilbert cube I^ω and $f : Y \to S$ is an essential mapping of Y into the circumference S, then $\inf_{s\in S} \rho\left(f^{-1}(s), f^{-1}(-s)\right) \leq \sigma(Y)$.*

To simplify our exposition, we define and use the following notation. Let W be a subset of either R^2 or R^3. Let

$$R_W = \{w \in W \mid p_1(w) \geq 0\} \quad \text{and}$$
$$L_W = \{w \in W \mid p_1(w) \leq 0\}.$$

Let W be a subset of R^2 and let $w \in W$. We use C_w to denote the circle in R^3 generated by w when W is rotated about the y-axis. We let D_w represent the disc corresponding to C_w. Let J be an arc in the plane such that J and the x-axis intersect in a single point, w. By J_{-s} we denote the arc generated by rotating J about the y-axis, which is the copy of J that goes through the point $-s$ where $s \in C_w$ and $-s$ is the point on C_w antipodal to s. By $\mathrm{cl}(W)$ we mean the closure of W in the space under consideration. We let θ represent the origin either in R^2 or R^3. Let W be a subset of R^2 (or R^3) such that $R^2 - W$ $(R^3 - W)$ consists of two components. Let $H(W)$ represent the closure of the bounded component in R^2 (or R^3), if there is one. For points a, b in R^2 (or R^3) we let $\overline{ab}$ represent the line connecting these two points.

3. MAIN RESULTS

Let f be a concave upward function where $f : [0,p] \to [0,q]$ and $f(0) = q$ and $f(p) = 0$. Let G denote the graph of f in R^2. Let G_y denote the surface generated by rotating G about the y-axis. Let $P = (p, 0)$ and $Q = (0, q)$. Let G_s denote the copy of G in G_y through the point s, where $s \in C_P$.

THEOREM 3.1. *Let G_y be as defined above. Then $\sigma(G_y) = \sigma_0(G_y) = 2p$.*

PROOF. Let $C = \{(s, -s) \mid s \in C_P\}$. The set C is connected, $p_1(C) = p_2(C)$, and $d(s, -s) = 2p$ for each $s \in C_P$. So, $\sigma(G_y) \geq 2p$. Suppose $D \subset G_y \times G_y$ is a closed connected set such that $p_1(D) \subseteq p_2(D)$. Let $p : R^3 \to R^3$ be defined by $p(x, y, z) = (0, y, 0)$. So, $p \circ p_1(D) \subseteq \{0\} \times [0, q] \times \{0\}$.

Since D is closed and connected, $p\circ p_1$ and $p\circ p_2$ are continuous functions, $\{0\}\times[0,q]\times\{0\}$ is an arc, and the semi span of an arc is zero, there is a $d' \in D$ such that $p\circ p_1(d') = p\circ p_2(d')$. So, $p_1(d')$ and $p_2(d')$ are both elements of $C_{p'}$ for some $p' \in G$, where $p' = (x', y', z')$. Hence $d(p_1(d'), p_2(d')) \leq 2x' \leq 2p$. So, $2p \leq \sigma(G_y) \leq \sigma_0(G_y) \leq 2p$, and hence $\sigma(G_y) = \sigma_0(G_y) = 2p$. ∎

Let $C(r, h)$ denote the right circular cone, where r is the radius of the base and h is the height.

COROLLARY 3.2. *For the right circular cone $C(r, h)$, $\sigma(C(r, h)) = \sigma_0(C(r, h)) = 2r$.*

THEOREM 3.3. *Let G_y be as given above, then*

$$\sigma^*(G_y) = \sigma_0^*(G_y) = d(-P, G).$$

PROOF. Let

$$D = \bigcup_{s \in C_P} (\{-s\} \times G_s) \cup (G_s \times \{-s\}).$$

Clearly, D is connected, $p_1(D) = p_2(D) = G_y$ and for all $(x, y) \in D$ $d(x, y) \geq d(-P, S) = d(-P, G)$, where $S \in G$ such that $d(-P, S) = d(-P, G)$. Hence, $\sigma^*(G_y) \geq d(-P, G)$.

