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OSCILLATORY AND ASYMPTOTIC PROPERTIES OF SOLUTIONS OF NONLINEAR FOURTH ORDER DIFFERENCE EQUATIONS

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Abstract. The authors consider the difference equation

$$\Delta^2(r_n\Delta^2 y_n) \pm q_n f(y_n) = Q_n; \quad n = 1, 2, 3, \dots \quad (*)$$

where $r_n > 0$, $q_n > 0$, for all $n \geq 1$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous such that $uf(u) > 0$ for $u \neq 0$. Dividing the solutions of $(*)$ into several classes for the cases $Q_n = 0$ and $Q_n \neq 0$, the authors obtain conditions for the existence/nonexistence of solutions of $(*)$ in these classes. Examples are inserted to illustrate the results.

1. Introduction

In this paper we are concerned with the oscillatory and nonoscillatory behavior of solutions of the nonlinear nonhomogeneous fourth order difference equations

$$(\mathrm{E}_{\pm}) \qquad \Delta^2(r_n\Delta^2 y_n) \pm q_n f(y_n) = Q_n; \quad n = 1, 2, 3, \dots$$

where Δ is the forward difference operator defined by $\Delta y_n = y_{n+1} - y_n$ and the real sequences $\{r_n\}, \{q_n\}, \{Q_n\}$ and the function f satisfies the following conditions:

- (c1) $\{r_n\}$ is a positive real sequence such that $0 < m \leq r_n \leq M$, for all $n \geq 1$;
- (c2) $q_n > 0$ for all $n \geq 1$ and $Q_n \not\equiv 0$ for all $n \geq 1$;
- (c3) $f : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing such that $uf(u) > 0$ for $u \neq 0$.

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By a solution of equation $(E_\pm)$, we always mean a real sequence $\{y_n\}$ satisfying equation $(E_\pm)$ for all $n \geq 1$ and for which $\sup\{| y_n |: n \geq s\} > 0$ for any $s \geq 1$. A solution of $(E_\pm)$ is nonoscillatory if it is either eventually positive or eventually negative, and oscillatory otherwise.

The problem of oscillation and nonoscillation of solutions of difference equations has received a great deal of attention in the last few years, for example, see [1,2,5], which cover a large number of recent papers. Compared to second order difference equations, the study of higher order equations and in particular fourth order equations has received considerably less attention, see, for example [3,4,6,9–14] and the references cited therein. Therefore, in this paper we study the oscillatory and asymptotic properties of solutions of the equation $(E_\pm)$.

Before considering the nonhomogeneous equations $(E_\pm)$, we first study the oscillatory and asymptotic properties of the associated homogeneous equations

$$\Delta^2(r_n\Delta^2 y_n) \pm q_n f(y_n) = 0. \tag{H$_\pm$}$$

In Section 2, we classify all nonoscillatory solutions of $(H_\pm)$ into several classes according to their asymptotic behavior and obtain conditions for the existence/nonexistence of solutions in these classes. In Section 3, we first transform the equation $(E_\pm)$ into $(H^\pm_\pm)$ and then classify the nonoscillatory solutions into several classes as in Section 2. Using the results obtained in Section 2 we establish conditions for the existence/nonexistence of solutions of $(E_\pm)$ in these classes. Results obtained here are motivated by some of the results obtained in [3]. Examples are inserted to illustrate the results.

2. Classification of solutions of equation $(\mathrm{H}_\pm)$

Following Yan and Liu [14] and Graef and Thandapani [3], we say that a solution $\{y_n\}$ is of type I or M_1 if for n sufficiently large

$$y_n > 0,\ \Delta y_n > 0,\ r_n\Delta^2 y_n < 0 \ \text{ and } \ \Delta(r_n\Delta^2 y_n) > 0;$$

and it is of type II or M_2 if for n sufficiently large

$$y_n > 0,\ \Delta y_n > 0,\ r_n\Delta^2 y_n > 0 \ \text{ and } \ \Delta(r_n\Delta^2 y_n) > 0.$$

Further it has been shown in [3] and [11] under the condition (c_1) that a positive solution of (H_+) is necessarily of type M_1 or M_2. Now we establish conditions for the nonexistence of solutions of (H_+) in the classes M_1 and M_2.

