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ON CERTAIN SUBCLASSES OF p-VALENT FUNCTIONS DEFINED IN TERMS OF CERTAIN FRACTIONAL DERIVATIVE OPERATORS

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Abstract. In the present paper we study certain subclasses $H_{\lambda,\mu,\eta}(a,b,p\sigma)$ and $K_{\lambda,\mu,\eta}(a,b,p\sigma)$ of analytic and p-valent functions. The results presented include coefficient estimates and distortion properties for functions belonging to such classes. Further results giving closure theorems, various properties and modified Hadamard products of several functions belonging to above classes are also mentioned.

1. Introduction and Preliminaries

Denote by A_p the class of functions defined by

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad (a_{p+n} \geq 0;\ p \in \mathbf{N}), \tag{1.1}$$

which are analytic and p-valent in the unit disk $U = \{z : |z| < 1\}$. A function $f(z) \in A_p$ is said to be in the class $H_{\lambda,\mu,\eta}(a,b,p\sigma)$ if and only if

$$\left| \frac{G^p_{\lambda,\mu,\eta} f(z) - 1}{b G^p_{\lambda,\mu,\eta} f(z) - a} \right| < \sigma, \quad (z \in U), \tag{1.2}$$

$$(-1 \leq a < b \leq 1; \quad 0 < b \leq 1; \quad 0 < \sigma \leq 1; \quad \lambda \geq 0;$$
$$\mu < p+1; \quad \eta > \max(\lambda,\mu) - p - 1,\ \forall\, p \in \mathbf{N})$$

where $G^p_{\lambda,\mu,\eta} f(z)$ denotes Saigo's modified (normalized) operator defined by

$$G^p_{\lambda,\mu,\eta} f(z) = \phi_p(\lambda,\mu,\eta) z^{\mu-p} J^{\lambda,\mu,\eta}_{0,z} f(z), \tag{1.3}$$

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with

$$\phi_p(\lambda, \mu, \eta) = \frac{\Gamma(1-\mu+p)\Gamma(1+\eta-\lambda+p)}{\Gamma(1+p)\Gamma(1+\eta-\mu+p)}. \tag{1.4}$$

Further, $f(z) \in A_p$ is said to be in the class $K_{\lambda,\mu,\eta}(a, b, p\sigma)$ if and only if

$$\frac{zf'(z)}{p} \in H_{\lambda,\mu,\eta}(a, b, p\sigma). \tag{1.5}$$

The Saigo fractional integral operator of a function $f(z)$ is defined by ([17])

$$I_{0,z}^{\lambda,\mu,\eta} f(z) = \frac{z^{-\lambda-\mu}}{\Gamma(\lambda)} \int_0^z (z-t)^{\lambda-1} \cdot {}_2F_1\left(\lambda+\mu, -\eta; \lambda; 1-\frac{t}{z}\right) dt. \tag{1.6}$$

The operator $J_{0,z}^{\lambda,\mu,\eta}$ occurring in R.H.S. of (1.3) is the Saigo fractional derivative operator defined as follows [17] (see also [8], [11]):

$$\begin{aligned} J_{0,z}^{\lambda,\mu,\eta} f(z) &= \frac{d^m}{dz^m}(I_{0,z}^{m-\lambda,\mu-m,\eta-m}) \\ &= \frac{d^m}{dz^m}\Bigg\{\frac{z^{\lambda-\mu}}{\Gamma(m-\lambda)} \int_0^z (z-t)^{m-\lambda-1} \cdot {}_2 \\ &\qquad \cdot {}_2F_1\left(\mu-\lambda, m-\eta; m-\lambda; 1-\frac{t}{z}\right) f(t)dt\Bigg\}. \end{aligned} \tag{1.7}$$

$$(m-1 \leq \lambda < m; \quad m \in \mathbf{N}; \quad \mu, \eta \in \mathbf{R})$$

where the function $f(z)$ is analytic in a simply-connected region of the z-plane containing the origin, with the order