Suppose $f, g : C \to G_y$ are continuous functions from a connected set C into G_y such that $f(C) \subseteq g(C) = G_y$ and for each $c \in C$ $d(f(c), g(c)) \geq \sigma_0^*(G)$. Let $p : R^3 \to R^3$ be given by $p(x, y, z) = (0, y, 0)$. Let

$$r : \{0\} \times [0, q] \times \{0\} \to G_P$$
$$l : \{0\} \times [0, q] \times \{0\} \to G_{-P}$$

be defined by $r(0, y, 0) = (x, y, z)$ where (x, y, z) is the corresponding point on G_P and $l(0, y, 0) = (x, y, z)$ where (x, y, z) is the corresponding point on G_{-P}.

Let $m : C \to R$ be defined by $m(c) = m(r \circ p \circ g(c), l \circ p \circ f(c))$ where $m(q_1, q_2)$ is the slope of the line segment between q_1 and q_2 in R^2. Clearly, the line segment between $r \circ p \circ g(c)$ and $l \circ p \circ f(c)$ is never vertical. This line segment is never degenerate since the only way this could be the case is if $r \circ p \circ g(c) = l \circ p \circ f(c) = Q$ for some $c \in C$. But, then $g(c) = f(c)$ and $d(f(c), g(c)) = 0 < \sigma_0^*(G_y)$ which is contrary to our assumption that $d(f(c), g(c)) \geq \sigma_0^*(G_y)$ since $\sigma_0^*(G_y) \geq \sigma^*(G_y) \geq d(-P, G) > 0$. Consequently, m is a well defined function. There is a $c' \in C$ such that $g(c') = Q = p \circ g(c') = r \circ p \circ g(c')$. So, $m(r \circ p \circ g(c'), l \circ p \circ f(c')) > m(-P, S)$, since G_{-P} lies "below" the line in R^2 through Q of slope $m(-P, S)$. Also, there is a $c'' \in C$ such that $g(c'') = P$, $p \circ g(c'') = 0$, $r \circ p \circ g(c'') = P$. In this case $m(r \circ p \circ g(c''), l \circ p \circ f(c'')) \leq 0$. Since C is connected, $[m(c''), m(c')] \subseteq m(C)$. Hence, there is a $c^* \in C$ such that $m(c^*) = m(r \circ p \circ g(c^*), l \circ p \circ f(c^*)) = m(-P, S)$. Let $r \circ p \circ g(c^*) = q_1$ and $l \circ p \circ f(c^*) = q_2$. Since $m(q_1, q_2) = m(-P, S)$, $d(q_1, q_2) \leq d(-P, S)$, because of the construction of G. Hence, $g(c^*) \in C_{q_1}$, and $f(c^*) \in C_{q_2}$, and $d(g(c^*), f(c^*)) \leq d(-P, S)$. Since $d(-P, S) \leq \sigma^*(G_y) \leq \sigma_0^*(G_y) \leq d(-P, S)$, we see that $\sigma^*(G_y) = \sigma_0^*(G_y) = d(-P, S)$. ∎

COROLLARY 3.4. *For the right circular cone $C(r, h)$, when $h \leq r$,*

$$\sigma^*(C(r, h)) = \sigma_0^*(C(r, h)) = \sqrt{r^2 + h^2}$$

and when $h > r$,

$$\sigma^*(C(r, h)) = \sigma_0^*(C(r, h)) = \frac{2rh}{\sqrt{r^2 + h^2}}.$$

THEOREM 3.5. *For the space $G_y \cup D_P$ defined above,*

$$\sigma(G_y \cup D_P) = \sigma_0(G_y \cup D_P) = 2p.$$

PROOF. Same as proof of Theorem 3.1. ∎

THEOREM 3.6. *For the space $G_y \cup D_P$ defined above,*

$$\sigma^*(G_y \cup D_P) = \sigma_0^*(G_y \cup D_P) = \max\{\min\{q, d(-p, G)\}, p\}.$$

PROOF. We consider two cases.

CASE 1: $q \le p$

Let

$$C = (\{-P\} \times R_H) \cup (L_H \times \{P\}) \cup \{(-s, s)|s \in C_P\} \\ \cup (R_H \times \{-P\}) \cup (\{P\} \times L_H),$$

where $H = G_y \cup D_P$. The set C is connected, $p_1(C) = p_2(C) = G_y \cup D_P$, and for all $(x, y) \in C$, $d(x, y) \ge p$. So, $\sigma^*(G_y \cup D_P) \ge p$. Also, for all $x \in G_y$, $d(x, \theta) \le p$. So, $\sigma_0^*(G_y \cup D_P) \le p$. Note that in this case $q \le p \le d(-P, G)$. So, $p = \max\{\min\{q, d(-P, G)\}, p\}$.