Theorem 2.1. *With respect to the difference equation (H_+), assume that*

$$\sum_{n=n_0}^{\infty} q_n = \infty. \tag{1}$$

Then $M_1 = \emptyset$.

Proof. Let $\{y_n\}$ be a M_1–type solution of equation (H_+). Without loss of generality we may assume that

$$y_n > 0,\ \Delta y_n > 0,\ r_n\Delta^2 y_n < 0 \ \text{ and } \ \Delta(r_n\Delta^2 y_n) > 0 \ \text{ for all } \ n \geq N \geq 1.$$

Let $w_n = \dfrac{\Delta(r_n\Delta^2 y_n)}{f(y_n)}$. Then from equation (H_+), we have

$$\begin{aligned}\Delta w_n &\leq -q_n - \frac{\Delta(r_{n+1}\Delta^2 y_{n+1})\Delta f(y_n)}{f(y_n)f(y_{n+1})}\\ &\leq -q_n.\end{aligned}$$

Summing the last inequality we see that

$$\sum_{s=N}^{n} q_s \leq w_N$$

a contradiction to (1). This completes the proof of the theorem. ∎

Example 2.2. Consider the equation

$$\Delta^2\left(\frac{n+1}{n+2}\Delta^2 y_n\right) + \frac{2n^2(4n+9)}{(n-1)^3(n+1)(n+3)^3(n+2)^2}y_n^3 = 0,\ n \geq 2. \tag{2}$$

All conditions of Theorem 2.1 are satisfied except condition (1). Hence the equation (2) has a solution $\{y_n\} = \left\{\frac{n-1}{n}\right\}$ belonging to the class M_1. To prove our next result we require the following lemma.

Lemma 2.3. *Let $\{y_n\}$ be a M_2–type solution of equation (H_+). Then, for all sufficiently large n,*

(a) $\Delta(r_n\Delta^2 y_n) \leq \dfrac{2r_n\Delta^2 y_n}{n}$;

(b) $r_n\Delta^2 y_n \leq \dfrac{6M\Delta y_n}{n}$;

(c) $\Delta y_n \leq \frac{2}{n}\left(\dfrac{6M}{m}+1\right)y_n$.

Proof. Since $\{y_n\}$ is a M_2–type solution of (H_+), there is an integer $N \geq 1$ such that $\{y_n\}$, $\{\Delta y_n\}$, $\{r_n\Delta^2 y_n\}$ and $\{\Delta(r_n\Delta^2 y_n)\}$ are all positive for $n \geq N$. From equation (H_+), we have $\Delta^2(r_n\Delta^2 y_n) < 0$ for $n \geq N$, so that $\Delta(r_n\Delta^2 y_n)$ is decreasing for $n \geq N$. Hence

$$r_n\Delta^2 y_n \geq r_n\Delta^2 y_n - r_N\Delta^2 y_N = \sum_{s=N}^{n-1}\Delta(r_s\Delta^2 y_s) \geq \Delta(r_n\Delta^2 y_n)(n-N).$$

Since $(n-N) \geq \frac{n}{2}$ for $n \geq 2N$, we have $r_n\Delta^2 y_n \geq \dfrac{n\Delta(r_n\Delta^2 y_n)}{2}$ for $n \geq 2N$, which proves (a).

For $n \geq N_1 \geq 2N$ we have from summation by parts formula

$$r_s\Delta^2 y_s(s-N)\Big|_{N_1}^{n} - \sum_{s=N_1}^{n-1} r_s\Delta^2 y_s = \sum_{s=N_1}^{n-1}(s-N_1+1)\Delta(r_s\Delta^2 y_s).$$

Now using the result (a) in the last equation, we obtain

$$(n-N_1)r_n\Delta^2 y_n \leq 3\sum_{s=N_1}^{n-1} r_s\Delta^2 y_s. \tag{3}$$

Since $0 < m \leq r_n \leq M$ and $\Delta^2 y_n > 0$ for $n \geq N_1$, (3) implies that

$$(n-N_1)r_n\Delta^2 y_n \leq 3M(\Delta y_n - \Delta y_{N_1}) \leq 3M\Delta y_n \quad \text{for} \quad n \geq N_1$$

and $\dfrac{nr_n\Delta^2 y_n}{2} \leq 3M\Delta y_n$ for $n \geq 2N_1$. This proves (b).

