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GROWTH OF MAXIMUM MODULUS OF POLYNOMIALS WITH PRESCRIBED ZEROS

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Abstract. Let $P(z)$ be a polynomial of degree n not vanishing in $|z| < k$ where $k \geq 1$. It is shown that

$$\max_{|z|=R>1} |P(z)| < \frac{(R+k)^n}{(R+k)^n + (1+Rk)^n} \times \left\{ (R^n+1) \max_{|z|=1} |P(z)| - \left(R^n - \left(\frac{1+Rk}{R+k} \right)^n \right) \min_{|z|=k} |P(z)| \right\}.$$

Among other things our result includes a refinement of a theorem due to Ankeny and Rivilin as a special case. We shall also prove an another result of similar nature.

Let $P(z)$ be a polynomial of degree n, then

$$\max_{|z|=R>1} |P(z)| \leq R^n \max_{|z|=1} |P(z)|. \tag{1}$$

Inequality (1) is a simple deduction from Maximum Modulus Principle (see [6, vol. 1, p. 137, problem III 269] or [7, p. 346]). It was shown by Ankeny and Rivilin [1] (see also [5, p. 442]), that if $P(z) \neq 0$ in $|z| < 1$, then (1) can be replaced by

$$\max_{|z|=R>1} |P(z)| \leq \frac{R^n+1}{2} \max_{|z|=1} |P(z)|. \tag{2}$$

Inequality (2) is sharp, with equality for $P(z) = \alpha z^n + \beta$, $|\alpha| = |\beta| = 1$. For the class of polynomials not vanishing in the disk $|z| < k$, $k \geq 1$, Aziz and Mohammad [4] proved the following generalization of inequality (2).

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THEOREM 1. *Let $P(z)$ be a polynomial of degree n having no zeros in the disk $|z| < k$, where $k > 1$, then*

$$\max_{|z|=R>1} |P(z)| \le \frac{(R^n+1)(R+k)^n}{(R+k)^n+(1+Rk)^n} \max_{|z|=1} |P(z)|.$$

Theorem 1 does not appear to be sharp for $k > 1$ with the exception $n = 1$. However Aziz [2] (see also [3]) have proved the following sharp result which is an interesting generalization of inequality (2).

THEOREM 2. *Let $P(z)$ be a polynomial of degree n which does not vanish in the disk $|z| < 1$, then*

$$\max_{|z|=R>1} |P(z)| \le \left(\frac{R^n+1}{2}\right) \max_{|z|=1} |P(z)| - \left(\frac{R^n-1}{2}\right) \min_{|z|=1} |P(z)|.$$

Here the result is best possible and equality holds for $P(z) = \alpha z^n + \beta$ where $|\beta| \ge |\alpha|$

In this paper we first prove the following more general result which provides a refinement of Theorem 1 and includes Theorem 2 as a special case.

THEOREM 3. *If $P(z)$ is a polynomial of degree n which does not vanish in $|z| < k$ where $k \ge 1$, then*

$$\max_{|z|=R>1} |P(z)| < \frac{(R+k)^n}{(R+k)^n+(1+Rk)^n} \times$$

$$\left\{(R^n+1)\max_{|z|=1}|P(z)| - \left(R^n - \left(\frac{1+Rk}{R+k}\right)^n\right)\min_{|z|=k}|P(z)|\right\}. \tag{3}$$

For $k = 1$, this reduces to Theorem 2.

If $P(z)$ does not vanish in $|z| < k$, where $k \ge 1$ then it is known (see [4, inequality (6)]) that

$$\max_{|z|=R} |P(z)| \le \left(\frac{R+k}{1+k}\right)^n \max_{|z|=1} |P(z)| \quad \text{for} \quad 1 \le R^2 \le k. \tag{4}$$

The result is best possible and equality in (4) holds for $P(z) = ((z+k)/(1+k))^n$. Here we present the following refinement of (4).

THEOREM 4. *If $P(z)$ is a polynomial of degree n having no zeros in the disk $|z| < k$ where $k \ge 1$ then for $1 \le R \le k^2$ we have*

$$\max_{|z|=R} |P(z)| \le \left(\frac{R+k}{1+k}\right)^n \max_{|z|=1} |P(z)| - \left\{\left(\frac{R+k}{1+k}\right)^n - 1\right\} \min_{|z|=k} |P(z)|.$$

REMARK 5. Theorem 3 in general provides much better information than Theorem 1 regarding $\max_{|z|=R>1} |P(z)|$. We illustrate this with the help of following examples.