$$f(z) = o\left(|z|^r\right) \quad (z \to 0), \tag{1.8}$$

for

$$r > \max(0, \mu-\eta) - 1. \tag{1.9}$$

It being understood that $(z-t)^{m-\lambda-1}$, $(m \in \mathbf{N})$ denotes the principal value for $0 \leq \arg(z-t) < 2\pi$. The operator $J_{o,z}^{\lambda,\mu,\eta}$ includes the well known Riemann-Liouville and Erdélyi-Kober operators of fractional calculus ([13]; see also [10]). Indeed, we have

$$J_{o,z}^{\lambda,\mu,\eta} f(z) = {}_0D_z^{\lambda} f(z), \quad (\lambda \geq 0), \tag{1.10}$$

and

$$J_{o,z}^{\lambda,\mu,\eta} f(z) = \frac{d^m}{dz^m}\left(E_{o,z}^{m-\lambda,\eta-\lambda} f(z)\right). \tag{1.11}$$

$$(m-1 \leq \lambda < m; \; m \in \mathbf{N}; \; \eta \in \mathbf{R}).$$

We shall denote by $(\lambda)_n$ the factorial function defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \left\{ \begin{array}{cc} \lambda(\lambda+1)\cdots(\lambda+n-1), & n \in \mathbf{N}; \\ 1, & n=0 \end{array} \right\} \tag{1.12}$$

$$(\lambda \neq 0, -1, -2, \dots).$$

For $\lambda = \mu = \sigma = 1$, we have

$$H_{1,1,\eta}(a, b, p, 1) = \tau^*(p, a, b), \tag{1.13}$$

and

$$K_{1,1,\eta}(a, b, p, 1) = C(p, a, b), \tag{1.14}$$

where $\tau^*(p, a, b)$ is the subclass of functions $f(z) \in A_p$ satisfying the condition:

$$\left| \left(\frac{f'(z)}{z^{p-1}} - p \right) \left(\frac{bf'(z)}{z^{p-1}} - ap \right)^{-1} \right| < 1. \tag{1.15}$$

$$(-1 \leq a < b \leq 1;\ 0 < b \leq 1;\ p \in \mathbf{N};\ z \in U).$$

The classes $\tau^*(p, a, b)$ and $C(p, a, b)$ were studied by Shukla and Dashrath [14] and Owa and Srivastava [2]. Similarly the subclasses $H_{\lambda,\mu,\eta}(a, b, p, \sigma)$ of A_p for $\mu = \lambda$, $a = 2\alpha - 1$, $b = 1$ reduces to

$$H_{\lambda,\lambda,\eta}(2\alpha - 1, 1, p, \sigma) = T_p(\alpha, \sigma, \lambda), \tag{1.16}$$

where $T_p(\alpha, \sigma, \lambda)$ is the subclass of functions satisfying the condition:

$$\left| \frac{\Omega_z^{(\lambda,p)} f(z) - 1}{\Omega_z^{(\lambda,p)} f(z) + 1 - 2\alpha} \right| < \sigma \tag{1.17}$$

$$(z \in U;\ 0 \leq \alpha < 1;\ 0 < \sigma \leq 1;\ \lambda \geq 0;\ p \in \mathbf{N}).$$

The class of functions $T_p(\alpha, \sigma, \lambda)$ was studied recently by Srivastava and Aouf ([15] and [16]), under rather restricted condition for λ, viz. $0 \leq \lambda < 1$. The Saigo operators (1.3), (1.6) and (1.7) have been invoked in several recent papers devoting to the study of certain classes of univalent (or multivalent) functions. One may refer to the papers [1] – [12], and [15] – [17].

Our purpose in the present paper is to obtain coefficient bounds and distortion properties for the functions belonging to the subclasses $H_{\lambda,\mu,\eta}(a, b, p, \sigma)$ and $K_{\lambda,\mu,\eta}(a, b, p, \sigma)$. Further results include distortion theorems (involving generalized fractional derivative operators), extremal properties, closure theorems, several properties of subclasses, and results involving modified Hadamard products of two functions belonging to the above classes of functions.

2. The class $H_{\lambda,\mu,\eta}(a,b,p,\sigma)$

Before stating and proving our first main result, we mention the following known result which shall be used in the sequel ([17]; see also [11]):

Lemma 2.1. *Let $\lambda,\mu,\eta \in \mathbf{R}$ such that $\lambda \geq 0$; $k > \max(0,\mu-\eta)-1$, then*

$$J_{0,z}^{\lambda,\mu,\eta} z^k = \frac{\Gamma(1+k)\Gamma(1+k-\mu+\eta)}{\Gamma(1+k-\mu)\Gamma(1+k-\lambda+\eta)} z^{k-\mu}. \tag{2.1}$$

We now prove the following result giving the coefficient bounds for a function $f(z)$ belonging to the class $H_{\lambda,\mu,\eta}(a,b,p,\sigma)$.

