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ON REDUNDANCE OF ONE OF THE AXIOMS OF A GENERALIZED NORMED SPACE

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Abstract. It is shown that one of the Zalar's axioms of a generalized normed space is redundant. More precisely, Jordan-von Neumann type theorem for a pre-Hilbert module is also valid if that axiom is omitted.

The concept of an H*-algebra was introduced by W. Ambrose in [1] in order to obtain an abstract characterization of Hilbert-Schmidt operators.

Definition 1. *Let* $(A, |\,.\,|)$ *be a complex Banach algebra with an inner product* $\langle\, .\, ,\, .\, \rangle$ *such that* $\langle a, a\rangle = |a|^2$ *for all* $a \in A$. *If for each* $a \in A$ *there exists some* $a^* \in A$ *such that*

$$\langle ab, c\rangle = \langle b, a^*c\rangle \text{ and } \langle ba, c\rangle = \langle b, ca^*\rangle \text{ for all } b, c \in A,$$

then A *is called an H*-algebra.*

Proper H*-algebra is an H*-algebra which satisfies the following equivalent conditions:

(i) $aA = 0 \Rightarrow a = 0$;
(ii) $Aa = 0 \Rightarrow a = 0$.

An H*-algebra A is proper if and only if each $a \in A$ has a unique adjoint $a^* \in A$. From now on, A will denote a proper H*-algebra.

An element $a \in A$ is called

(i) self-adjoint if $a = a^*$;
(ii) a projection if $a = a^* = a^2 \neq 0$;
(iii) normal if $a^*a = aa^*$;
(iv) positive ($a \geq 0$) if $\langle ax, x\rangle \geq 0$ for every $x \in A$.

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It is not difficult to establish that every $a \geq 0$ is self-adjoint. Projections $e, f \in A$ are called mutually orthogonal if $\langle e, f\rangle = \langle f, e\rangle = 0$ which is equivalent to $ef = fe = 0$. A primitive projection is a projection which can not be expressed as the sum of two mutually orthogonal projections. If $e \in A$ is a primitive projection, then $eAe = \mathbf{C}e$ holds. In this article, by a projection base in A we mean a maximal family of mutually orthogonal primitive projections. If $\{e_i\}$ is a projection base in A, then $|a|^2 = \sum_i |ae_i|^2$ holds for every $a \in A$. The existence of a projection base in A, as well as other recalled facts, was proved in the pioneering paper [1].

According to [3], every normal element $a \in A$ can be expressed as $\sum_i \mu_i e_i$, where $\{\mu_i\}$ is a family of complex numbers and $\{e_i\}$ is a projection base which will be called a projection base associated with a.

The trace-class for A was defined to be the set $\tau(A) = \{ab : a, b \in A\}$. It is a self-adjoint ideal of A which is dense in A. There is a norm $\tau(\,.\,)$ on the trace-class such that $(\tau(A), \tau(\,.\,))$ is a Banach algebra. The inequality $\tau(ab) \leq |a||b|$ holds for all $a, b \in A$.

LEMMA 2. *For all* $a, b \in A,\ \tau(a^2 - b^2) \leq |a - b||a + b|$.

PROOF. The expression (used in [4])

$$a^2 - b^2 = \frac{1}{2}(a-b)(a+b) + \frac{1}{2}(a+b)(a-b)$$

implies

$$\begin{aligned} \tau(a^2 - b^2) &\leq \frac{1}{2}\tau\big((a-b)(a+b)\big) + \frac{1}{2}\tau\big((a+b)(a-b)\big) \leq \\ &\leq \frac{1}{2}|a-b||a+b| + \frac{1}{2}|a+b||a-b| = |a-b||a+b|. \end{aligned}$$

□

If $a \in \tau(A)$ is positive, then there is a unique positive element $b \in A$ such that $b^2 = a$; b is called a square root of a. For every $a \in A$, a square root of a^*a is denoted by $[a]$.

LEMMA 3. *For every* $a \in A,\ |[a]| = |a|$.

PROOF.

$$|[a]|^2 = \sum_i |[a]e_i|^2 = \sum_i \langle [a]^2 e_i, e_i\rangle = \sum_i \langle a^*ae_i, e_i\rangle = \sum_i |ae_i|^2 = |a|^2$$

□

Hilbert A-modules were first defined by P. P. Saworotnow in [2]. A Hilbert A-module arises as a generalization of a complex Hilbert space when the complex field is replaced by a proper H*-algebra. We shall omit the axiom of completeness and consider pre-Hilbert A-modules.