Example 6. Let

$$P(z) = (z^2 + 9)(z - 19).$$

Then $P(z)$ is a polynomial of degree 3 which does not vanish in $|z| < t$, where $0 < t \leq 3$. Clearly

$$|P(z)| \geq \left\{9 - |z|^2\right\} \{19 - |z|\}$$

which in particular gives

$$\min_{|z|=2} |P(z)| \geq 85 \quad \text{and} \quad \max_{|z|=1} |P(z)| = 200$$

Using Theorem 1 with $k = t = 3$, $R = 2$, it follows that

$$\max_{|z|=2} |P(z)| \leq 480.8 \tag{5}$$

where as using Theorem 3 with $k = 2$, and $R = 2$, we get

$$\max_{|z|=2} |P(z)| \leq 435.5$$

which is much better than (5).

Example 7. Let

$$P(z) = z^3 + 3^3,$$

then $P(z)$ does not vanish in $|z| < t$, where $0 < t \leq 3$. Clearly

$$\min_{|z|=2} |P(z)| \geq 19 \quad \text{and} \quad \max_{|z|=1} |P(z)| = 28.$$

Using Theorem 1 with $k = t = 3$, $R = 2$, it follows that

$$\max_{|z|=2} |P(z)| \leq 67.4. \tag{6}$$

We use Theorem 3 with $k = t = 2$, $R = 2$, we get

$$\max_{|z|=2} |P(z)| \leq 46.5$$

which is much better than (6).

Similar remarks apply to Theorem 4 also. For the proof of Theorem 3 we need the following lemma.

Lemma 8. *If $P(z)$ is a polynomial of degree n which does not vanish for $|z| < k$, $k > 0$ then for all $R \geq 1$, $r \leq k$ and for every θ, $0 \leq \theta < 2\pi$*

$$|P(Rre^{i\theta})| < \left(\frac{Rr + k}{r + Rk}\right)^n \left|R^n P\left(\frac{re^{i\theta}}{R}\right)\right| - \left\{\left(\frac{Rr + k}{r + Rk}\right)^n\right\} \min_{|z|=k} |P(z)|. \tag{7}$$

Proof. The result is obvious for $R = 1$. So we assume $R > 1$. By hypothesis, the polynomial $P(z)$ has all its zeros in $|z| \geq k$ and $m = \min_{|z|=k} |P(z)|$, therefore, $m \leq |P(z)|$ for $|z| \leq k$. We show for any given complex number α with $|\alpha| \leq 1$, the polynomial $F(z) = P(z) + \alpha m$ has all its zeros in $|z| \geq k$. This is obvious if $m = 0$ that is if $P(z)$ has a zero on $|z| = k$. We now suppose that all the zeros of $P(z)$ lie in $|z| > k$ so that $m = \min_{|z|=k} |P(z)| > 0$. Hence $\frac{m}{P(z)}$ is analytic for $|z| \leq k$ and $\left|\frac{m}{P(z)}\right| \leq 1$ for $|z| = k$. Since $\frac{m}{P(z)}$ is not a constant, it follows by Maximum Modulus Principle that

$$m < |P(z)| \quad \text{for} \quad |z| < k. \tag{8}$$

Now assume that $F(z) = P(z) + \alpha m$ has a zero in $|z| < k$, say at $z = z_0$ with $|z_0| < k$, then

$$P(z_0) + \alpha m = F(z_0) = 0.$$

This implies

$$|P(z_0)| = |\alpha m| \leq m,$$

which is a contradiction to (8). Hence we conclude that in any case $F(z) = P(z) + \alpha m$ has all its zeros in $|z| \geq k$. Let

$$R_1 e^{i\theta_1}, R_2 e^{i\theta_2}, \ldots, R_n e^{i\theta_n}$$

be the zeros of $F(z)$. Then $R_j \geq k$, $j = 1, 2, \ldots, n$ and we have

$$F(z) = \prod_{j=1}^{n} (z - R_j e^{i\theta_j}),$$

therefore, for all $R \geq 1$, $r \leq k$ and for every θ, $0 \leq \theta < 2\pi$, we have

$$\begin{aligned} \left|\frac{F(Rre^{i\theta})}{R^n F\left(\frac{re^{i\theta}}{R}\right)}\right| &= \prod_{j=1}^{n} \left|\frac{Rre^{i\theta} - R_j e^{i\theta_j}}{re^{i\theta} - RR_j e^{i\theta_j}}\right| \\ &= \prod_{j=1}^{n} \left|\frac{Rre^{i(\theta-\theta_j)} - R_j}{re^{i(\theta-\theta_j)} - RR_j}\right|. \end{aligned} \tag{9}$$