Again from summation by parts formula for $n \geq N_2 \geq 2N_1$, we have

$$m(s-N_2)\Delta y_s\Big|_{N_2}^{n} - m\sum_{s=N_2}^{n-1}\Delta y_s = m\sum_{s=N_2}^{n-1}(s-N_2+1)\Delta^2 y_s \leq \sum_{s=N_2}^{n-1} s r_s\Delta^2 y_s.$$

Now using the result (b) in the last inequality, we obtain

$$m(n-N_2)\Delta y_n - m\sum_{s=N_2}^{n-1}\Delta y_s \leq 6M\sum_{s=N_2}^{n-1}\Delta y_n.$$

Thus for $n \geq 2N_2$, we have

$$\frac{mn}{2}\Delta y_n \leq (6M+m)\sum_{s=N_2}^{n-1}\Delta y_s = (6M+m)(y_n - y_{N_2}) \leq (6M+m)y_n,$$

from which (c) follows. This completes the proof of the lemma. □

REMARK 2.4. If $\{y_n\}$ is a M_1–type solution of (H_+), then a similar argument yields $n\Delta y_n \leq 2y_n$ for all large n.

THEOREM 2.5. *With respect to the difference equation* (H_+) *assume that*

$$\frac{f(u)}{u} \geq M_1 \quad \textit{for all} \quad u \neq 0 \tag{4}$$

and

$$\sum_{n=n_0}^{\infty} n^2 q_n = \infty. \tag{5}$$

Then $M_2 = \emptyset$.

PROOF. Let $\{y_n\}$ be a M_2–type solution of (H_+). Without loss of generality, we may assume that $y_n > 0$, $\Delta y_n > 0$, $r_n\Delta^2 y_n > 0$ and $\Delta(r_n\Delta^2 y_n) > 0$ for all $n \geq N \geq 1$. Define

$$w_n = \frac{\Delta(r_n\Delta^2 y_n)}{r_n\Delta^2 y_n}.$$

Then from equation (H_+) we have

$$\Delta w_n \leq -\frac{q_n f(y_n)}{r_n\Delta^2 y_n} \leq 0,\ n \geq N$$

or

$$\Delta w_n + \frac{q_n f(y_n)}{r_n\Delta^2 y_n} \leq 0,\ n \geq N. \tag{6}$$

From Lemma 2.3, we have

$$r_n\Delta^2 y_n \leq \frac{12M}{n^2}\left(\frac{6M}{m}+1\right) y_n,\ n \geq 2N_2. \tag{7}$$

For $n \geq 2N_2 + N = N_3$, we have from (6) and (7)

$$\Delta w_n + \frac{n^2 q_n f(y_n)}{12M\left(\frac{6M}{m}+1\right) y_n} \leq 0,\ n \geq N_3. \tag{8}$$

In view of condition (4), (8) implies that

$$\Delta w_n + \frac{M_1}{12M\left(\frac{6M}{m}+1\right)} n^2 q_n \leq 0,\ n \geq N_3.$$

Now summing the last inequality from N_3 to n and then using the condition (5), we see that $\{w_n\}$ is eventually negative, which is absurd. This completes the proof of the theorem. □

EXAMPLE 2.6. Consider the difference equation

$$\Delta^2\left(\frac{n-1}{n}\Delta^2 y_n\right) + \frac{4}{n^3(n+1)(n+2)(1+n^4)}(y_n + y_n^3) = 0,\ n \geq 2. \tag{9}$$

It is easy to see that all conditions of Theorem 2.5 are satisfied except condition (5). Hence the equation (9) has a solution $\{y_n\} = \{n^2\}$ which belongs to the class M_2.

