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2–ISOMETRIC OPERATORS

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Abstract. An operator T on a complex Hilbert space is called a 2–isometry if $T^{*2}T^2 - 2T^*T + I = 0$. Our underlying purpose in this article is to investigate some algebraic and spectral properties of 2–isometries.

1. Introduction

Let H be a complex Hilbert space. By an operator on H, we shall mean a bounded linear transformation from H to H. Let $\sigma(T)$, $\pi(T)$, $\pi_0(T)$, $\pi_{00}(T)$ and $w(T)$, respectively denote the spectrum, the approximate point spectrum, the point spectrum, the set of eigenvalues with finite multiplicity and the Weyl spectrum of an operator T. We use the symbol $\partial\sigma(T)$ for the boundary of $\sigma(T)$. If for an operator T, $w(T) = \sigma(T) \sim \pi_{00}(T)$, then we say that the Weyl's theorem holds for T.The spectral radius and the numerical radius of T will be denoted by $r(T)$ and $|W(T)|$ respectively. If $r(T) = |W(T)|$, then T is called a spectraloid operator. By saying that an operator T is power bounded, we mean that there exists some $M > 0$ such that $\|T^n\| \leq M$ for each positive integer n. According to [1], an operator T is defined to be a 2–isometry if $T^{*2}T^2 - 2T^*T + I = 0$. In the present note, we explore some properties of 2–isometries.

Clearly every isometry is a 2–isometry. According to [1, Proposition 1.23], an invertible 2–isometry turns out to be a unitary operator. It is obvious from the definition that every 2–isometry is left invertible. In particular if both T and T^* are 2–isometries then T is invertible and so must be unitary.

2. Results

Theorem 2.1. *A power of a 2–isometry is again a 2–isometry.*

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PROOF. Let T be a 2–isometry. We prove the assertion by using the mathematical induction. Since T is a 2–isometry, the result is true for $n = 1$. Now assume that the result is true for $n = k$, i.e.,

$$T^{*2k}T^{2k} - 2T^{*k}T^k + I = 0. \tag{2.1}$$

Then

$$\begin{aligned}
& T^{*2(k+1)}T^{2(k+1)} - 2T^{*k+1}T^{k+1} + I \\
= \;& T^{*2}(T^{*2k}T^{2k})T^2 - 2T^{*k+1}T^{k+1} + I \\
= \;& T^{*2}(2T^{*k}T^k - I)T^2 - 2T^{*k+1}T^{k+1} + I \qquad \text{(by (2.1))} \\
= \;& 2T^{*k+2}T^{k+2} - T^{*2}T^2 - 2T^{*k+1}T^{k+1} + I \\
= \;& 2T^{*k}(T^{*2}T^2 - T^*T)T^k - T^{*2}T^2 + I \\
= \;& 2T^{*k}(T^*T - I)T^k - T^{*2}T^2 + I \qquad (T \text{ is a 2–isometry}) \\
= \;& (2T^{*k+1}T^{k+1} - 2T^{*k}T^k) - T^{*2}T^2 + I \\
= \;& 2(T^{*2}T^2 - T^*T) - T^{*2}T^2 + I \qquad \text{(by (2.1))} \\
= \;& T^{*2}T^2 - 2T^*T + I \\
= \;& 0.
\end{aligned}$$

This shows that the result is true for $n = k + 1$: thus T^n is a 2–isometry for each n. ∎

It is well known and obvious that a unilateral weighted shift is an isometry iff all its weights lie on the unit circle. In the next result, we obtain a necessary and sufficient condition under which a non–isometric unilateral weighted shift is a 2–isometry.