Since $R_j \geq k \geq r$ and $R \geq 1$, therefore, it can be easily verified after a short calculation that

$$\begin{aligned} \left|\frac{Rre^{i(\theta-\theta_j)} - R_j}{re^{i(\theta-\theta_j)} - RR_j}\right| &= \left(\frac{R^2r^2 + R_j^2 - 2RrR_j \cos(\theta-\theta_j)}{r^2 + R^2R_j^2 - 2RrR_j \cos(\theta-\theta_j)}\right)^{1/2} \\ &\leq \left(\frac{Rr + R}{r + RR_j}\right) \leq \left(\frac{Rr + k}{r + Rk}\right). \end{aligned} \tag{10}$$

The first estimate is obtained by observing that the function

$$f(t) = \frac{Rr^2 + R_j^2 - 2RrR_j t}{r^2 + R^2R_j^2 - 2RrR_j t}$$

is a decreasing function of t on $[-1, 1]$, which follows from taking a derivative and using the hypothesis $R_j \geq r$. The function f, therefore, has a maximum at $t = -1$ and the first estimate follows. The estimate (10) also follows by noting that the function

$$g(R_j) = \frac{Rr + R_j}{R + RR_j}$$

is a decreasing function of R_j which can be verified by using derivative again and the fact that $R_j \geq k$. Thus $g(k)$ is maximum. Using (10) in (9), it follows that

$$|F(Rre^{i\theta})| \leq \left(\frac{Rr + k}{r + Rk}\right)^n R^n F\left(\frac{re^{i\theta}}{R}\right)$$

for every θ, $0 \leq \theta < 2\pi$, $R > 1$, $k \geq r$. Replacing $F(z)$ by $P(z) + \alpha m$, we get

$$|P(Rre^{i\theta}) + \alpha m| \leq \left(\frac{Rr + k}{r + Rk}\right)^n \left|R^n P\left(\frac{re^{i\theta}}{R}\right) + R^n \alpha m\right| \tag{11}$$

for every α with $|\alpha| \leq 1$, $0 \leq \theta < 2\pi$, $R > 1$ and $k \geq r$. Since $r/R \leq k$, we choose argument of α with $|\alpha| = 1$ on the R. H. S of (11) such that for $|z| = 1$,

$$\left|P\left(\frac{rz}{R}\right) + \alpha m\right| = \left|P\left(\frac{rz}{R}\right)\right| - m \tag{12}$$

which is possible by (8). Using (12) in (11), we abtain for $|z| = 1$, $R > 1$ and $k > r$,

$$|P(Rrz)| - m \leq \left(\frac{Rr + k}{r + Rk}\right)^n \left|R^n P\left(\frac{r}{R}\right)\right| - \left(\frac{Rr + k}{r + Rk}\right)^n R^n m.$$

This implies

$$\begin{aligned} |P(Rrz)| \quad \leq \quad & \left(\frac{Rr + k}{r + Rk}\right)^n \left|R^n P\left(\frac{rz}{R}\right)\right| \\ & - \left\{\left(\frac{Rr + k}{r + Rk}\right)^n R^n - 1\right\} \min_{|z|=k} |P(z)| \end{aligned} \tag{13}$$

for $|z| = 1$, $R \geq 1$ and $r \leq k$, which is the desired result. This completes the proof of Lemma 8. □

We also need the following lemma:

Lemma 9. *If $P(z)$ is a polynomial of degree n, then*

$$|P(Re^{i\theta})| + |Q(Re^{i\theta})| \leq (R^n + 1) \max_{|z|=1} |P(z)|, \quad 0 \leq \theta \leq 2\pi,$$

where

$$Q(z) = z^n \overline{P(1/\overline{z})} \quad \text{and} \quad R \geq 1.$$

Lemma 9 is due to Aziz and Mohammad [4]. However, for the sake of completeness, we give here a brief outline of the proof. In fact, we deduce it from Lemma 8, and thereby present an independent proof of Lemma 9. Let $M = \max_{|z|=1} |P(z)|$, then