From Theorem 2.1 and Theorem 2.5, we obtain the following oscillation criterion for equation (H_+).

THEOREM 2.7. *With respect to the difference equation* (H_+) *assume conditions (4) and (5) hold. Then all solutions of* (H_+) *are oscillatory.*

Next we shall give an improved version of Theorem 2.5.

Theorem 2.8. *With respect to the difference equation* (H_+) *assume that for all* $c > 0$

$$\sum_{n=n_0}^{\infty} q_n f(cn^2) = \infty. \tag{10}$$

Then $M_2 = \emptyset$.

Proof. Let $\{y_n\}$ be a solution of type M_2. Then, as in Theorem 2.5, we may assume y_n, Δy_n, $r_n\Delta^2 y_n$ and $\Delta(r_n\Delta^2 y_n)$ are all positive for $n \geq N$. Since $\{r_n\Delta^2 y_n\}$ is positive and increasing for $n \geq N$, there is a constant $k > 0$ such that $r_n\Delta^2 y_n > k$ for all $n \geq N$. It then follows that

$$y_n > c(n-N)(n-N-1) \quad \text{for all} \quad n \geq N, \quad \text{where } c = \frac{k}{2M}.$$

From equation (H_+) we have

$$\Delta(r_N\Delta^2 y_N) = \Delta(r_n\Delta^2 y_n) + \sum_{s=N}^{n-1} q_s f(y_s) > \sum_{s=N}^{n-1} q_s f(c(s-N)(s-N+1)$$

for all $n \geq N$. Thus,

$$\sum_{n=N}^{\infty} q_n f(cn^2) < \infty$$

which contradicts (10). This completes the proof. ∎

In an analogous manner we may define a solution $\{y_n\}$ of equation (H_-) to be of type M_1 if for n sufficiently large

$$y_n > 0,\ \Delta y_n < 0,\ r_n\Delta^2 y_n > 0 \ \text{ and } \ \Delta(r_n\Delta^2 y_n) < 0;$$

a solution $\{y_n\}$ is of type M_2 if for n sufficiently large

$$y_n > 0,\ \Delta y_n > 0,\ r_n\Delta^2 y_n > 0 \ \text{ and } \ \Delta(r_n\Delta^2 y_n) < 0;$$

and a solution $\{y_n\}$ is of type M_3 if for n sufficiently large

$$y_n > 0,\ \Delta y_n > 0,\ r_n\Delta^2 y_n > 0 \ \text{ and } \ \Delta(r_n\Delta^2 y_n) > 0.$$

It is easily seen that a positive solution of equation (H_-) is necessarily of type M_1, M_2 or M_3 and the following analogue of Lemma 2.3 can be derived for M_2–type solutions.

Lemma 2.9. *Let* $\{y_n\}$ *be a* M_2*–type solution of equation* (H_-)*. Then for* $n \geq N_1$,

(i) $nr_n\Delta^2 y_n \leq 3M\Delta y_n$,

(ii) $n\Delta y_n \leq 2\left(\frac{6M}{m} + 1\right) y_n$.

In the following we obtain criteria for the nonexistence of M_1, M_2 and M_3–type solution for the equation (H_-).

THEOREM 2.10. (a) *Equation (H_-) has no M_2–type solutions if*

$$\sum_{n=n_0}^{\infty} n q_n = \infty. \tag{11}$$

(b) *Any M_1–type solution of equation (H_-) tends to zero as $n \to \infty$ if*

$$\sum_{n=n_0}^{\infty} n^{(3)} q_n = \infty \tag{12}$$

where $n^{(3)}$ is the usual factorial notation.