Definition 4. *Let H be a right module over A. If a $\tau(A)$-valued mapping $(\,.\,,\,.\,)$ on $H\times H$ has the following properties:*

(i) $(f,g+h)=(f,g)+(f,h)$ *for all* $f,g,h\in H$;
(ii) $(f,ga)=(f,g)a$ *for all* $f,g\in H$ and $a\in A$;
(iii) $(f,g)^*=(g,f)$ *for all* $f,g\in H$;
(iv) $(f,f)\geq 0$ *for every* $f\in H$ *and* $(f,f)=0\Rightarrow f=0$;

then it is called a generalized inner product and $(H,(\,.\,,\,.\,))$ is called a pre-Hilbert A-module.

Naturally, the question of analogous generalization of a complex normed space appeared. The notion of a generalized normed space was introduced by B. Zalar in [4].

Definition 5. *Let H be a right module over A. Let $N:H\to A$ be a mapping with the following properties:*

(i) $N(f)\geq 0$ *for every* $f\in H$;
(ii) $N(f)=0\Rightarrow f=0$;
(iii) $N(fa)=[N(f)a]$ *for all* $f\in H$ *and* $a\in A$;
(iv) $|N(f+g)|\leq |N(f)|+|N(g)|$ *for all* $f,g\in H$;
(v) *If $\{f_\alpha\}$ is a generalized sequence in H such that for all $\varepsilon>0$ there exists α_0 such that for all $\alpha,\beta\geq\alpha_0$ we have $|N(f_\alpha-f_\beta)|<\varepsilon$, then $\{N(f_\alpha)\}$ is a generalized Cauchy sequence in A.*

Then N is called an A-valued generalized norm (A-norm) and (H,N) is called a generalized normed space or a normed A-module.

If A is the H*-algebra of complex numbers, then H is a usual complex normed space.

The main result in [4] is a generalization of the well-known Jordan-von Neumann theorem which characterizes pre-Hilbert spaces among normed spaces:

Theorem 6. *A normed A-module (H,N) satisfies the parallelogram law*

$$N(f+g)^2+N(f-g)^2=2N(f)^2+2N(g)^2 \text{ for all } f,g\in H$$

if and only if H is a pre-Hilbert A-module with respect to the generalized inner product $(\,.\,,\,.\,)$ such that $N(f)^2=(f,f)$ holds for all $f\in H$.

Let us note that axiom (v) represents some sort of A-norm continuity. Zalar questioned if that axiom was redundant. The aim of this article is to show that axiom (v) can be omitted, by proving

Theorem 7. *Let H be a right module over A. If $N:H\to A$ is a mapping that satisfies the axioms (i)-(iv) in Definition 5 and the parallelogram law, then it also satisfies (v).*

The proof of this theorem consists of several steps.

STEP 1. *If $f \in H$ and $e \in A$ is a primitive projection, then there exists a nonnegative number λ such that $N(fe) = \lambda e$.*

PROOF. Axiom (iii) implies

$$N(fe)^2 = [N(f)e]^2 = eN(f)^2 e \in eAe = \mathbf{C}e.$$

Hence there exists $\mu \in \mathbf{C}$ such that $N(fe)^2 = \mu e$. But

$$\mu |e|^2 = \langle \mu e, e\rangle = \langle N(fe)^2 e, e\rangle = |N(fe)e|^2$$

implies $\mu \geq 0$. If we put $\lambda = \sqrt{\mu}$, then we have $N(fe)^2 = \lambda^2 e = (\lambda e)^2$. Since $N(fe) \geq 0$ and $\lambda e \geq 0$, it follows that $N(fe) = \lambda e$. ∎

STEP 2. *For all $f \in H$ and $a \in A$, $|N(fa)| = |N(f)a|$.*

PROOF. Directly from axiom (iii) and Lemma 3. ∎

STEP 3. *If $\{e_i\}$ is a projection base in A, then $\sum_i N(fe_i) \in A$ for every $f \in H$ and $|\sum_i N(fe_i)| = |N(f)|$ holds.*