CASE 2: $q > p$

Let

$$C = \bigcup_{s \in C_P} ((\{s\} \times G_{-s}) \cup (G_{-s} \times \{s\})) \cup (\{Q\} \times D_P) \cup (D_P \times \{Q\}).$$

The set C is connected, $p_1(C) = p_2(C) = G_y \cup D_P$, and for all $(x, y) \in C$, $d(x, y) \ge \min\{q, d(-p, G)\}$. Hence, $\sigma^*(G_y \cup D_P) \ge \min\{q, d(-P, G)\}$.

It can be shown that $\sigma_0^*(G_y \cup D_P) \le d(-P, G)$ by a proof almost identical to the proof in Theorem 3.3 showing that $\sigma_0^*(G_y) \le d(-P, G)$. Also, for all $x \in G_y \cup D_P$, $d(x, \theta) \le q$. So, $\sigma_0^*(G_y \cup D_P) \le \min\{q, d(-P, G)\} = \max\{\min\{q, d(-P, G)\}, p\}$, and

$$\sigma^*(G_y \cup D_P) = \sigma_0^*(G_y \cup D_P) = \max\{\min\{q, d(-P, G)\}, p\}.$$

∎

THEOREM 3.7. *For the space $H(G_y \cup D_P)$,*

$$\sigma(H(G_y \cup D_P)) = \sigma_0(H(G_y \cup D_P)) = 2p.$$

PROOF. Same as proof of Theorem 3.1. ∎

THEOREM 3.8. *For the space $H(G_y \cup D_P)$,*

$$\sigma^*(H(G_y \cup D_P)) = \sigma_0^*(H(G_y \cup D_P)) = p$$

when $q \le p$, and

$$\sigma^*(H(G_y \cup D_P)) = \sigma_0^*(H(G_y \cup D_P)) = \min\{\frac{p^2 + q^2}{2q}, d(-P, G)\}$$

when $q > p$.

PROOF. We consider two cases. Let $H = H(G_y \cup D_P)$.

CASE 1: $q \leq p$

Let

$$C = (\{-P\} \times R_H) \cup (L_H \times \{P\}) \cup \{(-s,s) | s \in C_P\} \\ \cup (R_H \times \{-P\}) \cup (\{P\} \times L_H)\,.$$

The set C is connected, $p_1(C) = p_2(C) = H(G_y \cup D_P)$, and for all $(x,y) \in C$, $d(x,y) \geq p$. Hence $\sigma^*(H(G_y \cup D_P)) \geq p$. Clearly, $\sigma_0^*(H(G_y \cup D_P)) \leq p$. Since for all $x \in H(G_y \cup D_P)$, $d(x,\theta) \leq p$. Hence $\sigma^*(H(G_y \cup D_P)) = \sigma_0^*(H(G_y \cup D_P)) = p$.

CASE 2: $q > p$

Let

$$\begin{aligned} C = \quad & (\{-P\} \times \{(x,y,z) \in R_H \mid y \geq \tfrac{q^2-p^2}{2q}\}) \\ & \cup (\{-P\} \times G_P) \\ & \cup (G_{-P} \times \{P\}) \\ & \cup (\{(x,y,z) \in L_H \mid y \geq \tfrac{q^2-p^2}{2q}\} \times \{P\}) \\ & \cup \{(s,-s) \mid s \in C_p\} \\ & \cup (G_P \times \{-P\}) \\ & \cup (\{P\} \times G_{-P}) \\ & \cup (\{P\} \times \{(x,y,z) \in L_H \mid y \geq \tfrac{q^2-p^2}{2q}\}) \\ & \cup (\{(x,y,z) \in R_H \mid y \geq \tfrac{q^2-p^2}{2q}\} \times \{-P\}) \\ & \cup (\{Q\} \times \{(x,y,z) \in H \mid y \leq \tfrac{q^2-p^2}{2q}\}) \\ & \cup (\{(x,y,z) \in H \mid y \leq \tfrac{q^2-p^2}{2q}\} \times \{Q\}). \end{aligned}$$

