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KOEBE DOMAIN OF STARLIKE FUNCTIONS OF COMPLEX ORDER WITH MONTEL NORMALIZATION

Yaşar Polatoğlu, Metin Bolcal and Arzu Şen
University of Kültür, Turkey

Abstract. Let $S^*(1-b)$, ($b \neq 0$ complex) denote the class of functions $f(z) = z + \alpha_2 z^2 + ...$ analytic in $D = \{z| \ |z| < 1\}$ which satisfies, for $z = e^{i\theta} \in D$, $\frac{f(z)}{z} \neq 0$ in D, and

$$Re\left[1+\frac{1}{b}\left(z\frac{f'(z)}{f(z)}-1\right)\right] > 0.$$

The aim of this paper is to give the Koebe domain of the above mentioned class.

1. Introduction

Let F denote the class of functions $f(z) = z+\alpha_2 z^2+...$ which are analytic in D. We shall need the following definitions.

Definition 1.1. *A function $f(z) \in F$ is said to be starlike function of complex order $(1-b)$, ($b \neq 0$ complex), that is $f(z) \in S^*(1-b)$ if and only if $\frac{f(z)}{z} \neq 0$ in D, and*

$$Re\left[1+\frac{1}{b}\left(z\frac{f'(z)}{f(z)}-1\right)\right] > 0, \quad z \in D. \tag{1.1}$$

It should be noticed that by giving specific values to b we obtain the following important subclasses, [5].

1. $b = 1$, $S^*(1-b) = S(0) = S^*$ is the well known class of starlike functions.

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2. $b = 1-\alpha$, $(0 \le \alpha < 1)$, $S^*(1-b) = S(\alpha)$ is the class of starlike functions of order α.
3. $b = \cos\lambda e^{-i\lambda}$, $S^*(1-b) = S^*(1-\cos\lambda e^{-i\lambda}) = S^*_\lambda$, $|\lambda| < \frac{\pi}{2}$, is the class of spirallike functions.
4. $b = (1-\alpha)\cos\lambda e^{-i\lambda}$, $|\lambda| < \frac{\pi}{2}$, $0 \le \alpha < 1$, $S^*(1-b) = S^*(1-(1-\alpha)\cos\lambda e^{-i\lambda}) = S^*_{\lambda,\alpha}$ is the class of spirallike functions of order α.

Definition 1.2. *The Koebe domain for the family F is denoted by $K(F)$ and by the definition this is the largest domain contained in $f(D)$ for every function $f(z)$ in F.*

From this definition the Koebe domain has the following properties.

1. $K(F)$ is the collection of points w such that w is in $f(D)$ for every function $f(z)$ in F,

$$K(F) = \bigcap_{f(z)\in F} f(D). \tag{1.2}$$

2. Supposing the set F is invariant under the rotation, so $e^{i\alpha}f(e^{-i\alpha}z)$ is in F whenever $f(z)$ in F. Then the Koebe Domain will be either the single point $w = 0$ or an open disc $|w| < R$. In the second case R is often easy to find indeed, supposing that we have a sharp lover bound $M(r)$ for $f(re^{i\theta})$ for all functions in F, and F contains only

$$R = \lim_{r\to 1^-} M(r) \tag{1.3}$$

gives the disc $|w| < R$ as the Koebe Domain for the set F. (see 1.2)

Montel type normalization. We can also impose a Montel type normalization. This means that for some fixed r_0 with $0 < r_0 < 1$, we consider the family of functions $f(z)$ regular and univalent in D with $f(0) = 0$, $f'(0) = 1$, $f(r_0) = r_0$.

We note that: If the class of starlike functions is normalized by the Montel type normalization, then the class is denoted by

$$S^*_{montel}(1-b)$$

2. Koebe domain for the class of starlike functions of complex order with Montel normalization

In this section we shall give the Koebe Domain for the class of starlike functions of complex order with Montel normalization.

Theorem 2.1. *Let $f(z) \in S^*(1-b)$, then*

$$\frac{2|u|\cdot(1-|v|^2)^{2b}}{(1+|b|)|v||1-u\overline{v}|^{2b-2}[|1-u\overline{v}|+|u-v|]^2} \le \left|\frac{f(u)}{f(v)}\right| \le \frac{2|u|(1-|v|^2)^{2b}}{(1+|b|)|v||1-u\overline{v}|^{2b-2}[|1-u\overline{v}|-|u-v|]^2}$$

holds for $u, v \in D$, $u \neq v$. This bound is sharp, because the extremal function is $w = f(z)$ defined for $|z| < 1$ by

$$w = f_*(z) = \frac{z}{(1-z)^{2b}}$$

PROOF. We consider the Möbius transformation

$$u = \frac{z+v}{1+\overline{v}z} \Longleftrightarrow z = \frac{u-v}{1-u\overline{v}} \tag{2.1}$$

which is analytic and univalent in D, and this Möbius transformation maps the unit disc on to the itself. Now we define the function:

$$F(z) = \frac{\alpha z f\left(\frac{z+\alpha}{1+\overline{\alpha}z}\right)}{f(z)(z+\alpha)(1+\overline{\alpha}z)^{2b-1}} \Longleftrightarrow F(z) = \frac{v(1-u\overline{v})^{2b-1}(u-v)f(u)}{u(1-|v|^2)^{2b}f(v)}, \tag{2.2}$$

for $\alpha = v$. This function is starlike function of complex order in the unit disc D. On the other hand, if $g(z)$ is starlike function of complex order, then

$$\frac{2|z|}{(1+|b|)(1+|z|)^2} \leq F(z) \leq \frac{2|z|}{(1+|b|)(1-|z|)^2} \tag{2.3}$$

holds (see [5]). Therefore applying the inequality (2.3) to the function $F(z)$ which is defined by (2.2) and using the relation (2.1) we get

$$\begin{aligned} &\frac{2\frac{u-v}{1-u\overline{v}}}{(1+|b|)\left(1+\left|\frac{u-v}{1-u\overline{v}}\right|\right)^2} \\ &\leq \left|\frac{v(1-u\overline{v})^{2b-1}(u-v)f(u)}{u(1-|v|^2)^{2b}f(v)}\right| \leq \frac{2\frac{u-v}{1-u\overline{v}}}{(1+|b|)\left(1-\left|\frac{u-v}{1-u\overline{v}}\right|\right)^2} \end{aligned} \tag{2.4}$$

Simple calculations from (2.4) show that this teorem is true.

COROLLARY 2.2. *The Koebe Domain of starlike function of complex order with Montel type normalization is:*

$$R = \frac{(1-r_0^2)^{2b}}{2(1+|b|)(1-2r_0\cos\theta + r_0^2)^{2b}}, \qquad 0 \leq \theta \leq 2\pi$$

Indeed, if we take $v = f(r_0) = r_0$, $0 < r_0 < 1$, $u = z = re^{i\theta}$ in Theorem 2.1, we obtain

$$\begin{aligned} &\frac{2|re^{i\theta}|(1-r_0^2)^{2b}}{(1+|b|)|1-r_0re^{i\theta}|^{2b-2}\left[|1-r_0e^{i\theta}| + |re^{i\theta}-r_0|\right]^2} \leq |f(z)| \leq \\ &\frac{2|re^{i\theta}|(1-r_0^2)^{2b}}{(1+|b|)|1-r_0re^{i\theta}|^{2b-2}\left[|1-r_0e^{i\theta}| - |re^{i\theta}-r_0|\right]^2} \end{aligned} \tag{2.5}$$

Therefore we have

$$M(r) = \frac{2|re^{i\theta}|(1-r_0^2)^{2b}}{(1+|b|)\cdot|1-re^{i\theta}|^{2b-2}\left[|1-re^{i\theta}|+|re^{i\theta}|\right]^2}.$$

If we take $\lim\limits_{r\to 1^-} M(r)$, we get

$$R = K(S^*_{montel}(1-b)) = \frac{(1-r_0^2)^{2b}}{2(1+|b|)\cdot(1-2r_0\cos\theta+r_0^2)^b}. \tag{2.6}$$

Giving the specific values to b we obtain the following results.

1. For $b=1$, $R = K(S^*_{montel}) = \dfrac{(1-r_0^2)^2}{4\cdot(1-2r_0\cos\theta+r_0^2)}$
2. For $b = 1-\alpha$, $0\le\alpha<1$,
$$R = K(S^*_{montel}(\alpha)) = \frac{(1-r_0^2)^{2(1-\alpha)}}{2\alpha(1-2r_0\cos\theta+r_0^2)^{1-\alpha}}$$
3. For $b=\cos\lambda e^{-i\lambda}$, ($|\lambda|<\frac{\pi}{2}$)
$$\begin{aligned} R =& K\left(S^*_{montel}(1-\cos\lambda\cdot e^{-i\lambda})\right) = \\ =& \frac{(1-r_0^2)^{2\cos\lambda e^{-i\lambda}}}{2(1+\cos\lambda)(1-2r_0\cos\theta+r_0^2)^{\cos\lambda e^{-i\lambda}}} \end{aligned}$$
4. For $b=(1-\alpha)\cos\lambda e^{-i\lambda}$, $|\lambda|<\frac{\pi}{2}$, $0\le\alpha<1$,
$$\begin{aligned} R &= K\left(S^*_{montel}(1-(1-\alpha)\cos\lambda\cdot e^{-i\lambda})\right) \\ &= \frac{(1-r_0^2)^{(1-\alpha)\cos\lambda e^{-i\lambda}}}{(1+(1-\alpha)\cos\lambda)(1-2r_0\cos\theta+r_0^2)^{(1-\alpha)\cos\lambda e^{-i\lambda}}} \end{aligned}$$

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Department of Mathematics,
Faculty of Science,
University of Kültür, Turkey
E-mail: m.bolcal@iku.edu.tr

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