PROOF. Step 1 implies the existence of a family $\{\lambda_i\}$ of nonnegative numbers such that $N(fe_i) = \lambda_i e_i$ holds for every i. Consequently, the family $\{N(fe_i)\}$ is orthogonal. Moreover,

$$\sum_i \left| N(fe_i) \right|^2 \overset{\text{Step 2}}{=} \sum_i |N(f)e_i|^2 = |N(f)|^2.$$

Thus, $\sum_i N(fe_i) \in A$ and $|\sum_i N(fe_i)| = |N(f)|$. ∎

STEP 4. *If $f, g \in H$ and $e \in A$ is a primitive projection, then the inequality*

$$|N(fe + ge)| \leq |N(fe) + N(ge)|$$

holds.

PROOF. According to axiom (iv),

$$|N(fe + ge)| \leq |N(fe)| + |N(ge)|.$$

Let $\alpha, \beta \geq 0$ be such that $N(fe) = \alpha e$ and $N(ge) = \beta e$ hold. We have

$$\begin{aligned} |N(fe)| + |N(ge)| &= |\alpha e| + |\beta e| = \alpha |e| + \beta |e| = \\ &= |(\alpha + \beta)e| = |\alpha e + \beta e| = |N(fe) + N(ge)|. \end{aligned}$$

∎

STEP 5. *For all $f, g \in H$,*

$$|N(f+g)|^2 + |N(f-g)|^2 = |N(f) + N(g)|^2 + |N(f) - N(g)|^2.$$

PROOF. Let $\{e_i\}$ be a projection base in A. For every i, we have

$$\begin{aligned} |N(f+g)e_i|^2 + |N(f-g)e_i|^2 &= \langle N(f+g)^2 e_i, e_i\rangle + \langle N(f-g)^2 e_i, e_i\rangle = \\ &= \langle (N(f+g)^2 + N(f-g)^2)e_i, e_i\rangle = \langle (2N(f)^2 + 2N(g)^2)e_i, e_i\rangle = \\ &= 2\langle N(f)^2 e_i, e_i\rangle + 2\langle N(g)^2 e_i, e_i\rangle = 2|N(f)e_i|^2 + 2|N(g)e_i|^2. \end{aligned}$$

Summing over the set of all i gives

$$|N(f+g)|^2 + |N(f-g)|^2 = 2|N(f)|^2 + 2|N(g)|^2.$$

Since the norm $|\,.\,|$ in A satisfies the parallelogram law, the right side is equal to $|N(f)+N(g)|^2 + |N(f)-N(g)|^2$. ∎

STEP 6. *If $\{e_i\}$ is a projection base in A and $f, g \in H$, then*

$$\Big|\sum_i N(fe_i) - \sum_i N(ge_i)\Big| \le |N(f-g)|$$

holds.

PROOF. Since Step 5 implies

$$|N(fe_i + ge_i)|^2 + |N(fe_i - ge_i)|^2 = |N(fe_i) + N(ge_i)|^2 + |N(fe_i) - N(ge_i)|^2$$

and Step 4

$$|N(fe_i + ge_i)| \le |N(fe_i) + N(ge_i)|,$$

therefore for every i we have

$$|N(fe_i) - N(ge_i)| \le |N(fe_i - ge_i)| = |N((f-g)e_i)| \overset{\text{Step 2}}{=} |N(f-g)e_i|$$

and hence

$$\sum_i |N(fe_i) - N(ge_i)|^2 \le \sum_i |N(f-g)e_i|^2 = |N(f-g)|^2.$$

According to Step 3, $\sum_i N(fe_i) \in A$ and $\sum_i N(ge_i) \in A$. Applying Step 1, we conclude that for every i there exist $\alpha_i \ge 0$ and $\beta_i \ge 0$ such that $N(fe_i) = \alpha_i e_i$ and $N(ge_i) = \beta_i e_i$. Now we have

$$\begin{aligned} \Big|\sum_i N(fe_i) - \sum_i N(ge_i)\Big|^2 &= \Big|\sum_i \alpha_i e_i - \sum_i \beta_i e_i\Big|^2 = \\ &= \Big|\sum_i (\alpha_i - \beta_i)e_i\Big|^2 = \sum_i |\alpha_i e_i - \beta_i e_i|^2 = \\ &= \sum_i |N(fe_i) - N(ge_i)|^2 \le |N(f-g)|^2. \end{aligned}$$