Theorem 2.2. *Let the function $f(z)$ be defined by (1.1). Then $f(z) \in H_{\lambda,\mu,\eta}(a,b,p,\sigma)$ if and only if*

$$\sum_{n=1}^{\infty} \delta_n(\lambda,\mu,\eta,p)(1+b\sigma)a_{p+n} \leq (b-a)\sigma, \tag{2.2}$$

where

$$\delta_n(\lambda,\mu,\eta,p) = \frac{\phi_p(\lambda,\mu,\eta)}{\phi_{p+n}(\lambda,\mu,\eta)} = \frac{(1+p)_n(1+\eta-\mu+p)_n}{(1-\mu+p)_n(1+\eta-\lambda+p)_n}, \tag{2.3}$$

and $\pi_p(\lambda,\mu,\eta)$ is given by (1.4).
The condition (2.2) is sharp.

Proof. Let $f(z) \in H_{\lambda,\mu,\eta}(a,b,p,\sigma)$. Then in view of (1.2), we have

$$\left|\frac{G^p_{\lambda,\mu,\eta}f(z)-1}{bG^p_{\lambda,\mu,\eta}f(z)-a}\right| = \left|\frac{\sum_{n=1}^{\infty}\delta_n(\lambda,\mu,\eta,p)a_{p+n}z^n}{(b-a)-b\sum_{n=1}^{\infty}\delta_n(\lambda,\mu,p)a_{p+n}z^n}\right| < \sigma. \tag{2.4}$$

Since $|\Re(z)| \leq |z|$, we get from (2.4) that

$$\Re\left\{\frac{\sum_{n=1}^{\infty}\delta_n(\lambda,\mu,\eta,p)a_{p+n}z^n}{(b-a)-b\sum_{n=1}^{\infty}\delta_n(\lambda,\mu,p)a_{p+n}z^n}\right\} < \sigma. \tag{2.5}$$

Choosing values of z on the real axis, and letting $z \to 1_-$ through real values, we arrive at the assertion (2.2).

Conversely, let the inequality (2.2) hold true. Then

$$\begin{aligned}&\left|G^p_{\lambda,\mu,\eta}f(z)-1\right| - \sigma\left|bG^p_{\lambda,\mu,\eta}f(z)-a\right| \\ &\qquad < \sum_{n=1}^{\infty}\delta_n(\lambda,\mu,\eta,p)(1+b\sigma)a_{p+n}-(b-a)\sigma \leq 0,\end{aligned}$$

by the assumption. This implies that $f(z) \in H_{\lambda,\mu,\eta}(a,b,p,\sigma)$.

We note that the assertion (2.2) of Theorem 2.2 is sharp, and the extremal function is given by

$$f(z) = z^p \frac{(b-a)\sigma}{(1+b\sigma)\delta_n(\lambda,\mu,\eta,p)} z^{p+n}. \tag{2.6}$$

□

THEOREM 2.3. *Let the function $f(z)$ be defined by (1.1). Then $f(z) \in K_{\lambda,\mu,\eta}(a,b,p,\sigma)$ if and only if*

$$\sum_{n=1}^{\infty} \delta_n(\lambda,\mu,\eta,p)(1+b\sigma)(p+n)a_{p+n} \leq (b-a)p\sigma. \tag{2.7}$$

The condition (2.7) is sharp.

PROOF. The desired assertion (2.7) follows easily on using (1.5) and (2.2). The result (2.7) is sharp, and the extremal function is given by

$$f(z) = z^p \frac{(b-a)p\sigma}{(p+n)(1+b\sigma)\delta_n(\lambda,\mu,\eta,p)} z^{p+n}. \tag{2.8}$$

□

3. Distortion Properties

We prove two results giving distortion properties of $f(z)$ belonging to the classes $H_{\lambda,\mu,\eta}(a,b,p,\sigma)$ and $K_{\lambda,\mu,\eta}(a,b,p,\sigma)$.