THEOREM 2.2. *A non–isometric unilateral weighted shift T with weights $\{\alpha_n\}$ is a 2–isometry if and only if*

(i) $|\alpha_n|^2|\alpha_{n+1}|^2 - 2|\alpha_n|^2 + 1 = 0$ *for each n;*
(ii) $|\alpha_n| \neq 1$ *for each n.*

PROOF. Suppose T is a 2–isometry. If $\{e_n\}$ is an orthonormal base for H, then $Te_n = \alpha_n e_{n+1}$ and hence (i) follows. Suppose (ii) is false. Select the least positive integer k such that $|\alpha_k| = 1$. If $k > 1$, then (i) gives $|\alpha_{k-1} = 1$ which is contrary to the selection of k. Therefore $|\alpha_1 = 1$. Using the induction argument and (i), one can show that $|\alpha_n| = 1$ for each positive integer n. But this will contradict our assumption that T is non–isometric. Hence we conclude that (ii) is true. The converse assertion is obvious. ∎

COROLLARY 2.3. *Let T be a non–isometric unilateral weighted shift with weights $\{\alpha_n\}$. If T is a 2–isometry, then the following assertions hold.*

(i) $\{|\alpha_n|\}$ *is a strictly decreasing sequence of real numbers converging to 1.*
(ii) $\sqrt{2} > |\alpha_n| > 1$ *for each $n > 1$.*

PROOF. (i) Suppose $|\alpha_{n+1}| \geq |\alpha_n|$ for some n. Then by Theorem 2.2 (i), we find $0 \geq (1 - |\alpha_n|^2)^2$ or $|\alpha_n| = 1$. But this contradicts Theorem 2.2 (ii). Thus $\{|\alpha_n|\}$ is a strictly decreasing sequence of real numbers and so must be convergent. By Theorem 2.2 (i), we infer that $|\alpha_n| \to 1$.

(ii) Rewriting equality (i) of Theorem 2.2 as

$$|\alpha_{n+1}|^2 - 2 + 1/|\alpha_n|^2 = 0 \tag{2.2}$$

we get $\sqrt{2} > |\alpha_n|$ for each $n > 1$. By (i) and Theorem 2.2 (ii), $|\alpha_n| > 1$. This finishes the proof of (ii). □

THEOREM 2.4. *A power bounded 2–isometry is an isometry.*

PROOF. Let T be a power bounded 2–isometry. Then there exists a positive real number M such that

$$\|T^n\| \leq M \tag{2.3}$$

for $n = 1, 2, 3, \ldots$. The definition of a 2–isometry yields

$$\|T^2\|^2 + 1 = 2\|T\|^2. \tag{2.4}$$

Since T^n is also a 2–isometry by Theorem 2.1, an induction argument shows that

$$\|T^{2^n}\|^2 = 2^n\|T\|^2 - (2^n - 1) \tag{2.5}$$

for every positive integer n. Now (2.3) and (2.5) will give

$$M^2/2^n \geq \|T\|^2 - 1 + 1/2^n \geq 0.$$

Letting $n \to \infty$, we find $\|T\| = 1$. In particular, $I \geq T^*T$. Since $T^*T \geq I$ [1, Proposition 1.5], we conclude $T^*T = I$. □

REMARK 2.5. Above theorem can be used to show that unlike isometries, the class of 2–isometries is not bounded. To see this, use Theorem 2.2 to construct a 2–isometry T, which is not an isometry. Then by Theorem 2.4, we see that for each $M > 0$, there corresponds a positive integer n such that $\|T^n\| > M$. Since Theorem 2.1 says that T^n is also a 2–isometry, we conclude that the class of 2–isometries contains operators with arbitrarily large norm.

COROLLARY 2.6. *A 2–isometry similar to a spectraloid operator is an isometry.*

PROOF. Let T be a 2–isometry. Suppose it is similar to a spectraloid operator A. Then $r(T^n) = r(A^n) = |W(A^n)|$ for $n = 1, 2, 3, \ldots$. Since $r(T) = 1$, [1], we find $1 = |W(A^n)|$ and hence $\|A^n\| \leq 2$ for each n. Now the similarity of T and A shows that T is power bounded; thus the result follows from the preceding theorem. □

REMARK 2.7. Above corollary shows that unlike the class of isometries, the class of 2–isometries fails to be a subclass of spectraloid operators.