PROOF. To prove (b), let $\{y_n\}$ be a positive solution of equation (H_-) of the type M_1. Then there is an integer $N \geq 1$ such that

$$y_n > 0,\ \Delta y_n < 0,\ r_n \Delta^2 y_n > 0 \ \text{ and } \ \Delta(r_n \Delta^2 y_n)) < 0 \ \text{ for } \ n \geq N.$$

Let $n \geq N_1$ and $j > 2n = N$. Assume that $\lim\limits_{n\to\infty} y_n = c > 0$.
Summing equation (H_-) twice and using the estimate

$$r_n \Delta^2 y_n < M \Delta^2 y_n,$$

then sum twice the resulting inequality, we have

$$y_n \geq y_j - \Delta y_j (j-n) + M^{-1} r_j \Delta^2 y_j \frac{(j-n)^{(2)}}{2!} -$$
$$M^{-1} \Delta(r_j \Delta^2 y_j) \frac{(j-n)^{(3)}}{3!} + \frac{1}{M3!} \sum_{s=n}^{j-3} (s-n)^{(3)} q_s f(y_s),$$

from which we may obtain

$$\sum_{s=N_*}^{\infty} s^{(3)} q_s < 3! 2^3 M c^{-1} y_N.$$

This contradicts the condition (12). The proof of (a) is similar and hence the details are omitted. This completes the proof of the theorem. □

EXAMPLE 2.11. Consider the difference equation

$$\Delta^2 \left(\frac{n+2}{n+1} \Delta^2 y_n \right) - \frac{4}{n^3 (n+1)^4 (n+2)(n+3)} y_n^3 = 0,\ n \geq 2. \tag{13}$$

All conditions of Theorem 2.10 (a) are satisfied except condition (11). Hence the equation (13) has a solution $\{y_n\} = \{n(n+1)\}$ which belongs to the class M_2. The difference equation

$$\Delta^4 y_n - \frac{24 n^2}{(n+1)(n+2)(n+3)(n+4)} y_n^3 = 0 \tag{14}$$

satisfies all conditions of Theorem 2.10 (b) and hence any M_1–type solution of equation (14) tends to zero as $n \to \infty$. One such solution of (14) is $\{y_n\} = \{\frac{1}{n}\}$.

3. Classification of solutions of equation $(E_\pm)$

Let $u_n = y_n - R_n$ where $\{y_n\}$ is a positive solution of $(E_\pm)$ and $\{R_n\}$ is a solution of

$$\Delta^2(r_n\Delta^2 R_n) = Q_n. \tag{E}$$

Then we may transform (E_+) to homogeneous difference equation for which the results of Section 2 may be applied. Since the resulting equation does not have precise form (H_+) or (H_-), the arguments have to be modified accordingly. Specifically

$$\Delta^2(r_n\Delta^2 u_n) = \Delta^2(r_n\Delta^2 y_n) - \Delta^2(r_n\Delta^2 R_n) = -q_n f(y_n);$$

eliminating y_n, we see that $\{u_n\}$ is a solution of the homogeneous equation

$$\Delta^2(r_n\Delta^2 u_n) + q_n f(u_n + R_n) = 0. \tag{H$_+^+$}$$

Since $y_n > 0$ for $n \geq N \geq 1$, we have $u_n + R_n > 0$ and $\Delta^2(r_n\Delta^2 u_n) < 0$ for all $n \geq N$. Therefore $\{\Delta(r_n\Delta^2 u_n)\}$, $\{r_n\Delta^2 u_n\}$, $\{\Delta u_n\}$ and $\{u_n\}$ are monotonic and one-signed. If $u_n < 0$, that is, $0 < y_n < R_n$, we may further transform the equation by assuming $v_n = -u_n$. Then

$$\Delta^2(r_n\Delta^2 v_n) = -\Delta^2(r_n\Delta^2 u_n) = q_n f(R_n - v_n).$$

Thus $v_n = R_n - y_n$ is positive solution of the equation

$$\Delta(r_n\Delta^2 v_n) - q_n f(R_n - v_n) = 0. \tag{H$_+^-$}$$

If $\{u_n\}$ is a positive solution of (H_+^+) of type M_1 or M_2, we say that $\{y_n\}$ is a positive solution of (E_+) of type M_1^R or M_2^R.
If $\{v_n\}$ is a positive solution of (H_+^-) of type M_1, M_2 or M_3, we say that $\{y_n\}$ is a positive solution of (E_+) of M_1^R, M_2^R or M_3^R.