$$|P(z)| \leq M \qquad |z| = 1.$$

By Rouches theorem, it follows that for every real or complex number λ, with $|\lambda| > 1$, the polynomial

$$F(z) = P(z) - \lambda M$$

does not vanish in $|z| < 1$. Applying Lemma 8, to the polynomial $F(z)$ with $k = 1 = r$, it follows that for every θ, $0 \leq \theta < 2\pi$, $R > 1$,

$$\begin{aligned} |F(Re^{i\theta})| &\leq R^n \left| F\left(\frac{e^{i\theta}}{R}\right) \right| - (R^n - 1) \min_{|z|=1} |F(z)| \\ &\leq \left| R^n F\left(\frac{e^{i\theta}}{R}\right) \right|. \end{aligned} \tag{14}$$

If $G(z) = z^n \overline{F(1/\overline{z})}$, then we have $G(z) = Q(z) - \overline{\lambda} z^n M$ and

$$|G(Re^{i\theta})| = \left| R^n e^{in\theta} \overline{F\left(\frac{e^{i\theta}}{R}\right)} \right| = \left| R^n F\left(\frac{e^{i\theta}}{R}\right) \right|.$$

Using this in (14), it follows that for every $R \geq 1$, and $0 \leq \theta < 2\pi$,

$$|P(Re^{i\theta}) - \lambda M| = |F(Re^{i\theta})| \leq |G(Re^{i\theta})| = |Q(Re^{i\theta}) - \overline{\lambda} R^n e^{in\theta} M|$$

choosing the argument of λ in R. H. S of this inequality suitably, we get

$$|P(Re^{i\theta})| - |\lambda| M \leq |\lambda| R^n - |Q(Re^{i\theta})|.$$

Or

$$|P(Re^{i\theta})| + |Q(Re^{i\theta})| \leq (R^n + 1)|\lambda| M$$

for every θ, $0 \leq \theta < 2\pi$, and $k \geq 1$, letting $|\lambda| \to 1$, we get the assertion of Lemma 9.

Proof of Theorem 3. Since all the zeros of $P(z)$ lie in $|z| \geq k \geq 1$, using Lemma 8, it follows from (7) with $r = 1$, that

$$|P(Re^{i\theta})| \leq \left(\frac{R+k}{1+Rk}\right)^n \left| R^n P\left(\frac{e^{i\theta}}{R}\right) \right| - \left\{ \left(\frac{R+k}{1+Rk}\right)^n R^n - 1 \right\} m \tag{15}$$

for every θ, $0 \leq \theta \leq 2\pi$ and $R \geq 1$. Since

$$Q(z) = z^n P\overline{(1/\overline{z})}$$

therefore,

$$|Q(Re^{i\theta})| = |R^n P\left(\frac{e^{i\theta}}{R}\right)|. \tag{16}$$

Using (16) in (15), we get

$$|P(Re^{i\theta})| \leq \left(\frac{R+k}{1+Rk}\right)^n |Q(Re^{i\theta})| - \left\{ \left(\frac{R+k}{1+Rk}\right)^n R^n - 1 \right\} m.$$

This implies

$$(17)\quad \begin{aligned} &\frac{(1+Rk)^n+(R+k)^n}{(1+Rk)^n}|P(Re^{i\theta})| \le \\ &\left(\frac{R+k}{1+Rk}\right)^n \left\{|P(Re^{i\theta})|+|Q(Re^{i\theta})|\right\} - \left\{\left(\frac{R+k}{1+Rk}\right)^n R^n - 1\right\} m. \end{aligned}$$

Inequality (17) yields with the help of Lemma 9 that

$$(18)\quad \begin{aligned} &\frac{(1+Rk)^n+(R+k)^n}{(1+Rk)^n}|P(Re^{i\theta})| \le \\ &\frac{(R+k)^n(R^n+1)}{(1+Rk)^n}\max_{|z|=1}|P(z)| - \left\{\left(\frac{R+k}{1+Rk}\right)^n R^n - 1\right\}\min_{|z|=1}|P(z)| = \\ &\left(\frac{R+k}{1+Rk}\right)^n \left[(R^n+1)\max_{|z|=1}|P(z)| - \left\{R^n - \left(\frac{1+Rk}{R+k}\right)^n\right\}\min_{|z|=k}|P(z)|\right]. \end{aligned}$$