THEOREM 3.1. *Let $\lambda,\mu,\eta \in \mathbf{R}$ satisfy the inequalities*

$$\lambda \geq 0;\ \mu < p+1;\ \eta \geq \frac{\lambda(\mu-2-p)}{\mu};\ p \in \mathbf{N}. \tag{3.1}$$

Also, let the function $f(z)$ defined by (1.1) be in the class $H_{\lambda,\mu,\eta}(a,b,p,\sigma)$. Then

$$|f(z)| \geq |z|^p - \frac{\sigma(b-a)(1+p-\mu)(1+p+\eta-\lambda)}{(1+b\sigma)(1+p)(1+p+\eta-\mu)}|z|^{p+1}, \tag{3.2}$$

and

$$|f(z)| \leq |z|^p + \frac{\sigma(b-a)(1+p-\mu)(1+p+\eta-\lambda)}{(1+b\sigma)(1+p)(1+p+\eta-\mu)}|z|^{p+1}, \tag{3.3}$$

for $z \in U$, provided that $-1 \leq a < b \leq 1$; $0 < b \leq 1$; $0 < \sigma \leq 1$.

PROOF. We observe that $\delta_n(\lambda,\mu,\eta,p) \leq \delta_{n+1}(\lambda,\mu,\eta,p)$, $(n \in \mathbf{N})$ is satisfied when $\eta \geq \frac{\lambda(\mu-2-p)}{\mu}$ where $\delta_n(\lambda,\mu,\eta,p)$ is defined by (2.3). Thus, under the conditions stated in (3.1), it is observed that $\delta_n(\lambda,\mu,\eta,p)$ is non-decreasing for $n \geq 1$, and we have

$$0 < \frac{(1+p)(1+p+\eta-\mu)}{(1+p-\mu)(1+p+\eta-\lambda)} = \delta_1(\lambda,\mu,\eta,p) \leq \delta_n(\lambda,\mu,\eta,p),\ \forall\, n \in \mathbf{N}. \tag{3.4}$$

Since $f(z) \in H_{\lambda,\mu,\eta}(a,b,p,\sigma)$, therefore, in view of Theorem 2.2 and (3.4), we have

$$(1+b\sigma)\delta_1(\lambda,\mu,\eta,p)\sum_{n=1}^{\infty} a_{p+n} \leq \sum_{n=1}^{\infty}\delta_n(\lambda,\mu,\eta,p)(1+b\sigma)a_{p+n} \leq (b-a)\sigma. \tag{3.5}$$

This implies that

$$\sum_{n=1}^{\infty} a_{p+n} \leq \frac{(b-a)\sigma}{(1+b\sigma)\delta_1(\lambda,\mu,\eta,p)}. \tag{3.6}$$

On using (1.1), (3.4) and (3.6), we easily arrive at the desired results (3.2) and (3.3). ∎

Corollary 3.2. *Under the hypothesis of Theorem 3.1, $f(z)$ is included in a disc with centre at the origin and radius r given by*

$$r = 1 + \frac{\sigma(b-a)(1+p-\mu)(1+p+\eta-\lambda)}{(1+b\sigma)(1+p)(1+p+\eta-\mu)}. \tag{3.7}$$

Similarly, we can prove the following:

Theorem 3.3. *Let $\lambda,\mu,\eta \in \mathbf{R}$ satisfy the inequalities $\lambda \geq 0$; $\mu < p+1$; $\eta \geq \frac{\lambda(\mu-2-p)}{\mu}$; $p \in \mathbf{N}$. Also, let the function $f(z)$ defined by (1.1) be in the class $K_{\lambda,\mu,\eta}(a,b,p,\sigma)$. Then*

$$|f(z)| \geq |z|^p - \frac{p\sigma(b-\sigma)(1+p-\mu)(1+p+\eta-\lambda)}{(1+b\sigma)(1+p)^2(1+p+\eta-\mu)}|z|^{p+1}, \tag{3.8}$$

and

$$|f(z)| \geq |z|^p + \frac{p\sigma(b-\sigma)(1+p-\mu)(1+p+\eta-\lambda)}{(1+b\sigma)(1+p)^2(1+p+\eta-\mu)}|z|^{p+1}, \tag{3.9}$$

for $z \in U$, provided that $-1 \leq a < b \leq 1$; $0 < b \leq 1$; $0 < \sigma \leq 1$.