Similarly for the equation (E_-), we may let $u_n = y_n - R_n$. As above, it follows that $\{u_n\}$ is a nonoscillatory solution of

$$\Delta(r_n\Delta^2 u_n) - q_n f(u_n + R_n) = 0. \tag{H$_-^-$}$$

If $u_n < 0$, we may let, $v_n = -u_n$ then $\{v_n\}$ is a positive solution of

$$\Delta^2(r_n\Delta^2 v_n) + q_n f(R_n - v_n) = 0. \tag{H$_-^+$}$$

If $\{u_n\}$ is a positive solution of (H_-^-) of type M_1, M_2 or M_3, we say that $\{y_n\}$ is a positive solution of (E_-) of type M_1^R, M_2^R or M_3^R; if $\{v_n\}$ is a positive solution of (H_-^+) of type M_1 or M_2, then we say that $\{y_n\}$ a positive solution of (E_-) of type M_1^R or M_2^R.

In this section we obtain conditions for the nonexistence of positive solutions of (E_+) of types M_j^R $(j = 1, 2)$ and M_j^R $(j = 1, 2, 3)$. Similar results are also obtained for the equation (E_-).

Theorem 3.1. *Let* $\{R_n\}$ *be a bounded solution of* (E)*. Assume that*

$$\sum_{n=n_0}^{\infty} n^2 q_n = \infty \tag{15}$$

and

$$\frac{f(u)}{u} \geq B > 0 \quad \text{for} \quad u \neq 0, \tag{16}$$

then equation (E_+) *has no* M_2^R*–type solutions.*

Proof. Assume that $\{y_n\}$ is a positive solution of (E_+) of M_2^R–type for $n \geq N$. Then $u_n = y_n - R_n$ is a M_2–type solution of (H_+^+) for $n \geq N$. Let

$$z_n = \frac{\Delta(r_n \Delta^2 u_n)}{r_n \Delta^2 u_n}, \ n \geq N.$$

Then

$$\begin{aligned} \Delta z_n &= -\frac{q_n f(u_n + R_n)}{r_n \Delta^2 u_n} - \frac{\Delta(r_{n+1}\Delta^2 u_{n+1})\Delta(r_n \Delta^2 u_n)}{r_n \Delta^2 u_n r_{n+1} \Delta^2 u_{n+1}} \\ &\leq -\frac{q_n f(u_n + R_n)}{r_n \Delta^2 u_n}, \ n \geq N. \end{aligned} \tag{17}$$

By Lemma 2.3, we have $r_n \Delta^2 u_n \leq \frac{12M}{n^2}\left(\frac{6M}{m} + 1\right) u_n$ and using this in (17), we obtain

$$\Delta z_n \leq -\frac{q_n n^2 f(u_n + R_n)}{12M\left(\frac{6M}{m} + 1\right) u_n} = -A q_n n^2 \frac{f(u_n + R_n)}{u_n}, \tag{18}$$

where A is constant. From (16) and (18), we have

$$\Delta z_n + AB q_n n^2 \left(\frac{u_n + R_n}{u_n}\right) \leq 0. \tag{19}$$

Since $\{R_n\}$ is bounded and $\{y_n\}$ is of type M_2^R, $\{u_n\}$ is unbounded, which implies $\frac{R_n}{u_n} \to 0$ as $n \to \infty$. So for any $\varepsilon > 0$ with $0 < \varepsilon < 1$, $1 + \frac{R_n}{u_n} \geq 1 - \varepsilon$ for n sufficiently large ($n > N_1 \geq N$). Substituting this estimate in (19) and summing the resulting inequality, we obtain

$$AB(1-\varepsilon)\sum_{s=N_1}^{n-1} s^2 q_s < \infty,$$

which contradicts (13). This completes the proof of the theorem. ∎

Remark 3.2. If $\{R_n\}$ is oscillatory or eventually negative, then $u_n > 0$ and the conclusion of Theorem 3.1 becomes: A positive solution of (E_+) is of type M_1^R.