The set C is closed, $p_1(C) = p_2(C) = H(G_y \cup D_P)$, and for all $(x,y) \in C$, $d(x,y) \geq \min\{\frac{p^2+q^2}{2q}, d(-P,G)\}$. So, $\sigma^*(H(G_y \cup D_P)) \geq \min\{\frac{p^2+q^2}{2q}, d(-P,G)\}$. For all $x \in H(G_y \cup D_P)$, $d(x,(0,\frac{q^2-p^2}{2q},0)) \leq \frac{q^2+p^2}{2q}$. Hence, $\sigma_0^*(H(G_y \cup D_P)) \leq \frac{q^2+p^2}{2q}$. By a proof similar to the one in Theorem 3.3 showing that $\sigma_0^*(G_y) \leq d(-P,G)$, we can see that $\sigma_0^*(H(G_y \cup D_P)) \leq d(-P,G)$. Hence, $\sigma^*(H(G_y \cup D_P)) = \sigma_0^*(H(G_y \cup D_P)) \leq \min\{\frac{p^2+q^2}{2q}, d(-P,G)\}$. In this case, $\sigma^*(H(G_y \cup D_P)) = \sigma_0^*(H(G_y \cup D_P)) = \min\{\frac{p^2+q^2}{2q}, d(-P,G)\}$. ∎

THEOREM 3.9. *Let $X = G_P \cup G_{-P} \cup \overline{P(-P)}$, then*

$$\sigma(X) = \sigma_0(X) = \sigma^*(X) = \sigma_0^*(X) = \min\{q, d(-P,G)\}.$$

PROOF. Let

$$C = \{-P\} \times G_P \cup (\overline{P(-P)} \times Q) \cup (P \times G_{-P}) \cup (G_P \times \{-P\}) \cup (Q \times \overline{P(-P)}) \cup (G_{-P} \times \{P\}).$$

The set C is closed, connected, $p_1(C) = p_2(C) = X$, and for all $(x,y) \in C$, $d(x,y) \geq \min\{q, d(-P,G)\}$. Hence, $\sigma^*(X) \geq \min\{q, d(-P,G)\}$.

Let $D \subseteq X \times X$ be connected and closed, such that $P_1(D) \subseteq P_2(D) = X$. Let $p : R^2 \to [-P,P]$ be given by $p(x,y) = x$. So there is a $d' \in D$ such that $p \circ p_1(d') = p \circ p_2(d') = x'$, $p_1(d') = (x', y_1)$, $p_2(d') = (x', y_2)$, $d(p_1(d'), p_2(d')) = |y_1 - y_2| \leq q$.

Let L be the line through the origin O which is perpendicular to the line segment joining $-P$ and S. Note that L is not the y-$axis$, since $P \neq S$. Let $p : X \to L$ be the continuous function that projects points of X perpendicularly onto L. Now , consider $p \circ p_1, p \circ p_2 : D \to L$. Consider the ordering on L given by $(x_1, y_1) \leq (x_2, y_2)$ if and only if $x_1 \leq x_2$, for any $(x_1, y_1), (x_2, y_2) \in L$. Let $A = \{t \in D \mid p \circ p_1(t) \leq p \circ p_2(t)\}$ and $B = \{t \in D | p \circ p_1(t) \geq p \circ p_2(t)\}$. Since, $D = A \cup B$, D is connected, A and B are both closed, it must be that $A \cap B \neq \emptyset$. So, there is a $t' \in D$ such that $p \circ p_1(t') = p \circ p_2(t')$ and $p_1(t')$ and $p_2(t')$ must both be on a line segment S^* which is perpendicular to L and parallel to $\overline{(-P)S}$. Hence, the length of $S^* \leq d(-P,S)$. So, $\sigma(X) = \sigma^*(X) = \sigma_0(X) = \sigma_0^*(X) = \min\{q, d(-P,S)\}$. ∎

THEOREM 3.10. *Let Y be a continuum such that $Y \subseteq H(X)$ and $X = G_P \cup G_{-P} \cup \overline{P(-P)}$. Then, $\tau(Y) \leq \tau(X)$, where $\tau = \sigma, \sigma_0, \sigma^*, \sigma_0^*$.*

PROOF. Similar to the second part of the proof of Theorem 3.9. ∎

THEOREM 3.11. *Let X be as defined above. Then*

$$\sigma(H(X)) = \sigma_0(H(X)) = \min\{q, d(-P,S)\}.$$

PROOF. Similar to the proof of Theorem 3.9. ∎

THEOREM 3.12. *Let X be as defined above. When $p \geq q$,*

$$\sigma^*(H(X)) = \sigma_0^*(H(X)) = q.$$

When $p < q$,

$$\sigma^*(H(X)) = \sigma_0^*(H(X)) = \min\{\frac{q^2+p^2}{2q}, d(-P,G)\}.$$

PROOF. We consider two cases. Let $H = H(X)$.