Corollary 3.4. *Under the hypothesis of Theorem 3.3, $f(z)$ is included in a disc with centre at the origin and radius r' given by*

$$r' = 1 + \frac{p\sigma(b-a)(1+p-\mu)(1+p+\eta-\lambda)}{(1+b\sigma)(1+p)^2(1+p+\eta-\mu)}. \tag{3.10}$$

For convenience sake, we denote

$$M^p_{\lambda,\mu,\eta}f(z) = z^p G^p_{\lambda,\mu,\eta}f(z) = \phi_p(\lambda,\mu,\eta)z^\mu J^{\lambda,\mu,\eta}_{o,z}f(z), \tag{3.11}$$

involving the generalized fractional derivative operator (1.7), where $\phi_p(\lambda,\mu,\eta)$ is defined by (1.4).

Using Lemma 2.1 and [1, Theorem 1, pp. 28-29], we can easily prove the following result:

THEOREM 3.5. *Let $\lambda \geq 0$; $\mu < p+1$; $\eta > \max(\lambda,\mu) - p - 1$, $\forall\, p \in \mathbf{N}$. Then, the generalized fractional derivative operator $M^p_{\lambda,\mu,\eta}$ maps the class A_p into itself, and the image of a power series (1.1) has the form*

$$M^p_{\lambda,\mu,\eta} f(z) = z^p - \sum_{n=1}^{\infty} \delta_n(\lambda,\mu,\eta,p) a_{p+n} z^{p+n}, \tag{3.12}$$

and belongs to A_p, where $\delta_n(\lambda,\mu,\eta,p)$ (>0) is given by (2.3).

4. Further Distortion Properties

We next prove two further distortion theorems involving generalized fractional derivative operator $M^p_{\lambda,\mu,\eta}$.

THEOREM 4.1. *Let $\lambda \geq 0$; $\mu < p+1$; $\eta > \max(\lambda,\mu) - p - 1$, $\forall\, p \in \mathbf{N}$, and let the function $f(z)$ defined by (1.1) belong to the class $H_{\lambda,\mu,\eta}(a,b,p,\sigma)$. Then*

$$|M^p_{\lambda,\mu,\eta} f(z)| \geq |z|^p - \frac{(b-a)\sigma}{(1+b\sigma)} |z|^{p+1}, \tag{4.1}$$

and

$$|M^p_{\lambda,\mu,\eta} f(z)| \leq |z|^p + \frac{(b-a)\sigma}{(1+b\sigma)} |z|^{p+1}, \tag{4.2}$$

for $z \in U$, where $M^p_{\lambda,\mu,\eta} f(z)$ is defined by (3.12).

PROOF. Using Theorem 2.2 and (3.12), we can easily obtain the desired results (4.1) and (4.2). We omit further details. □

Similarly we can establish the following result:

THEOREM 4.2. *Let $\lambda \geq 0$; $\mu < p+1$; $\eta > \max(\lambda,\mu) - p - 1$, $\forall\, p \in \mathbf{N}$, and let the function $f(z)$ defined by (1.1) belong to the class $K_{\lambda,\mu,\eta}(a,b,p,\sigma)$. Then*

$$|M^p_{\lambda,\mu,\eta} f(z)| \geq |z|^p - \frac{\sigma p(b-a)}{(1+p)(1+b\sigma)} |z|^{p+1}, \tag{4.3}$$

and

$$|M^p_{\lambda,\mu,\eta} f(z)| \leq |z|^p + \frac{\sigma p(b-a)}{(1+p)(1+b\sigma)} |z|^{p+1}, \tag{4.4}$$

for $z \in U$, where $M^p_{\lambda,\mu,\eta} f(z)$ is defined by (3.12).

5. Properties of classes $H_{\lambda,\mu,\eta}(a,b,p,\sigma)$ and $K_{\lambda,\mu,\eta}(a,b,p,\sigma)$

The proofs of the following assertions are analogous to the similar results proved in [2], and we omit details involved.