Theorem 3.3. *Let $\{R_n\}$ be a solution of equation (E). Equation (E_+) has no positive solutions of types M_1^R or M_2^R if*

$$\sum_{n=N}^{\infty} q_n f(R_n + c) = \infty \tag{20}$$

for all positive constant c.

Proof. Suppose that $\{y_n\}$ is a positive solution of (E_+) of type M_1^R or M_2^R for $n \geq N$. Then $u_n = y_n - R_n$ is a M_1 or M_2–type solution of (E_+^+) for $n \geq N_2 \geq N$. Summing (E_+^+) we obtain

$$\sum_{s=N_2}^{n-1} q_s f(u_s + R_s) = \Delta(r_{N_2}\Delta^2 u_{N_2}) - \Delta(r_n \Delta^2 u_n)$$

or

$$\sum_{s=N_2}^{n-1} q_s f(u_s + R_s) \leq \Delta(r_{N_2}\Delta^2 u_{N_2}).$$

Since $\{u_n\}$ is M_1 or M_2–type solution, we have $\Delta u_n > 0$ for $n \geq N_1$ and hence there is a constant $c > 0$ such that $u_n \geq c$ for $n \geq N_2$ and

$$\sum_{s=N_2}^{n-1} q_s f(R_s + c) < \infty$$

a contradiction to (20). This completes the proof. ∎

Example 3.4. Consider the difference equation

$$\begin{aligned} &\Delta^4 y_n + \frac{24}{n^2(n+1)^2(n+2)(n+3)(n+4)(1+n(n+1))} y_n(1+ \mid y_n \mid) \\ &= \frac{24}{n(n+1)(n+2)(n+3)(n+4)}. \end{aligned} \tag{21}$$

With $R_n = \dfrac{1}{n}$ all conditions of Theorem 3.1 are satisfied except condition (15) and hence the equation (21) has a solution $\{y_n\} = \{n(n+1)\}$ which belongs to the class M_2^R.

Remark 3.5. If $\{R_n\}$ is oscillatory or eventually negative, the conclusion of Theorem 3.3 may be strengthened to: Equation (E_+) has no positive solutions.

Theorem 3.6. *Let $\{R_n\}$ be a solution of equation (E).*

(i) *Equation (E_+) has no positive solution of type M_2^R such that $r_n\Delta^2(y_n - R_n)$ is bounded if for all positive constants c*

$$\sum_{n=N}^{\infty} n q_n f(R_n + c) = \infty. \tag{22}$$

(ii) *Equation* (E_+) *has no positive solution of type* M_1^R *such that* $y_n - R_n$ *is bounded if for all positive constants* c

$$\sum_{n=N}^{\infty} n^{(3)} q_n f(R_n + c) = \infty, \tag{23}$$

where $n^{(3)}$ *is the usual factorial notation.*

Proof. We prove part (ii) of the theorem since the proof of part (i) is similar and hence the details are omitted. Let $\{y_n\}$ be a M_1^R–type solution of (E_+) for $n \geq N$. Then $u_n = y_n - R_n$ is a M_1–type solution of (H_+^+) for $n \geq N_1 \geq N$, and $\Delta^2(r_n \Delta^2 u_n)$ for $n \geq N_1 \geq N$. Multiplying (H_+^+) by $n^{(3)}$ and summing from N_1 to $n-1$, we obtain

$$\begin{aligned}\sum_{s=N_1}^{n-1} s^{(3)} q_s f(u_s + R_s) &= -\sum_{s=N_1}^{n-1} s^{(3)} \Delta^2(r_s \Delta^2 u_s) \\ &\leq -[\rho(s)]_{N_1}^{n} - 6mu_{N_1+3} + 6mu_{n+3}\end{aligned}$$

where $\rho(s) = s^{(3)}\Delta(r_s\Delta^2 u_s) - 3s^{(2)} r_{s+1}\Delta^2 u_{s+1} + 6ms\Delta u_{s+2}$.
This contradicts (23) if $\{u_n\}$ is bounded for large n. This completes the proof. □

Corollary 3.7. *Let* $\{R_n\}$ *be a bounded solution of* (E). *Then the equation* (E_+) *has no bounded positive solutions of type* M_1^R *or* M_2^R *if (23) holds for all positive constants* c.