CASE 1: $p \geq q$

Let

$$D = (\{-P\} \times R_H) \cup (L_H \times \{P\}) \cup \{Q\} \times \overline{(-P)P} \cup (\{P\} \times L_H) \cup (R_H \times \{-P\}).$$

The set D is closed, connected, $p_1(D) = p_2(D) = H(X)$, and for all $(x,y) \in D$, $d(x,y) \geq q$. Hence, $\sigma^*(H(X)) \geq q$. Suppose $D \subseteq H(X) \times H(X)$ is a closed, connected set, such that $p_1(D) \subseteq p_2(D) = H(X)$. Consider the sets $p \circ p_1(D)$ and $p \circ p_2(D)$ where $p(x,y) = x$. They are closed, connected subsets such that $p \circ p_1(D) \subseteq p \circ p_2(D) = J \subset [-P, P]$. So, there exists a $d' \in D$ such that $p \circ p_1(d') = p \circ p_2(d')$. So, $p_1(d') = (x', y_1)$ and $p_2(d') = (x', y_2)$ and $d(p_1(d'), p_2(d')) = d((x', y_1), (x', y_2)) \leq d(y_1, y_2) \leq q$.

Case 2: $p < q$

Let

$$\begin{aligned} C = \ & (\{-P\} \times \{(x,y,z) \in R_H \mid y \geq \tfrac{q^2-p^2}{2q}\}) \\ & \cup(\{-P\} \times G_P) \cup (G_{-P} \times \{P\}) \\ & \cup(\{(x,y,z) \in L_H \mid y \geq \tfrac{q^2-p^2}{2q}\} \times \{P\}) \\ & \cup(\{Q\} \times \{(x,y,z) \in H(X) \mid y \leq \tfrac{q^2-p^2}{2q}\}) \\ & \cup(\{(x,y,z) \in H(X) \mid y \leq \tfrac{q^2-p^2}{2q}\} \times \{Q\}) \\ & \cup(\{(x,y,z) \in R_H \mid y \geq \tfrac{q^2-p^2}{2q}\} \times \{-P\}) \\ & \cup(\{P\} \times \{(x,y,z) \in L_H \mid y \geq \tfrac{q^2-p^2}{2q}\}) \end{aligned}$$

The set C is closed, connected, $p_1(C) = p_2(C) = H(X)$, and for all $(x,y) \in C$, $d(x,y) \geq \min\{d(-P,G), \frac{q^2+p^2}{2q}\}$. So, $\sigma^* H(X) \geq \min\left\{d(-P,G), \frac{q^2+p^2}{2q}\right\}$.

For all $(x,y) \in H(X)$, $d((x,y),(0,\frac{q^2-p^2}{2q})) \leq \frac{q^2+p^2}{2q}$. So, $\sigma_0^*(H(X)) \leq \frac{q^2+p^2}{2q}$. By a proof similar to the one given in Theorem 3.9, we can show that $\sigma_0^*(H(X)) \leq d(-P,G)$. Consequently, $\sigma^*(H(X)) = \sigma_0^*(H(X)) = \min\{\frac{q^2+p^2}{2q}, d(-P,G)\}$. ∎

Theorem 3.13. *Let $X = G_P \cup G_{-P} \cup \overline{P(-P)}$. Let Y be a plane separating continuum such that $X \subseteq \operatorname{cl} B$ where B is a bounded component of $R^2 - Y$, then $\sigma(Y) \geq \sigma(X)$.*