Theorem 5.1. *Let $\lambda \geq 0$; $\mu < p+1$; $\eta > \max(\lambda,\mu) - p - 1$, $\forall p \in \mathbf{N}$, $-1 \leq a_1 \leq a_2 \leq 1$; $0 < b_1 \leq b_2 \leq 1$; $a_2 < b_1$; $0 < \sigma \leq 1$. Then*

$$H_{\lambda,\mu,\eta}(a_1,b_2,p,\sigma) \supset H_{\lambda,\mu,\eta}(a_2,b_1,p,\sigma). \tag{5.1}$$

Theorem 5.2. *Let $\lambda \geq 0$; $\mu < p+1$; $\eta > \max(\lambda,\mu) - p - 1$, $\forall p \in \mathbf{N}$, $-1 \leq a_1 \leq a_2 \leq 1$; $0 < b_1 \leq b_2 \leq 1$; $a_2 < b_1$; $0 < \sigma \leq 1$. Then*

$$K_{\lambda,\mu,\eta}(a_1,b_2,p,\sigma) \supset K_{\lambda,\mu,\eta}(a_2,b_1,p,\sigma). \tag{5.2}$$

Theorem 5.3. *Under constraints stated with (1.2), let the function $f(z)$ defined by (1.1) be in the class $K_{\lambda,\mu,\eta}(a,b,p,\sigma)$. Then*

$$f(z) \in H_{\lambda,\mu,\eta}\left(\frac{bp+a}{1+p}, b, p, \sigma\right). \tag{5.3}$$

6. Closure Theorems

Let the functions $f_i(z)$ be defined for $i = 1,\ldots,m$ by

$$f_i(z) = z^p - \sum_{n=1}^{\infty} a_{i,p+n} z^{p+n}, \quad (a_{i,p+n} \geq 0,\ p \in \mathbf{N}), \tag{6.1}$$

for $z \in U$.

Now, we state some results for the closure under "arithmetic mean" of functions in the classes $H_{\lambda,\mu,\eta}(a,b,p,\sigma)$ and $K_{\lambda,\mu,\eta}(a,b,p,\sigma)$. The proofs run parallel to similar such assertions proved in [2] and [14], and we omit details.

Theorem 6.1. *Let the functions $f_i(z)$ defined by (6.1) be in the class $H_{\lambda,\mu,\eta}(a_i,b_i,p,\sigma)$ for each $i = 1,\ldots,m$. Then the function $h(z)$ defined by*

$$h(z) = z^p - \frac{1}{m}\sum_{n=1}^{\infty}\left(\sum_{i=1}^{m} a_{i,p+n}\right) z^{p+n}, \tag{6.2}$$

is in the class $H_{\lambda,\mu,\eta}(a,b,p,\sigma)$, where

$$a = \min_{1\leq i\leq m}(a_i), \quad b = \max_{1\leq j\leq m}(b_j). \tag{6.3}$$

Theorem 6.2. *Let the functions $f_i(z)$ defined by (6.1) be in the class $K_{\lambda,\mu,\eta}(a_i,b_i,p,\sigma)$ for each $i = 1,\ldots,m$. Then the function $h(z)$ defined by (6.2) is in the class $K_{\lambda,\mu,\eta}(a,b,p,\sigma)$, where a and b are defined by (6.3).*

Theorem 6.3. *Let the function $f(z)$ defined by (1.1) and the function $g(z)$ defined by*

$$g(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n}, \quad (b_{p+n} \geq 0,\ p \in \mathbf{N}), \tag{6.4}$$

be in the classes $H_{\lambda,\mu,\eta}(a,b,p,\sigma)$ *and* $K_{\lambda,\mu,\eta}(a,b,p,\sigma)$*, respectively. Then the function* $F(z)$ *defined by*

$$F(z) = z^p - \left(\frac{p+1}{2p+1}\right)\sum_{n=1}^{\infty}(a_{p+n} + b_{p+n})z^{p+n} \tag{6.5}$$

is in the class $H_{\lambda,\mu,\eta}(a,b,p,\sigma)$*.*

7. Modified Hadamard products

Let the function $f(z)$ be defined by (1.1) and the function $g(z)$ be defined by (6.4). Define the usual modified Hadamard product of two functions $f(z)$ and $g(z)$ by

$$f * g(z) = z^p - \sum_{n=1}^{\infty} a_{p+n}b_{p+n}z^{p+n}. \tag{7.1}$$

The proofs of the following assertions are analogous to the corresponding similar results proved in [2], and we omit details involved.