The proof follows by observing that since $\{R_n\}$ is bounded, a M_1^R or M_2^R solution $\{y_n\}$ is bounded if and only if $u_n = y_n - R_n$ is bounded. A M_2^R solution is unbounded by Lemma 2.3. A bounded M_1^R solution is excluded by Theorem 3.6 (ii).

Remark 3.8. In view of Remarks 3.2 and 3.5 we may assume, without loss of generality, that $R_n > 0$ in considering the behavior of positive solutions of (E_+) of types M_1^R, M_2^R and M_3^R.

Applying the proof of Theorem 2.10 to the equations (H_+^-) one can obtain information regarding the behavior of M_1^R or M_2^R type solutions of (E_+) for large n.

Theorem 3.9. *Let* $\{R_n\}$ *be a positive solution of equation* (E).

(i) *Equation* (E_+) *has no* M_2^R*–type solution which is bounded away from zero as* $n \to \infty$ *if*

$$\sum_{n=n_0}^{\infty} n q_n = \infty. \tag{24}$$

(ii) *Equation* (E_+) *has no* M_1^R*–type solution which is bounded away from zero as* $n \to \infty$ *if*

$$\sum_{n=n_0}^{\infty} n^{(3)} q_n = \infty. \tag{25}$$

PROOF. We shall prove part (ii) since the proof of part (i) is similar and hence the details are omitted. Let $\{y_n\}$ be a M_1^R–type solution of (E_+) for $n \geq N$. Then $v_n = R_n - y_n$ is a positive solution of (H_+^-) of type M_1 for $n \geq N$ and $\Delta^2(r_n \Delta^2 v_n) > 0$ for $n \geq N_1 \geq N$. If we assume that $\{y_n\}$ is bounded away from zero as $n \to \infty$, then there exists a positive constant c such that $(R_n - v_n) \geq c$ for $n \geq N_2 \geq N_*$. We may suppose $n \geq N_2$ and if $j \geq 2n = N_1$. As in the proof of Theorem 2.10, we may obtain via summation of equation (H_+^-) from n to $j-1$ and elementary estimates

$$\sum_{s=N_*}^{\infty} s^{(3)} q_s \leq 3! 2^3 M C^{-1} y_{N_1},$$

which contradicts (25). □

Finally we consider the nonhomogeneous equation (E_-) under the assumption that $\{R_n\}$ is a solution of (E). We first provide conditions for the nonexistence of M_1^R and M_2^R type solutions.

THEOREM 3.10. *Let* $\{R_n\}$ *be a solution of equation* (E).

(i) *Equation* (E_-) *has no* M_2^R*–type solutions if for all positive constants* c

$$\sum_{n=n_0}^{\infty} n q_n f(R_n + c) = \infty.$$

(ii) *Suppose that for all positive constants* c

$$\sum_{n=n_0}^{\infty} n^{(3)} q_n f(R_n + c) = \infty,$$

then any M_1^R *type solution* $\{y_n\}$ *of* (E_-) *satisfies*

$$\lim_{n \to \infty} (y_n - R_n) = 0.$$

PROOF. The proof is similar to that of Theorems 2.10 and 3.9 and hence the details are omitted. □

Now letting $w_n = \dfrac{\Delta(r_n \Delta v_n)}{v_n}$ and $z_n = \dfrac{\Delta(r_n \Delta v_n)}{r_n \Delta v_n}$ and repeating the procedures which led to Theorem 3.1, we may obtain the following analogue.

THEOREM 3.11. *Let $\{R_n\}$ be a bounded solution of equation (E).*

(i) *Equation (E_-) has no positive M_2^R solution which is bounded away from zero if*

$$\sum_{n=n_0}^{\infty} n^{(2)} q_n = \infty.$$

(ii) *Equation (E_-) has no positive M_1^R solution which is bounded away from zero as $n \to \infty$ if*

$$\sum_{n=n_0}^{\infty} q_n = \infty.$$

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