Proof. To simplify the proof, consider that X and Y have been translated into the plane by the translation t where $t : R^2 \to R^2$ is given by $t(x,y) = (x, y - \frac{q}{2})$. Clearly, the spans of X and Y are not affected by t. Let θ' be the acute angle formed by the positive *x-axis* and the ray $\overrightarrow{\theta t(P)}$. Let $0 < \varepsilon < \min\{\frac{\operatorname{diam} G}{4}, \frac{p}{4}\}$. Let $\theta \in (0, \min\{\frac{\pi}{8}, \frac{\theta'}{2}\})$ such that the portion of X contained in the wedge of angle 2θ formed by these pairs of rays, $\overrightarrow{\theta e^{i(\frac{\pi}{2}-\theta)}}$ and $\overrightarrow{\theta e^{i(\frac{\pi}{2}+\theta)}}$, $\overrightarrow{\theta e^{i(\pi+\theta'-\theta)}}$ and $\overrightarrow{\theta e^{i(\pi+\theta'+\theta)}}$, $\overrightarrow{\theta e^{i(2\pi-\theta'-\theta)}}$ and $\overrightarrow{\theta e^{i(2\pi-\theta'+\theta)}}$ is less than $\frac{\varepsilon}{2}$.

Let $q : Y \to S^1$ be given by $q\left(re^{i\gamma}\right) = e^{i\gamma}$. Since Y is a plane separating continuum and θ is in a bounded component of $R^2 - Y$ (this is true since we are considering Y in its new position under the translation t), q is an essential map. Let U be the unbounded component of $R^2 - X$ (where X in its new position under the translation t). We partition $\mathrm{cl}(U)$ into six sets as follows:

$$\begin{aligned}
A &= \{re^{i\alpha} \in \mathrm{cl}(U) \mid \pi/2 - \theta \le \alpha \le \pi/2 + \theta\},\\
B' &= \left\{re^{i\alpha} \in \mathrm{cl}(U) \mid \pi/2 + \theta \le \alpha \le \pi + \theta' - \theta\right\},\\
C &= \left\{re^{i\alpha} \in \mathrm{cl}(U) \mid \pi + \theta' - \theta \le \alpha \le \pi + \theta' + \theta\right\},\\
A' &= \left\{re^{i\alpha} \in \mathrm{cl}(U) \mid \pi + \theta' + \theta \le \alpha \le 2\pi - \theta' - \theta\right\},\\
B &= \left\{re^{i\alpha} \in \mathrm{cl}(U) \mid 2\pi - \theta' - \theta \le \alpha \le 2\pi - \theta' + \theta\right\}, \text{ and}\\
C' &= \left\{re^{i\alpha} \in \mathrm{cl}(U) \mid 2\pi - \theta' + \theta \le \alpha \le 2\pi \textit{ or } 0 \le \alpha \le \frac{\pi}{2} - \theta\right\}.
\end{aligned}$$

If $x \in A$ and $y \in A'$ then $d(x,y) \ge d\left(Q, \overline{P(-P)}\right) - \varepsilon = q - \varepsilon$. If $x \in B$ and $y \in B'$ then $d(x,y) \ge d(P, G_{-P}) - \varepsilon = d(-P, G_P) - \varepsilon$. If $x \in C$ and $y \in C'$ then $d(x,y) \ge d(-P,G) - \varepsilon$. In each of the three cases $d(x,y) \ge \min\{q, d(-P,G)\} - \varepsilon$.

Let $r : S^1 \to S^1$ be a one-to-one continuous function on S^1 such that:

$$\begin{aligned}
r\left(e^{i(2\pi - \theta' + \theta)}\right) &= e^{i0},\\
r\left(e^{i(\pi/2 - \theta)}\right) &= e^{i\pi/3},\\
r\left(e^{i(\pi/2 + \theta)}\right) &= e^{i2\pi/3},\\
r\left(e^{i\left(\pi + \theta' - \theta\right)}\right) &= e^{i\pi},\\
r\left(e^{i\left(\pi + \theta' + \theta\right)}\right) &= e^{i4\pi/3},\\
r\left(e^{i(2\pi - \theta' - \theta)}\right) &= e^{i5\pi/3},
\end{aligned}$$

Consider the function $r \circ q : Y \to S^1$. It is an essential map from Y onto S^1 such that $\inf_{s \in S^1} \left\{ d\left((r \circ q)^{-1}(s), \left(r \circ q^{-1}\right)(-s) \right) \right\} \ge \min\{q, d(-P,G)\} - \varepsilon$. Consequently, by Theorem 2.1, $\sigma(Y) \ge \min\{q, d(-P,G)\} - \varepsilon$. Since ε was arbitrary, we see that $\sigma(Y) \ge \min\{q, d(-P,G)\} = \sigma(X)$. □

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Department of Mathematics
University of Louisiana - Lafayette
LA 70504, USA

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