Theorem 7.1. *Under constraints stated with (1.2), let the function* $f(z)$ *defined by (1.1) be in the class* $H_{\lambda,\mu,\eta}(a_1,b_1,p,\sigma)$ *and let the function* $g(z)$ *defined by (6.4) be in the class* $H_{\lambda,\mu,\eta}(a_2,b_2,p,\sigma)$*. Then, the modified Hadamard product* $f*g(z)$ *defined by (7.1) is in the class* $H_{\lambda,\mu,\eta}(a(2b-a),b^2,p,\sigma)$*, where*

$$a = \min(a_1,a_2) \quad \textit{and} \quad b = \max(b_1,b_2). \tag{7.2}$$

Theorem 7.2. *Under constraints stated with (1.2), let the function* $f(z)$ *defined by (1.1) be in the class* $K_{\lambda,\mu,\eta}(a_1,b_1,p,\sigma)$*, and let the function* $g(z)$ *defined by (6.4) be in the class* $K_{\lambda,\mu,\eta}(a_2,b_2,p,\sigma)$*. Then, the modified Hadamard product* $f*g(z)$ *defined by (7.1) is in the class* $K_{\lambda,\mu,\eta}(a(2b-a),b^2,p,\sigma)$*, where*

$$a = \min(a_1,a_2) \quad \textit{and} \quad b = \max(b_1,b_2). \tag{7.3}$$

Theorem 7.3. *Under constraints stated with (1.2), let the function* $f(z)$ *defined by (1.1) and the function* $g(z)$ *defined by (6.4) be in the same class* $H_{\lambda,\mu,\eta}(a,b,p,\sigma)$*. Then, the modified Hadamard product* $f*g(z)$ *defined by (7.1) is in the class* $K_{\lambda,\mu,\eta}(a(2b-a),b^2,p,\sigma)$*.*

Theorem 7.4. *Under constraints stated with (1.2), let the function* $f(z)$ *defined by (1.1) and the function* $g(z)$ *defined by (6.4) be in the same class* $H_{\lambda,\mu,\eta}(a,b,p,\sigma)$*. Then, the modified Hadamard product* $f*g(z)$ *defined by (7.1) is in the class* $K_{\lambda,\mu,\eta}\left(\frac{(1+2a)b-a^2}{1-b},b,p,\sigma\right)$*.*

8. Extremal properties

Let the function $f(z)$ defined by (1.1), and the function $g(z)$ defined by (6.4) be in the class A_p, then the class A_p is said to be convex if

$$\alpha f(z) + (1-\alpha)g(z) \in A_p, \tag{8.1}$$

where $0 \le \alpha \le 1$.

Now, we prove the convexity of the classes $H_{\lambda,\mu,\eta}(a,b,p,\sigma)$ and $K_{\lambda,\mu,\eta}(a,b,p,\sigma)$.

Theorem 8.1. *The class $H_{\lambda,\mu,\eta}(a,b,p,\sigma)$ is convex.*

Proof. Let the function $f(z)$ defined by (1.1) and the function $g(z)$ defined by (6.4) be in the class $H_{\lambda,\mu,\eta}(a,b,p,\sigma)$, then

$$\alpha f(z) + (1-\alpha)g(z) = z^p - \sum_{n=1}^{\infty} \{\alpha a_{p+n} + (1-\alpha)b_{n+p}\} z^{p+n}. \tag{8.2}$$

Applying Theorem 2.2 for the functions $f(z)$ and $g(z)$, we get

$$\sum_{n=1}^{\infty}(1+b\sigma)\delta_n(\lambda,\mu,\eta,p)\{\alpha a_{p+n} + (1-\alpha)b_{p+n}\} \le (b-a)\sigma.$$

This implies that the class $H_{\lambda,\mu,\eta}(a,b,p,\sigma)$ is convex. ∎

Similarly, we can prove the following result:

Theorem 8.2. *The class $K_{\lambda,\mu,\eta}(a,b,p,\sigma)$ is convex.*

Theorem 8.3. *Let*

$$\psi_p(z) = z^p, \quad (p \in \mathbf{N}), \tag{8.3}$$

and

$$\psi_{p+n}(z) = z^p - \frac{(b-a)\sigma}{(1+b\sigma)\delta_n(\lambda,\mu,\eta,p)} z^{p+n}, \quad (n \in \mathbf{N}). \tag{8.4}$$

Then $f(z) \in H_{\lambda,\mu,\eta}(a,b,p,\sigma)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \alpha_{p+n}\psi_{p+n}(z), \tag{8.5}$$

where

$$\alpha_{p+n} \ge 0 \quad \textit{and} \quad \sum_{n=0}^{\infty} \alpha_{p+n} = 1. \tag{8.6}$$

Proof. Let $f(z) \in H_{\lambda,\mu,\eta}(a,b,p,\sigma)$. Then

$$a_{p+n} \le \frac{(b-a)\sigma}{(1+b\sigma)\delta_n(\lambda,\mu,\eta,p)}, \quad (n \in \mathbf{N}).$$

By taking

$$\alpha_{p+n} = \frac{(1+b\sigma)\delta_n(\lambda,\mu,\eta,p)}{(b-a)\sigma} a_{p+n}, \quad (n \in \mathbf{N}), \tag{8.7}$$

and

$$\alpha_p = 1 - \sum_{n=1}^{\infty} \alpha_{p+n}, \tag{8.8}$$

then from (8.3), (8.4), (8.7) and (8.8), we get the desired assertion (8.5).

Conversely, let

$$f(z) = \sum_{n=0}^{\infty} \alpha_{p+n}\psi_{p+n}(z) = z^p - \sum_{n=1}^{\infty} \frac{(b-a)\sigma}{(1+b\sigma)\delta_n(\lambda,\mu,\eta,p)} \alpha_{p+n} z^{p+n}. \tag{8.9}$$

Then, we have (in view of (8.6))

$$\begin{aligned}
&\sum_{n=1}^{\infty}(1+b\sigma)\delta_n(\lambda,\mu,\eta,p)\left\{\frac{(b-a)\sigma}{(1+b\sigma)\delta_n(\lambda,\mu,\eta,p)}\alpha_{p+n}\right\} \\
&\quad = (b-a)\sigma \sum_{n=1}^{\infty} \alpha_{p+n} = (b-a)\sigma(1-\alpha_p) \\
&\quad \leq (b-a)\sigma. \qquad (8.10)
\end{aligned}$$

It follows therefore from Theorem 2.2 that $f(z) \in H_{\lambda,\mu,\eta}(a,b,p,\sigma)$. □

Corollary 8.4. *The extreme points of the class $H_{\lambda,\mu,\eta}(a,b,p,\sigma)$ are the functions $\psi_{p+n}(z)$, $(n \in \mathbf{N} \cup \{0\})$ given by Theorem 8.3.*

Similarly, we can prove the following result:

Theorem 8.5. *Let*

$$\psi_p(z) = z^p, \quad (p \in \mathbf{N}), \tag{8.11}$$

and

$$\psi_{p+n}(z) = z^p - \frac{(b-a)p\sigma}{(p+n)(1+b\sigma)\delta_n(\lambda,\mu,\eta,p)} z^{p+n}, \quad (n \in \mathbf{N}). \tag{8.12}$$

Then $f(z) \in K_{\lambda,\mu,\eta}(a,b,p,\sigma)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \alpha_{p+n}\psi_{p+n}(z), \tag{8.13}$$

where

$$\alpha_{p+n} \geq 0 \quad \text{and} \quad \sum_{n=0}^{\infty} \alpha_{p+n} = 1. \tag{8.14}$$

Corollary 8.6. *The extreme points of the class $K_{\lambda,\mu,\eta}(a,b,p,\sigma)$ are the functions $\psi_{p+n}(z)$, $(n \in \mathbf{N} \cup \{0\})$ given by Theorem 8.5.*

We conclude this paper by remarking that several new (and known) results can be deduced from those incorporated in this paper by assigning appropriate special values to the parameters λ, μ, η, a, b, p, σ. In particular, in view of the relationship (1.10), the results of [2] and [14] follow from those given above.

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