From (18) it follows that

$$\begin{aligned} |P(Re^{i\theta})| \quad \le \quad & \frac{(R+k)^n}{(1+Rk)^n+(R+k)^n} \times \\ & \left[(R^n+1)\max_{|z|=1}|P(z)| - \left\{R^n - \left(\frac{1+Rk}{R+k}\right)^n\right\}\min_{|z|=1}|P(z)|\right] \end{aligned}$$

for every θ, $0 \le \theta < 2\pi$ and $R \ge 1$. Which is equivalent to the desired result. This completes the proof of Theorem 3. □

Proof of Theorem 4. Let $m = \min_{|z|=k}|P(z)|$, then we have

$$(19)\qquad m \le |P(z)| \quad \text{for} \quad |z| = k.$$

Since $P(z)$ does not vanish in $|z| < k$, and it follows as in the proof of Lemma 8 that for every real or complex number α with $|\alpha| \le 1$, the polynomial $F(z) = P(z) + \alpha m$ has all its zeros in $|z| \ge k$. If

$$R_1e^{i\theta_1}, R_2e^{i\theta_2}, \ldots, R_ne^{i\theta_n}$$

be the zeros of $F(z)$, then $R_j \ge k$, $j = 1, 2, \ldots, n$ and we have

$$F(z) = \prod_{j=1}^{n}(z - R_je^{i\theta_j}).$$

It can be easily seen for $1 \le R \le k^2$ and $0 \le \theta < 2\pi$

$$\begin{aligned} \left|\frac{P(Re^{i\theta})}{P(e^{i\theta})}\right| \quad &= \quad \prod_{j=1}^{n}\left|\frac{Re^{i\theta} - R_je^{i\theta_j}}{e^{i\theta} - R_je^{i\theta_j}}\right| \le \prod_{j=1}^{n}\left(\frac{R+R_j}{1+R_j}\right) \\ &\le \quad \prod_{j=1}^{n}\left(\frac{R+k}{1+k}\right) = \left(\frac{R+k}{1+k}\right)^n. \end{aligned}$$

This implies

$$|F(Re^{i\theta})| \leq \left(\frac{R+k}{1+k}\right)^n |F(e^{i\theta})| \tag{20}$$

for every θ, $0 \leq \theta < 2\pi$ and $1 \leq R \leq k^2$. Replacing $F(z)$ by $P(z) + \alpha m$ in (20), we get

$$|P(Re^{i\theta}) + \alpha m| \leq \left(\frac{R+k}{1+k}\right)^n |P(e^{i\theta}) + \alpha m| \tag{21}$$

for every α with $|\alpha| \leq 1$, $0 \leq \theta < 2\pi$ and $1 \leq R \leq k^2$. Since $P(z)$ does not vanish for $|z| < k$, by Maximum Modulus Principle it follows from (19) that

$$m \leq |P(z)| \quad \text{for} \quad |z| \leq k \quad \text{where} \quad k \geq 1. \tag{22}$$

Taking in particular $z = e^{i\theta}$, $0 \leq \theta < 2\pi$ in (22), then

$$|z| = |e^{i\theta}| = 1 \leq k$$

and we get

$$m \leq |P(e^{i\theta})| \quad \text{for} \quad 0 \leq \theta < 2\pi. \tag{23}$$

Choosing the argument α with $|\alpha| = 1$ on the R. H. S of (21) such that for $|z| = 1$,

$$|P(z) + \alpha m| = |P(z)| - m \tag{24}$$

which is possible by (23), we obtain from (21) that

$$|P(Re^{i\theta})| - m \leq \left(\frac{R+k}{1+k}\right)^n \left\{P(e^{i\theta}) - m\right\}$$

for every θ, $0 \leq \theta < 2\pi$, $1 \leq R \leq k^2$. This gives

$$|P(Rz)| \leq \left(\frac{R+k}{1+k}\right)^n |P(z)| - \left\{\left(\frac{R+k}{1+k}\right)^n - 1\right\} m$$

for $|z| = 1$ and $1 \leq R \leq k^2$, from which it immediately follows that

$$\max_{|z|=1} |P(z)| \leq \left(\frac{R+k}{1+k}\right)^n \max_{|z|=1} |P(z)| - \left\{\left(\frac{R+k}{1+k}\right)^n - 1\right\} \min_{|z|=k} |P(z)|$$

for $|z| = 1$ and $1 \leq R \leq k^2$. This completes the proof of Theorem 4. □

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