∎

STEP 7. *For all $f, g \in H$,*

$$|N(f) - N(g)|^2 \le |N(f-g)| \cdot \Big(|N(f)| + |N(g)|\Big).$$

Proof. Let $\{e_i\}$ be a projection base in A associated with $N(f)^2-N(g)^2$ and let $\{\mu_i\}$ be a family of complex numbers such that $N(f)^2-N(g)^2=\sum_i \mu_i e_i$. Step 1 implies the existence of families $\{\alpha_i\}$ and $\{\beta_i\}$ of nonnegative numbers such that $N(fe_i)=\alpha_i e_i$ and $N(ge_i)=\beta_i e_i$ hold for every i. Since

$$\begin{aligned}
\Big(\sum_i N(fe_i)\Big)^2-\Big(\sum_i N(ge_i)\Big)^2 &= \Big(\sum_i \alpha_i e_i\Big)^2-\Big(\sum_i \beta_i e_i\Big)^2 = \\
= \ & \sum_i(\alpha_i e_i)^2-\sum_i(\beta_i e_i)^2=\sum_i N(fe_i)^2-\sum_i N(ge_i)^2 = \\
= \ & \sum_i [N(f)e_i]^2-\sum_i [N(g)e_i]^2=\sum_i e_i N(f)^2 e_i-\sum_i e_i N(g)^2 e_i = \\
= \ & \sum_i e_i\Big(N(f)^2-N(g)^2\Big)e_i=\sum_i \mu_i e_i = N(f)^2-N(g)^2,
\end{aligned}$$

we have

$$\begin{aligned}
\tau\big(N(f)^2-N(g)^2\big) &= \tau\Big(\big(\sum_i N(fe_i)\big)^2-\big(\sum_i N(ge_i)\big)^2\Big) \overset{\text{Lemma 2}}{\leq} \\
\leq \ & \Big|\sum_i N(fe_i)-\sum_i N(ge_i)\Big|\cdot\Big|\sum_i N(fe_i)+\sum_i N(ge_i)\Big| \overset{\text{Step 6}}{\leq} \\
\leq \ & |N(f-g)|\cdot\Big|\sum_i N(fe_i)+\sum_i N(ge_i)\Big| \leq \\
\leq \ & |N(f-g)|\cdot\Big(\Big|\sum_i N(fe_i)\Big|+\Big|\sum_i N(ge_i)\Big|\Big) \overset{\text{Step 3}}{=} \\
= \ & |N(f-g)|\cdot\Big(|N(f)|+|N(g)|\Big).
\end{aligned}$$

Lemma 10(3) from [4] completes the proof. ∎

Step 8. *If $\{f_\alpha\}$ is a generalized sequence in H such that for all $\varepsilon>0$ there exists α_0 such that for all $\alpha,\beta\geq\alpha_0$ we have $|N(f_\alpha-f_\beta)|<\varepsilon$, then $\{N(f_\alpha)\}$ is a generalized Cauchy sequence in A.*

Proof. There exists α' such that

$$\begin{aligned}
\alpha\geq\alpha' \quad \Rightarrow \quad & |N(f_\alpha-f_{\alpha'})|<1 \Rightarrow |N(f_\alpha)| = \big|N\big((f_\alpha-f_{\alpha'})+f_{\alpha'}\big)\big| \leq \\
\leq \quad & |N(f_\alpha-f_{\alpha'})|+|N(f_{\alpha'})|<1+|N(f_{\alpha'})| \overset{def}{=} M.
\end{aligned}$$

For a given $\varepsilon>0$, there exists α'' with the property

$$\alpha,\beta\geq\alpha'' \Rightarrow |N(f_\alpha-f_\beta)|<\frac{\varepsilon^2}{2M}.$$

There is α_0 such that $\alpha'\leq\alpha_0$ and $\alpha''\leq\alpha_0$ hold. Step 7 now gives

$$\alpha,\beta\geq\alpha_0 \Rightarrow |N(f_\alpha)-N(f_\beta)|^2<\frac{\varepsilon^2}{2M}\cdot(M+M)=\varepsilon^2.$$

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