COROLLARY 2.8. *If T is a 2–isometry, then $1 \in \sigma(T^*T)$.*

PROOF. Suppose to the contrary that $1 \notin (T^*T)$. Then the operator $A = T^*T - I$ is invertible. Moreover $A \geq 0$ [1, Proposition 1.5]. From the definition of a 2–isometry it follows that $\sigma T^*AT = A$ or $(A^{1/2}TA^{-1/2})^*(A^{1/2}TA^{-1/2}) = I$ where $A^{1/2}$ denotes the positive square root of A. Thus T is similar to an isometry and so must be an isometry by virtue of Corollary 2.6. This contradicts our supposition that $1 \notin \sigma(T^*T)$. □

In the rest of the article, we shall obtain some spectral properties of 2-isometries.

THEOREM 2.9. *Let T be a 2–isometry. Then*

(i) *$z \in \pi(T)$ implies $z^* \in \pi(T^*)$.*
(ii) *$z \in \pi_0(T)$ implies $z^* \in \pi_0(T^*)$.*
(iii) *Eigenvectors of T corresponding to distinct eigen-values are orthogonal.*

PROOF. (i) Let $z \in \pi(T)$. Choose a sequence $\{x_n\}$ of unit vectors such that $(T - zI)x_n \to 0$. Then $(T^{*2}T^2 - z^2T^{*2})x_n \to 0$ and $T^*Tx_n - zT^*x_n \to 0$. The hypothesis that T is a 2–isometry yields $0 = T^{*2}T^2 - 2T^*T + I = T^{*2}T^2 - z^2T^{*2} - 2T^*T + 2zT^* + z^2T^{*2} - 2zT^* + I$. This will imply $z^2T^{*2}x_n - 2zT^*x_n + x_n \to 0$. Since $\pi(T)$ is a subset of the unit circle [1], we find $(T^* - z^*I)^2x_n \to 0$. From this it follows that $z^* \in \pi(T^*)$.

(ii) The argument is similar to one given in (i).

(iii) Let λ and μ be distinct eigen-values of T. Suppose $Tx = \lambda x$ and $Ty = \mu y$. Then $0 = \langle (T^{*2}T^2 - 2T^*T + I)x, y\rangle = \langle T^2x, T^2y\rangle - 2\langle Tx, Ty\rangle + \langle x, y\rangle = (\lambda^2\mu^{*2} - 2\lambda\mu^* + 1)\langle x, y\rangle$. Since $\lambda \neq \mu$ with $|\lambda| = 1 = |\mu|$, $\lambda^2\mu^{*2} - 2\lambda\mu^* + 1 = (\lambda/\mu - 1)^2 \neq 0$. This leads to $\langle x, y\rangle = 0$ which proves the assertion. □

THEOREM 2.10. *The spectrum of a 2–isometry is the closed unit disc provided it is non–unitary.*

PROOF. Let T be a non–unitary 2–isometry. Then $0 \in \sigma(T) \sim \pi(T)$. Since $\partial\sigma(T) \subseteq \pi(T)$, 0 turns out to be an interior point of $\sigma(T)$. Therefore we can find the largest positive number r such that $\{z : |z| \leq r\}$ is contained in $\sigma(T)$. It is possible to select a complex number z in $\partial\sigma(T)$ such that $r = |z|$. Since $\partial\sigma(T) \subseteq \pi(T) \subseteq \{z : |z| = 1\}$[1], $r = 1$. Consequently we find $\sigma(T) = \{z : |z| \leq 1\}$. □

COROLLARY 2.11. *If T is a 2–isometry, then each isolated point in its spectrum is an eigen–value.*

PROOF. If $\sigma(T)$ has an isolated point, then it is clear from the above theorem that T is unitary and hence the result follows. □

Corollary 2.12. *Let T be a 2–isometry. If the Lebesgue planar measure of $\sigma(T)$ is zero, then T is unitary.*

Corollary 2.13. *The Weyl's theorem holds for 2–isometries.*

Proof. The result holds if T is unitary. Assume that T is non-unitary. Then Theorem 2.10 shows that $\pi_{00}(T) = \emptyset$. Also by Theorem 2.9 (ii) and Lemma 3 of [2], $\sigma(T) \sim \pi_{00}(T) \subseteq w(T)$ and hence $\sigma(T) \subseteq w(T)$. This completes the argument. □

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